ASPECTS OF D-BRANE DYNAMICS IN SUPERSTRING THEORY

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Ph.D. Thesis

- SUPERSTRING THEORY AND D-BRANES.

  D-BRANE DYNAMICS.

  LARGE AND SHORT DISTANCES DESCRIPTIONS.

- POINT LIKE OBJECTS IN D=4 FROM D-BRANES.

  PHASE-SHIFT FOR CONSTANT VELOCITIES.

  RADIATION OF PARTICLES.

  F. Hussain, R. Iengo, C. Núñez: PLB 409, NPB 517

- DYONIC EXTREMAL BLACK HOLES IN D=4.

  M. Bertolini, P. Fré, R. Iengo: NPB 522, PLB 431

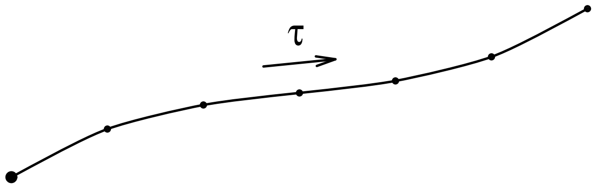
- SPIN EFFECTS IN D-BRANE DYNAMICS.

  SCALE INVARIANCE FOR SMALL VELOCITY.

  J.F. Morales, M. Serone: PLB 417, NPB 5xx

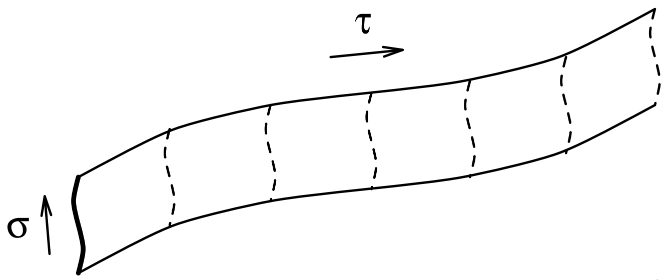
SUPERSTRINGS AND D-BRANES

Field theory describes 0-dimensional point-like objects.

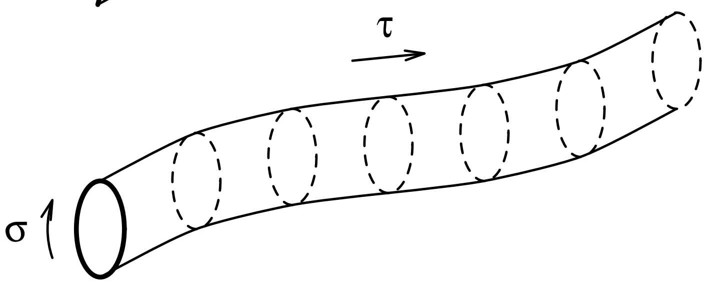


Point particle

String theory describes 1-dimensional extended objects.



Open strings



Closed strings

There are five consistent perturbative string theories, in D=10:

Type IIA,B.

N=2 SUSY. Closed strings

Type I.

N=1 SUSY. Unoriented open + closed strings.

Heterotic $SO(32)/E_8 \times E_8$.

N=1 SUSY. Closed strings

The oscillation modes can be interpreted as particles. The mass scale is the Plank mass, and at low energy only massless modes are relevant. The effective actions are various versions of SUGRA and SYM.

The five string theories are actually related by dualities, relating two theories through a specific map between their free parameters. For example:

S-dualities

- Type I $\Leftrightarrow$ Heterotic
- Type II on $K3$ $\Leftrightarrow$ Heterotic on T^4.

T-dualities

- Type IIA on S_1 $\Leftrightarrow$ Type IIB on S_1
- Heterotic $SO(32)$ on S_1 $\Leftrightarrow$ Heterotic $E_8 \times E_8$ on S_1.

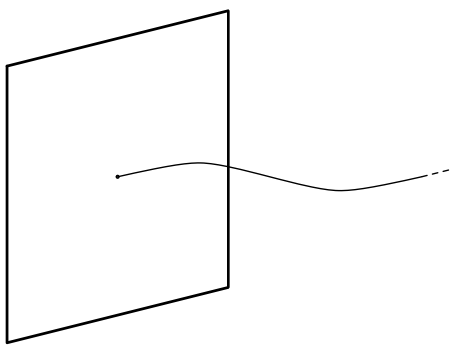
These dualities connecting the five string theories show that they are different limits of a fundamental D=11 supermembrane theory called M-theory. It is therefore important to study non-perturbative aspects of superstring theories.

D-branes

They arise as hyper-planes on which open strings can end. More precisely, two kinds of boundary conditions are allowed for open string end-points:

- Neumann: $\partial_n X = 0$, $\psi = \pm\tilde{\psi}$.
- Dirichlet: $\partial_t X = 0$, $\psi = \mp\tilde{\psi}$.

Dp-brane:



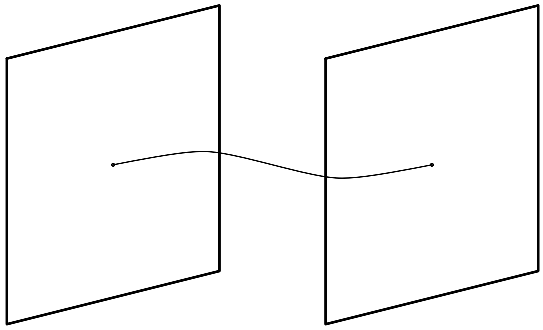
X^α, $\alpha = 0, 1, .., p$: Neumann

X^i, $i = p+1, ..., 9-p$: Dirichlet

Lorentz invariance: $SO(9,1) \to SO(p,1) \times SO(9-p)$.

SUSY: $Q, \tilde{Q} \to Q^+ = \frac{1}{\sqrt{2}}(Q + M\tilde{Q})$ with $M = \prod(\Gamma^i\Gamma^{11})$.

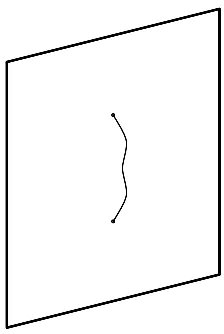
In general, the two end-points of an open string end on two D-branes.



In conventional open string theory, both ends lie on D9-branes, and one recovers ten-dimensional Lorentz invariance.

A generic Dp-brane corresponds to an exact BPS string backgrounds breaking Lorentz invariance and preserving $\frac{1}{2}$ of the SUSY.

D-brane are dynamical objects. Fluctuations of the world-volume are described by open strings starting and ending on the D-brane.

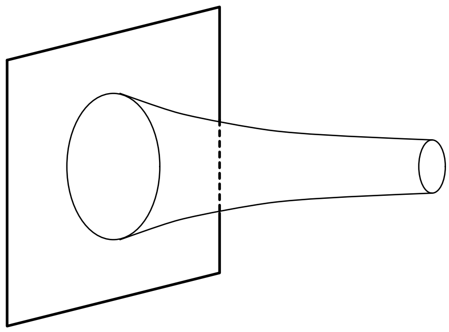


Massless modes of open strings:

D=10 N=1 SYM $\rightarrow$ D=p+1

$(A_\mu; \lambda) \rightarrow (A_\alpha, q^i; \lambda^a)$

D-brane are also sources of closed strings. In fact, the open strings living on the world-volume can close and then come out of the world-volume.



Massless modes of closed strings:

D=10 N=2 SUGRA

$(g_{\mu\nu}, b_{\mu\nu}, \phi; C_{\mu_1 \ldots \mu_n}; \lambda^1, \chi^1; \lambda^2, \chi^2)$

The effective action of a Dp-brane can be determined by requiring conformal invariance of the string action on a disk ending on it. One finds (Leigh):

$$S = -\hat{T}_p \int_{W_{p+1}} d^{p+1}\xi e^{-\phi}\sqrt{-\det\left(\hat{g}_{\mu\nu} + \mathcal{F}_{\mu\nu}\right)} - \hat{\mu}_p \int_{W_{p+1}} C \wedge e^{\mathcal{F}} \wedge \sqrt{\mathcal{A}}\Big|_{(p+1)}$$

Here $\mathcal{F}_{\mu\nu} = 2\pi\alpha' F_{\mu\nu} - \hat{b}_{\mu\nu}$ and $\mathcal{A}(\mathcal{R})$ is the Roof genus. Moreover:

$$\hat{T}_p = \hat{\mu}_p = \sqrt{\alpha'}^{-1}\left(2\pi\sqrt{\alpha'}\right)^{-p}$$

At low energy or large distances, D-branes appear therefore as sources of the massless modes of closed strings. Consider $\mathcal{F} = 0$ and trivial topology ($\mathcal{A} = 1$). In the Einstein frame and in units of $\sqrt{2}\kappa_{(10)} = (2\pi)^{7/2}\alpha'^2$:

$$T_p = \mu_p = \sqrt{2\pi}\left(2\pi\sqrt{\alpha'}\right)^{3-p}$$
$$a_p = \frac{p-3}{4}\sqrt{2\pi}\left(2\pi\sqrt{\alpha'}\right)^{3-p}$$

Since $H_{(p)} = {}^*H_{(10-p)}$, Dp and D(6-p)-branes are magnetically dual. The Dirac quantization condition $\mu_p\mu_{6-p} = 2\pi n$ is satisfied with $n = 1$.

D-branes carry therefore elementary quanta of RR charge. They are the exact quantum description of BPS SUGRA solutions called p-branes.

The p-brane solution is

$$\begin{cases} g_{\alpha\beta} = H_p^{-1/2}\eta_{\alpha\beta}\ , \quad g_{ij} = H_p^{1/2}\delta_{ij} \\ C_{\alpha_1\ldots\alpha_{p+1}} = \epsilon_{\alpha_1\ldots\alpha_{p+1}}\left(H_p^{-1} - 1\right) \\ \phi = \dfrac{3-p}{4}\ln H_p \end{cases}$$

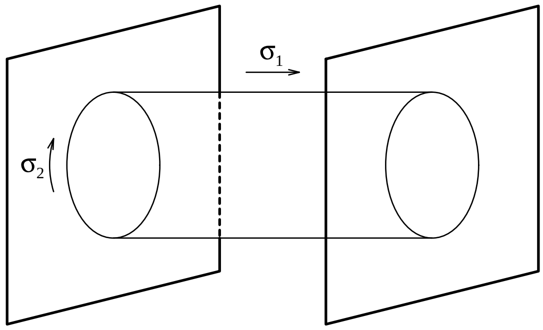
where

$$H_p(r) = 1 + 2\kappa_{(10)}^2\hat{T}_p\Delta_{(9-p)}(r)$$

The charges of the p-brane solution are proportional to those of the Dp-brane.

D-BRANE DYNAMICS

D-branes can also interact among themselves. The leading string diagram is a cylinder joining the two D-branes, and can be interpreted equivalently as an exchange of closed strings, or as a loop of open strings.



Phase-shift.

The interaction can be interpreted as a classical force mediated by closed strings or as a quantum Casimir-like force induced by vacuum fluctuations of open strings.

Open string channel

Take $\sigma_1 = \sigma \in [0, \pi]$ and $\sigma_2 = \tau \in [0, t]$. The boundary conditions are:

$$\partial X^\mu = M^\mu{}_\nu \bar\partial \bar X^\nu \; , \quad \psi^\mu = \eta M^\mu{}_\nu \tilde\psi^\nu$$

The diagonal matrix $M^\mu{}_\nu$ has entry ± 1 for Neumann and Dirichlet directions. Here $\eta = \pm 1$ and $\eta_1 \eta_2 = \pm 1$ correspond to the R and NS sectors.

The one-loop effective action is

$$\mathcal{A} = \int_0^\infty \frac{dt}{t} Z(t)$$

where $(P_{GSO} = \frac{1}{2}(1 + (-1)^F))$

$$Z(t) = \mathrm{STr}[P_{GSO}\, e^{-\frac{\pi}{2}tH}]$$
$$= \frac{1}{2}\Big(\mathrm{Tr}_{NS}[e^{-\frac{\pi}{2}tH}] + \mathrm{Tr}_{NS}[(-1)^F e^{-\frac{\pi}{2}tH}] - \mathrm{Tr}_R[e^{-\frac{\pi}{2}tH}] - \mathrm{Tr}_R[(-1)^F e^{-\frac{\pi}{2}tH}]\Big)$$

One finds (Polchinski)

$$\mathcal{A} = \frac{V_{p+1}}{(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dt}{t^{\frac{p+3}{2}}} e^{-\frac{r^2}{4\pi\alpha'}t} \frac{1}{2} \sum_{\alpha=2}^{4} (-1)^{1+\alpha} \frac{\vartheta_\alpha^4(0|\frac{it}{2})}{\eta^{12}(\frac{it}{2})} = 0$$

where r is the transverse distance.

For a Dp-brane moving with $v = \mathrm{th}\pi\epsilon$ the boundary conditions get rotated:

$$M^\mu{}_\nu \to M^\mu{}_\nu(\epsilon)$$

Repeating the computation for two moving Dp-branes one finds (Bachas)

$$\mathcal{A} = \frac{V_p}{(2\pi\sqrt{\alpha'})^p} \int_0^\infty \frac{dt}{t^{\frac{p+2}{2}}} e^{-\frac{b^2}{4\pi\alpha'}t} \frac{1}{2} \sum_{\alpha=2}^{4} (-1)^{1+\alpha} \frac{\vartheta_\alpha(\frac{\epsilon t}{2}|\frac{it}{2})\vartheta_\alpha^3(0|\frac{it}{2})}{\vartheta_1(\frac{\epsilon t}{2}|\frac{it}{2})\eta^9(\frac{it}{2})} \neq 0$$

where $\epsilon = \epsilon_1 - \epsilon_2$ and b is the impact parameter.

Closed string channel

Take $\sigma_1 = \tau \in [0, l]$ and $\sigma_2 = \sigma \in [0, 2\pi]$. The boundary conditions are

$$\partial X^\mu = -M^\mu{}_\nu \bar{\partial} \bar{X}^\nu \ , \quad \psi^\mu = i\eta M^\mu{}_\nu \tilde{\psi}^\nu$$

These define a unique closed string eigenstate $|B, \eta\rangle$ which represents the corresponding D-brane.

The solution for the boundary state is $|B, \eta\rangle = |B\rangle_B \otimes |B, \eta\rangle_F$ with

$$|B\rangle_B = \exp\left\{\sum_{n=1}^\infty \left(\frac{1}{n} M_{\mu\nu} a^\mu{}_{-n} \tilde{a}^\nu{}_{-n}\right)\right\} |\Omega\rangle_B$$

$$|B, \eta\rangle_F = \exp\left\{-i\eta \sum_{n>0}^\infty \left(M_{\mu\nu} \psi^\mu{}_{-n} \tilde{\psi}^\nu{}_{-n}\right)\right\} |\Omega, \eta\rangle_F$$

Fermions have integer or half-integer moding in the RR and NSNS sectors.

The vacua implement the zero mode boundary conditions and are

$$|\Omega\rangle_B = \delta^{(9-p)}\left(x^i - Y^i\right)|0\rangle = \int \frac{d^{9-p}k}{(2\pi)^{9-p}} e^{ik\cdot Y} |k^i\rangle$$

$$|\Omega, \eta\rangle_F = \begin{cases} |0\rangle \ , & \text{NSNS} \\ \mathcal{M}_{\alpha\beta}|\alpha\rangle|\tilde{\beta}\rangle \ , \quad \mathcal{M} = C\Gamma^0 ....\Gamma^p \dfrac{1 - i\eta\Gamma^{11}}{1 - i\eta} \ , & \text{RR} \end{cases}$$

The arbitrary sign η has to do with the GSO-projection. In fact, the overall sign is irrelevant and ($P_{GSO} = \frac{1}{2}(1+(-1)^F)$, $\tilde{P}_{GSO} = \frac{1}{2}(1+(-1)^{\tilde{F}})$)

$$|B\rangle = P_{GSO}\tilde{P}_{GSO}|B,+\rangle = \frac{1}{2}\left(|B,+\rangle - |B,-\rangle\right)$$

One has to choose a Type IIA or IIB projection for p even or odd.

The amplitude is obtained by inserting a closed string propagator

$$\Delta = \frac{1}{H} = \int_0^\infty dl e^{-lH}$$

between the two boundary states

$$\begin{aligned}\mathcal{A} &= \sum \langle B_1|P_{GSO}\tilde{P}_{GSO}\Delta|B_2\rangle \\ &= \frac{1}{2}\int_0^\infty dl \left\{\langle B_1,+|e^{-lH}|B_2,+\rangle_{NSNS} - \langle B_1,+|e^{-lH}|B_2,-\rangle_{NSNS}\right. \\ &\qquad\qquad \left. +\langle B_1,+|e^{-lH}|B_2,+\rangle_{RR} - \langle B_1,+|e^{-lH}|B_2,-\rangle_{RR}\right\}\end{aligned}$$

The result is

$$\mathcal{A} = \frac{V_{p+1}}{2^4(2\pi\sqrt{\alpha'})^{p+1}}\int_0^\infty \frac{dl}{l^{\frac{9-p}{2}}} e^{-\frac{r^2}{4\pi\alpha' l}} \frac{1}{2}\sum_{\alpha=2}^{4}(-1)^{1+\alpha}\frac{\vartheta_\alpha^4(0|2il)}{\eta^{12}(2il)} = 0$$

The boundary state for a Dp-brane moving with $v = \text{th}\pi\epsilon$ is obtained through a boost (Billó, Cangemi, Di Vecchia)

$$|B,\epsilon\rangle = e^{-i\epsilon_i J^{0i}}|B\rangle$$

In particular

$$M^\mu{}_\nu \longrightarrow M^\mu{}_\nu(\epsilon)$$

Repeating the computation one finds

$$\mathcal{A} = \frac{V_p}{2^3(2\pi\sqrt{\alpha'})^p}\int_0^\infty \frac{dl}{l^{\frac{8-p}{2}}} e^{-\frac{b^2}{4\pi\alpha' l}} \frac{1}{2}\sum_{\alpha=2}^{4}(-1)^{1+\alpha}\frac{\vartheta_\alpha(i\epsilon|2il)\vartheta_\alpha^3(0|2il)}{\vartheta_1(i\epsilon|2il)\eta^9(2il)} \neq 0$$

The expressions obtained in the two channels are identical ($l \leftrightarrow 1/t$). The two open and closed string descriptions are equivalent. Their truncations to the lightest modes are instead relevant in different regimes.

Short distance limit: $b \ll l_s = \sqrt{\alpha'}$

The amplitude is dominated by loops of light open strings

$$\mathcal{A}_{short} = \frac{V_p}{2(4\pi)^{\frac{p}{2}}} \int_0^\infty \frac{dt}{t^{1+\frac{p}{2}}} e^{-\left(\frac{b}{2\pi\alpha'}\right)^2 t} \frac{6 + 2\mathrm{ch}2\frac{\pi\epsilon}{2\pi\alpha'}t - 8\mathrm{ch}\frac{\pi\epsilon}{2\pi\alpha'}t}{\mathrm{sh}\frac{\pi\epsilon}{2\pi\alpha'}t}$$

This is the one-loop effective action for the N=1 SU(2) SYM theory reduced from D=10 to D=p+1 describing the open strings living on the two Dp-branes. For $b \neq 0$, the theory is broken to $U(1)$ in the Coulomb branch. By T-duality, the relative velocity corresponds to $E = \pi\epsilon$, and the particles running in the loop have $m = \frac{b}{2\pi\alpha'}$ and $e = \frac{1}{2\pi\alpha'}$.

For $v \to 0$, $\mathcal{A}_{short} \sim v^3/b^{6-p}$ by SUSY (non-renorm. theorem). Since

$$\text{Loop of spin } s \text{ particle} \Rightarrow \mathrm{ch}2s\frac{\pi\epsilon}{2\pi\alpha'}t$$

a cancellation occurs between loops of spin 0 and 1 bosons and spin $\frac{1}{2}$ fermions.

Large distance limit: $b \gg l_s = \sqrt{\alpha'}$

The amplitude is dominated by the exchange of massless closed strings and one finds

$$\mathcal{A}_{large} = V_p T_p^2 \frac{\frac{3}{4} + \frac{1}{4}\mathrm{ch}2\pi\epsilon - \mathrm{ch}\pi\epsilon}{\mathrm{sh}\pi\epsilon} \Delta_{(8-p)}(b)$$

This is the eikonal approximation of the phase-shift in SUGRA.

For $v \to 0$, $\mathcal{A}_{large} \sim v^3/b^{6-p}$ by SUSY (no-force condition). Since

$$\text{Exchange of spin } s \text{ particle} \Rightarrow \mathrm{ch}s\pi\epsilon$$

a cancellation occurs between the attractive dilaton and graviton exchange in the $NSNS$ sector and the repulsive (p+1)-form exchange in the RR sector.

Scale invariance for $v \to 0$

For $v \to 0$, the SYM and SUGRA descriptions agree, due to SUSY, and

$$V \sim \frac{v^4}{r^{7-p}}$$

POINT LIKE OBJECTS IN D=4 FROM D-BRANES

Consider a D-brane wrapped on some six-dimensional compact manifold. To obtain a point-like configuration in D=4, we take Neumann b.c. in time x^0 and Dirichlet in the 3 non-compact directions x^1, x^2, x^3. One can consider various possibilities for the b.c. in the 6 compact directions $x^4, ..., x^9$.

Organize the compact directions in 3 pairs (x^a, x^{a+1}), $a = 4, 6, 8$, corresponding to 3 tori T^2. Consider then three kinds of compactification:

- T^6: N=8 SUSY in D=4 (KK).
- $T^2 \times T^4/\mathbb{Z}_2$: N=4 SUSY in D=4 (K3).
- $T^6/\mathbb{Z}_3$: N=2 SUSY in D=4 (CY).

The $\mathbb{Z}_N$ action identifies points related by $\frac{2\pi}{N}$ rotations in N of the 3 T^2s.

There can be twisted sectors in which strings close only up to a $\mathbb{Z}_N$ rotation. One has to project onto $\mathbb{Z}_N$-invariant states with $P_N = \frac{1}{N}(1+g+...+g^{N-1})$. g is the generator of $\mathbb{Z}_N$. We can use

$$g = \exp\left\{i\sum_a z^a J^{aa+1}\right\}$$

with

- $\mathbb{T}$: $z^4 = z^6 = z^8 = 0$.
- $\mathbb{Z}_2$: $z^4 = 0, z^6 = z^8 = 0, \frac{2\pi}{2}$.
- $\mathbb{Z}_3$: $z^4 = z^6 = z^8 = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$.

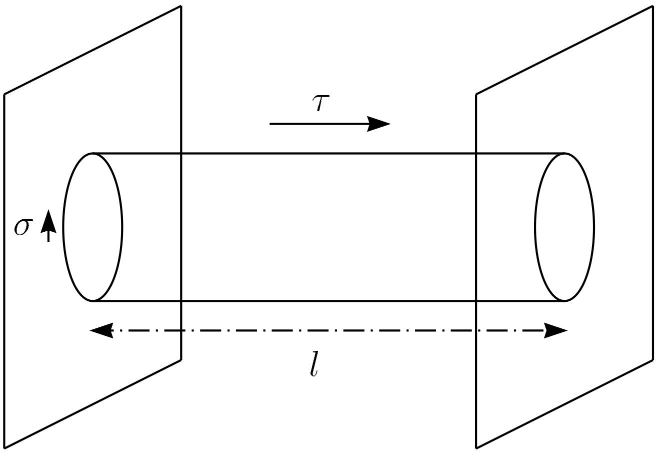
A wrapped D-brane is described by a projected boundary state in each sector:

$$|B\rangle_{inv} = P_N|B\rangle$$

Twisted sectors require equal b.c. for the directions x^a, x^{a+1} in each T^2.

Two interesting cases: the D0-brane (IIA) and the wrapped D3-brane (IIB).

Dynamics



The amplitude is

$$\mathcal{A} = \sum \int_0^\infty dl \langle B_1, \epsilon_1 | e^{-lH} P_N P_{GSO} \tilde{P}_{GSO} | B_2, \epsilon_2 \rangle$$

The result can be written as

$$\mathcal{A} = \frac{M^2}{2^4} \sum \int_0^\infty \frac{dl}{2\pi l} \, e^{-\frac{b^2}{2l}} \, \Gamma(l, \vec{b}_c, \vec{w}_c) \, Z(l, \epsilon)$$

where M is the mass, b the impact parameter and $\epsilon = \epsilon_1 - \epsilon_2$.

$\Gamma(l, \vec{b}_c, \vec{w}_c)$ is a lattice sum encoding the contribution of KK and winding modes. $Z(l, \epsilon)$ is the total partition function in each orbifold sector.

In the large distance limit $b \to \infty$, only very long world-sheets with $l \to \infty$ contribute. In this limit:

$$\Gamma(l, \vec{b}_c, \vec{w}_c) \underset{l \to \infty}{\longrightarrow} 1 \, , \quad Z(l, \epsilon) \underset{l \to \infty}{\longrightarrow} 8 \, \frac{\alpha + \beta \mathrm{ch} 2\pi\epsilon + \gamma \mathrm{ch} \pi\epsilon}{\mathrm{sh} \pi\epsilon}$$

The modular integral gives then the transverse propagator $\Delta_{(2)}(b)$. Finally:

$$\mathcal{A} \underset{b \to \infty}{\longrightarrow} M^2 \left(\alpha + \beta \mathrm{ch} 2\pi\epsilon + \gamma \mathrm{ch} \pi\epsilon \right) \int_0^\infty d\tau \, \Delta_{(3)} \left(r(\tau) \right)$$

where

$$r(\tau) = \sqrt{b^2 + \mathrm{sh}^2 \pi\epsilon \, \tau^2}$$

D0-brane

All the compact directions (x^a, x^{a+1}) are Dirichlet. If the position coincides with a fixed point, twisted sectors can contribute. The boundary state is rotation invariant and P_N acts trivially.

In the untwisted sector one finds:

$$Z(l,\epsilon) = \sum_{\alpha=2}^{4} (-1)^{1+\alpha} \frac{\vartheta_\alpha(i\epsilon|2il)\vartheta_\alpha^3(0|2il)}{\vartheta_1(i\epsilon|2il)\eta^9(2il)}$$

For $l \to \infty$ this reduces to

$$Z(l,\epsilon) \underset{l\to\infty}{\longrightarrow} \frac{6 + 2\mathrm{ch}2\pi\epsilon - 8\mathrm{ch}\pi\epsilon}{\mathrm{sh}\pi\epsilon}$$

and one finds in all cases

$$\mathcal{A}^{(un.)} \underset{b\to\infty}{\longrightarrow} M^2 \left(\frac{3}{4} + \frac{1}{4}\mathrm{ch}2\pi\epsilon - \mathrm{ch}\pi\epsilon\right) \int_{-\infty}^{\infty} d\tau \Delta_{(3)}(r(\tau))$$

For orbifold twisted sectors one finds:

$$Z(l,\epsilon) = \begin{cases} \sum_{a,b=0}^{\frac{1}{2}} (-1)^{2(a+b)} \frac{\vartheta\left[{a \atop b}\right](i\epsilon|2il)\vartheta\left[{a \atop b}\right](0|2il)\vartheta^2\left[{a-\frac{1}{2} \atop b}\right](0|2il)}{\vartheta\left[{\frac{1}{2} \atop \frac{1}{2}}\right](i\epsilon|2il)\vartheta^2\left[{0 \atop \frac{1}{2}}\right](0|2il)\eta^3(2il)} & , \quad \mathbb{Z}_2 \\ \sum_{a,b=0}^{\frac{1}{2}} (-1)^{2(a+b)} \frac{\vartheta\left[{a \atop b}\right](i\epsilon|2il)\vartheta^3\left[{a-\frac{1}{3} \atop b}\right](0|2il)}{\vartheta\left[{\frac{1}{2} \atop \frac{1}{2}}\right](i\epsilon|2il)\vartheta^3\left[{\frac{1}{6} \atop \frac{1}{2}}\right](0|2il)\eta^3(2il)} & , \quad \mathbb{Z}_3 \end{cases}$$

For $l \to \infty$ these reduce to

$$Z(l,\epsilon) \underset{l\to\infty}{\longrightarrow} \begin{cases} \frac{4 - 4\mathrm{ch}\pi\epsilon}{\mathrm{sh}\pi\epsilon} & , \quad \mathbb{Z}_2 \\ \frac{2 - 2\mathrm{ch}\pi\epsilon}{\mathrm{sh}\pi\epsilon} & , \quad \mathbb{Z}_3 \end{cases}$$

and one finds both for $\mathbb{Z}_2$ and $\mathbb{Z}_3$

$$\mathcal{A}^{(tw.)} \underset{b\to\infty}{\longrightarrow} \frac{M^2}{4}(1 - \mathrm{ch}\pi\epsilon) \int_{-\infty}^{\infty} d\tau \Delta_{(3)}(r(\tau))$$

Therefore

$$V^{(un.)} \sim \frac{v^4}{r} , \quad V^{(tw.)} \sim \frac{v^2}{r}$$

D3-brane

In each pair of compact directions (x^a, x^{a+1}), take x^a Dirichlet and x^{a+1} Neumann. The mixed b.c. in each T^2 are incompatible with twisted sectors. The boundary state is not rotation invariant and P_N acts non-trivially.

The partition function is

$$Z(l,\epsilon) = \frac{1}{N}\sum_{\{z^a\}} Z'(l,\epsilon,z^a)$$

with

$$Z'(l,\epsilon,z^a) = \sum_{\alpha=2}^{4}(-1)^{1+\alpha}\frac{\vartheta_\alpha(i\epsilon|2il)}{\vartheta_1(i\epsilon|2il)}\prod_a \frac{(2\sin z^a)\,\vartheta_\alpha(\frac{z^a}{\pi}|2il)}{\vartheta_1(\frac{z^a}{\pi}|2il)}$$

In the limit $l \to \infty$ this reduces to

$$Z'(l,\epsilon,z^a) \underset{l\to\infty}{\longrightarrow} \frac{2\sum_a \cos 2z^a + 2\mathrm{ch}2\pi\epsilon - 8\prod_a \cos z^a \mathrm{ch}\pi\epsilon}{\mathrm{sh}\pi\epsilon}$$

Averaging one finds:

$$\mathcal{A} \underset{b\to l_s}{\longrightarrow} \begin{cases} M^2\left(\frac{3}{4}+\frac{1}{4}\mathrm{ch}2\pi\epsilon - \mathrm{ch}\pi\epsilon\right)\int_{-\infty}^{\infty} d\tau\Delta_{(3)}(r(\tau)) & , \quad \mathbb{T}, \mathbb{Z}_2 \\ \frac{M^2}{4}\left(\mathrm{ch}2\pi\epsilon - \mathrm{ch}\pi\epsilon\right)\int_{-\infty}^{\infty} d\tau\Delta_{(3)}(r(\tau)) & , \quad \mathbb{Z}_3 \end{cases}$$

Therefore

$$V^{(un.)} \sim \begin{cases} \frac{v^4}{r} & , \quad \mathbb{T}, \mathbb{Z}_2 \\ \frac{v^2}{r} & , \quad \mathbb{Z}_3 \end{cases} \quad , \quad V^{(tw.)} = 0$$

Field theory interpretation

The field theory phase-shift in the eikonal approximation for two sources of scalars (a), gravitons (m) and vectors (e) is

$$\mathcal{A}^{(f.t.)} = \left(a^2 + \frac{m^2}{4}\mathrm{ch}2\pi\epsilon - e^2\mathrm{ch}\pi\epsilon\right)\int_{-\infty}^{\infty} d\tau\Delta_{(3)}(r(\tau))$$

One reads the following charges:

D0-brane

$$\begin{cases} a = \frac{\sqrt{3}}{2}M \;, \quad e = M \;, \quad m = M \quad , \quad \mathbb{T}, \mathbb{Z}_2, \mathbb{Z}_3 \\ a' = \frac{1}{2}M \;, \quad e' = \frac{1}{2}M \qquad\qquad\qquad , \quad \mathbb{Z}_2, \mathbb{Z}_3 \end{cases}$$

D3-brane

$$\begin{cases} a = \frac{\sqrt{3}}{2}M \;, \quad e = M \;, \quad m = M \quad , \quad \mathbb{T}, \mathbb{Z}_2 \\ a = 0 \;, \quad e = \frac{1}{2}M \;, \quad m = M \qquad , \quad \mathbb{Z}_3 \end{cases}$$

The scalar coupling distinguishes between two class of black holes.

Consider the typical D=4 action

$$\mathcal{S} = \frac{1}{2\kappa^2_{(4)}} \int d^4x \sqrt{g} \left(R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{2 \cdot 2!} e^{-\lambda\phi} F^2_{(2)} \right)$$

This has the BPS solution (Lü,Pope,Sezgin,Stelle)

$$\begin{cases} \phi = a \, \ln H(r) \\ ds^2 = -H(r)^{-b/2} dt^2 + H(r)^{b/2} d\vec{x} \cdot d\vec{x} \\ A_0 = c \left(H(r)^{-1} - 1 \right) \end{cases}$$

where

$$a(\lambda) = \frac{2\lambda}{1+\lambda^2} \;, \quad b(\lambda) = \frac{4}{1+\lambda^2} \;, \quad c(\lambda) = \frac{2}{\sqrt{1+\lambda^2}}$$

The charges are

$$a = aM \;, \quad m = bM \;, \quad e = cM \quad \text{and} \quad a^2 + \frac{m^2}{4} - e^2 = 0$$

Two cases: $\lambda = 0$ (IIB: $\mathbb{Z}_3$) and $\lambda \neq 0$ (IIA: $\mathbb{T}, \mathbb{Z}_2, \mathbb{Z}_3$, IIB: $\mathbb{T}, \mathbb{Z}_2$).

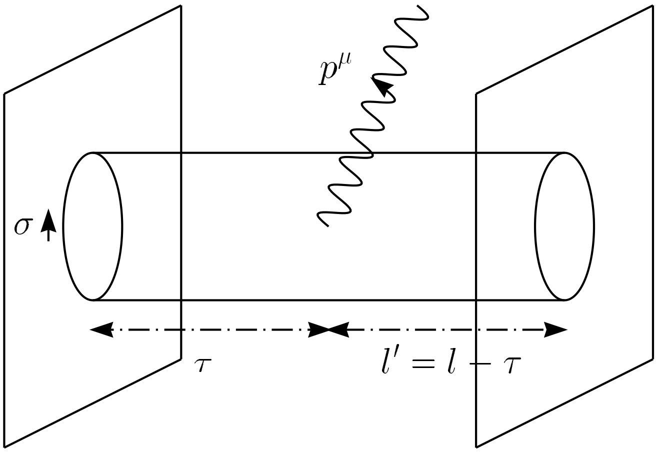
- Regular RN black hole: $\lambda = 0$, $a = 0, b = 4, c = 2$.

  D3 on $\mathbb{Z}_3$.

- Singular black brane: $\lambda = \sqrt{3}$: $a = \frac{\sqrt{3}}{2}, b = 1, c = 1$.

  D0 on $\mathbb{T}, \mathbb{Z}_2, \mathbb{Z}_3$ and D3 on $\mathbb{T}, \mathbb{Z}_2$.

Radiation of massless particles



The amplitude is

$$\mathcal{A} = \frac{M^2}{2^4} \sum \int_0^\infty dl \int dz \int d\bar{z} \langle\langle V(z,\bar{z})\rangle\rangle$$

where

$$\langle\langle V\rangle\rangle = \langle B,\epsilon_1|e^{-lH}V|B,\epsilon_2\rangle = \frac{\langle B,\epsilon_1|e^{-lH}V|B,\epsilon_2\rangle}{\langle B,\epsilon_1|e^{-lH}|B,\epsilon_2\rangle}\langle B,\epsilon_1|e^{-lH}|B,\epsilon_2\rangle$$
$$= \langle V\rangle Z$$

Nothing depends on σ so

$$\int_0^\infty dl \int dz \int d\bar{z} = \int_0^\infty dl \int_0^l d\tau = \int_0^\infty d\tau \int_0^\infty dl'$$

The kinematics is peculiar. The two D-branes can emit states with

$$k^\mu(\epsilon_1) = \left(\mathrm{sh}\,\pi\epsilon_1\, k^1, \mathrm{ch}\,\pi\epsilon_1\, k^1, \vec{k}_T\right)$$

$$q^\mu(\epsilon_2) = (\mathrm{sh}\,\pi\epsilon_2\, q^1, \mathrm{ch}\,\pi\epsilon_2\, q^1, \vec{q}_T)$$

The momentum of the out-going massless particle is

$$p^\mu = (p, \cos\theta\, p, \vec{p}_T = \vec{n}\sin\theta\, p)$$

Momentum conservation ($k^\mu - q^\mu = k^\mu$) completely fixes the energies and longitudinal momenta.

In particular

$$k^2 = \vec{k}_T^2 + \frac{p^{(2)2}}{\mathrm{sh}^2\pi\epsilon}\ , \quad q^2 = (\vec{k}_T - \vec{p}_T)^2 + \frac{p^{(1)2}}{\mathrm{sh}^2\pi\epsilon}$$

in terms of the boosted energies

$$p^{(1,2)} = (\mathrm{ch}\pi\epsilon_{1,2} - \sin\pi\epsilon_{1,2}\cos\theta)\, p$$

Consider massless NSNS states with p^μ and $\xi^{\mu\nu}$ in D=4:

$$V = \xi_{\mu\nu}(\partial X^\mu - \frac{1}{2}p\cdot\psi\psi^\mu)(\bar{\partial}X^\nu + \frac{1}{2}p\cdot\bar{\psi}\bar{\psi}^\nu)e^{ip\cdot X}$$

To compute $\mathcal{A}$ we need the partition function

$$Z(l,\epsilon) = \langle B_1,\epsilon_1|e^{-lH}|B_2,\epsilon_2\rangle$$

One can use Wick's theorem to reduce all the contractions to

$$P_X^{\mu\nu}(\tau,l',\epsilon) = \frac{\langle B,\epsilon_1|e^{-lH}X^\mu\bar{X}^\nu|B,\epsilon_2\rangle}{\langle B,\epsilon_1|e^{-lH}|B,\epsilon_2\rangle} = \langle X^\mu\bar{X}^\nu\rangle$$

$$P_\psi^{\mu\nu}(\tau,l',\epsilon) = \frac{\langle B,\epsilon_1|e^{-lH}\psi^\mu\tilde{\psi}^\nu|B,\epsilon_2\rangle}{\langle B,\epsilon_1|e^{-lH}|B,\epsilon_2\rangle} = \langle\psi^\mu\tilde{\psi}^\nu\rangle$$

Both the partition function and the correlators can be computed exactly, and one finds ϑ-functions. Using the notation $q = e^{-2\pi l}$ one finds

$$Z(l,\epsilon) = \sum_{n=0}^{\infty} d_n(\epsilon)q^{2n}$$

$$P^{\mu\nu}(\tau,l',\epsilon) = a^{\mu\nu}(\epsilon) + \sum_{n=0}^{\infty}\left[b_n(\epsilon)f_n(\tau) + c_n(\epsilon)f_n(l')\right]$$

where

$$f_n^{(B)}(x) = \ln\left(1 - q^{2n}e^{-4\pi x}\right)$$

$$f_n^{(NSNS)}(x) = \frac{q^{2n}e^{-4\pi x}}{1 - q^{2n}e^{-4\pi x}}\ , \quad f_n^{(RR)}(x) = \frac{q^n e^{-2\pi x}}{1 - q^{2n}e^{-4\pi x}}$$

In the large distance limit $b \to \infty$, again only world-sheets with $l \to \infty$ contribute, and the amplitude becomes

$$\mathcal{A} = \frac{M^2}{4\mathrm{sh}\pi\epsilon} \int \frac{d^2\vec{k}_T}{(2\pi)^2} e^{i\vec{k}_T\cdot\vec{b}} \int_0^\infty d\tau \int_0^\infty dl' e^{-\frac{q^2}{2}\tau} e^{-\frac{k^2}{2}l'} \times$$
$$\left[1 - e^{-4\pi\tau}\right]^{-\frac{p^{(2)2}}{2\pi}} \left[1 - e^{-4\pi l'}\right]^{-\frac{p^{(1)2}}{2\pi}} \mathcal{M}(\tau, l', \epsilon)$$

Double poles in $\mathcal{M}(\tau, l', \epsilon)$ cancel between bosons and fermions, and

$$\mathcal{M}(\tau, l', \epsilon) = F^{(bulk)}(\epsilon) + 4\mathrm{sh}\pi\, \epsilon_2\, p^{(2)}\, F^{(rad)}(\epsilon)\, \frac{e^{-4\pi\tau}}{1 - e^{-4\pi\tau}}$$
$$-4\mathrm{sh}\pi\epsilon_1\, p^{(1)}\, F^{(rad)}(\epsilon)\, \frac{e^{-4\pi l'}}{1 - e^{-4\pi l'}}$$

Integrating by parts in τ and l':

$$\frac{e^{-4\pi\tau}}{1 - e^{-4\pi\tau}} \doteq -\frac{1}{4}\frac{q^2}{p^{(2)2}}\,, \quad \frac{e^{-4\pi l'}}{1 - e^{-4\pi l'}} \doteq -\frac{1}{4}\frac{k^2}{p^{(1)2}}$$

Therefore $\mathcal{M}(\tau, l', \epsilon) = \mathcal{M}(\epsilon)$ and

$$\mathcal{A} = \frac{M^2}{\mathrm{sh}\pi\epsilon} \int \frac{d^2\vec{k}_T}{(2\pi)^2} e^{i\vec{k}_T\cdot\vec{b}} I_1 I_2 \left\{ F^{(bulk)} + \mathrm{sh}\pi\epsilon_1 \frac{k^2}{p^{(1)}} F^{(rad)} - \mathrm{sh}\pi\epsilon_2 \frac{q^2}{p^{(2)}} F^{(rad)} \right\}$$

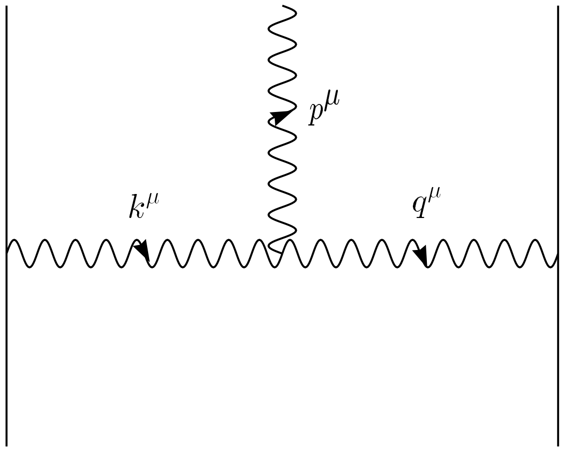
The proper time integrals can then be evaluated:

$$I_1(p, q) = \frac{1}{2}\int_0^\infty d\tau\, e^{-\frac{q^2}{2}\tau} \left[1 - e^{-4\pi\tau}\right]^{-\frac{p^{(2)2}}{2\pi}} = \frac{1}{8\pi} \frac{\Gamma\left[\frac{q^2}{8\pi}\right]\Gamma\left[-\frac{p^{(2)2}}{2\pi} + 1\right]}{\Gamma\left[\frac{q^2}{8\pi} - \frac{p^{(2)2}}{2\pi} + 1\right]} \xrightarrow[p\to 0]{} \frac{1}{q^2}$$

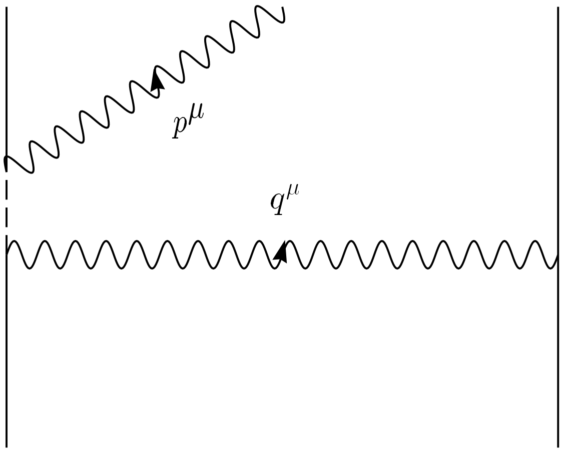
$$I_2(p, k) = \frac{1}{2}\int_0^\infty dl'\, e^{-\frac{k^2}{2}l'} \left[1 - e^{-4\pi l'}\right]^{-\frac{p^{(1)2}}{2\pi}} = \frac{1}{8\pi} \frac{\Gamma\left[\frac{k^2}{8\pi}\right]\Gamma\left[-\frac{p^{(1)2}}{2\pi} + 1\right]}{\Gamma\left[\frac{k^2}{8\pi} - \frac{p^{(1)2}}{2\pi} + 1\right]} \xrightarrow[p\to 0]{} \frac{1}{k^2}$$

Finally one finds

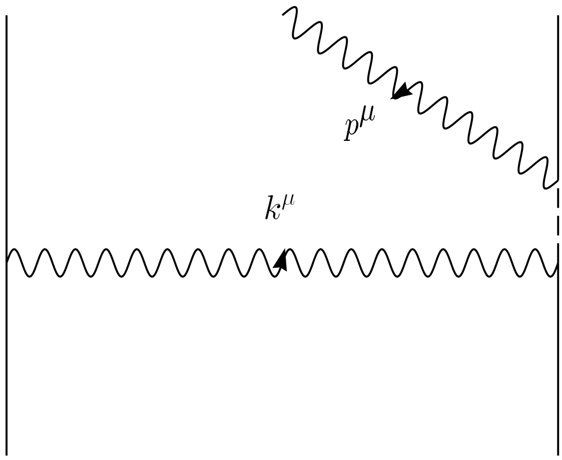
$$\mathcal{A} \xrightarrow[\substack{b\to\infty \\ p\to 0}]{} \frac{M^2}{\mathrm{sh}\pi\epsilon} \int \frac{d^2\vec{k}_T}{(2\pi)^2} e^{i\vec{k}_T\cdot\vec{b}} \left\{ \frac{F^{(bulk)}}{k^2 q^2} + \mathrm{sh}\pi\epsilon_1 \frac{F^{(rad)}}{p^{(1)} q^2} - \mathrm{sh}\pi\epsilon_2 \frac{F^{(rad)}}{p(2) k^2} \right\}$$



$$\sim \frac{1}{k^2 q^2}$$



$$\sim \frac{1}{p^{(1)} q^2}$$



$$\sim \frac{1}{p^{(2)} k^2}$$

For the dilaton ($\xi^{ij} \sim \delta^{ij} - p^i p^j / p^2$) and the axion ($\xi^{ij} \sim \epsilon^{ijk} p_k / p$), the total amplitude vanishes. For the graviton ($\xi^{ij} = h^{ij}, p_i h^{ij} = 0, h = 0$), one finds instead a non-vanishing result.

Untwisted sector for D0-branes in $\mathbb{T}, \mathbb{Z}_2, \mathbb{Z}_3$; D3-branes in $\mathbb{T}, \mathbb{Z}_2$:

$$F^{(bulk)} = \frac{3}{4}\left[h_{ij}k^ik^j\right] + \frac{1}{4}\left[\mathrm{ch}2\pi\epsilon\, h_{ij}k^ik^j - 2\mathrm{sh}2\pi\epsilon\, ph_{i1}k^i + 2\mathrm{sh}^2\pi\epsilon\, p^2h_{11}\right]$$
$$- \left[\mathrm{ch}\pi\epsilon\, h_{ij}k^ik^j - \mathrm{sh}\pi\epsilon\, ph_{i1}k^i\right]$$
$$F^{(rad)} = \frac{3}{4}\left[h_{i1}k^i\right] + \frac{1}{4}\left[\mathrm{ch}2\pi\epsilon\, h_{i1}k^i - \mathrm{sh}2\pi\epsilon\, ph_{i1}k^i\right]$$
$$- \left[\mathrm{ch}\pi\epsilon\, h_{i1}k^i - \frac{1}{2}\mathrm{sh}\pi\epsilon\, ph_{i1}k^i\right]$$

D3-branes in $\mathbb{Z}_3$:

$$F^{(bulk)} = \frac{1}{4}\left[\mathrm{ch}2\pi\epsilon\, h_{ij}k^ik^j - 2\mathrm{sh}2\pi\epsilon\, ph_{i1}k^i + 2\mathrm{sh}^2\pi\epsilon\, p^2h_{11}\right]$$
$$-\frac{1}{4}\left[\mathrm{ch}\pi\epsilon\, h_{ij}k^ik^j - \mathrm{sh}\pi\epsilon\, ph_{i1}k^i\right]$$
$$F^{(rad)} = \frac{1}{4}\left[\mathrm{ch}2\pi\epsilon\, h_{i1}k^i - \mathrm{sh}2\pi\epsilon\, ph_{i1}k^i\right]$$
$$-\frac{1}{4}\left[\mathrm{ch}\pi\epsilon\, h_{i1}k^i - \frac{1}{2}\mathrm{sh}\pi\epsilon\, ph_{i1}k^i\right]$$

Twisted sectors for D0-branes in $\mathbb{Z}_2, \mathbb{Z}_3$:

$$F^{(bulk)} = \frac{1}{4}\left[h_{ij}k^ik^j\right] - \frac{1}{4}\left[\mathrm{ch}\pi\epsilon\, h_{ij}k^ik^j - \mathrm{sh}\pi\epsilon\, ph_{i1}k^i\right]$$
$$F^{(rad)} = \frac{1}{4}\left[h_{i1}k^i\right] - \frac{1}{4}\left[\mathrm{ch}\pi\epsilon\, h_{i1}k^i - \frac{1}{2}\mathrm{sh}\pi\epsilon\, ph_{i1}k^i\right]$$

Field theory interpretation

In the eikonal approximation and for charges a, m and e:

$$F^{(bulk)} = a^2\left[h_{ij}k^ik^j\right] + \frac{m^2}{4}\left[\mathrm{ch}2\pi\epsilon\, h_{ij}k^ik^j - 2\mathrm{sh}2\pi\epsilon\, ph_{i1}k^i + 2\mathrm{sh}^2\pi\epsilon\, p^2h_{11}\right]$$
$$-e^2\left[\mathrm{ch}\pi\epsilon\, h_{ij}k^ik^j - \mathrm{sh}\pi\epsilon\, ph_{i1}k^i\right]$$
$$F^{(rad)} = a^2\left[h_{i1}k^i\right] + \frac{m^2}{4}\left[\mathrm{ch}2\pi\epsilon\, h_{i1}k^i - \mathrm{sh}2\pi\epsilon\, ph_{i1}k^i\right]$$
$$-e^2\left[\mathrm{ch}\pi\epsilon\, h_{i1}k^i - \frac{1}{2}\mathrm{sh}\pi\epsilon\, ph_{i1}k^i\right]$$

DYONIC BLACK HOLES IN $D = 4$.

Consider a wrapped D3-brane, but now with Neumann and Dirichlet directions forming an angle θ^a with the directions x^a, x^{a+1}.

Dynamics

The general interaction amplitude is

$$\mathcal{A} = \sum \int_0^\infty dl \langle B_1, \epsilon_1, \theta_1^a | e^{-lH} P_N P_{GSO} \tilde{P}_{GSO} | B_2, \epsilon_2, \theta_2^a \rangle$$

The result can be written as

$$\mathcal{A} = \frac{M^2}{2^4} \sum \int_0^\infty \frac{dl}{2\pi l} e^{-\frac{b^2}{2l}} Z(l, \epsilon, \theta^a)$$

where $Z(l, \epsilon, \theta^a)$ is the total partition function, $\epsilon = \epsilon_1 - \epsilon_2$ and $\theta^a = \theta_1^a - \theta_2^a$.

For $b \to \infty$, only world-sheets with $l \to \infty$ contribute. In this limit:

$$Z(l, \epsilon, \theta^a) \underset{l \to \infty}{\longrightarrow} 8 \frac{\alpha(\theta^a) + \beta(\theta^a) \mathrm{ch} 2\pi\epsilon + \gamma(\theta^a) \mathrm{ch} \pi\epsilon}{\mathrm{sh} \pi\epsilon} + 8\delta(\theta^a)$$

The modular integral gives then $\Delta_{(2)}(b)$ and finally:

$$\mathcal{A} \underset{b \to \infty}{\longrightarrow} M^2 \frac{\alpha + \beta \mathrm{ch} 2\pi\epsilon + \gamma \mathrm{ch} \pi\epsilon}{\mathrm{sh} \pi\epsilon} \Delta_{(2)}(b) + M^2 \delta \Delta_{(2)}(b)$$

For a Coulomb-like force, the non-relativistic phase-shift is

$$\vec{F}^{(Cou.)} = \frac{ee'}{4\pi} \frac{\vec{r}}{r^3} \;\Rightarrow\; \vec{\nabla} \mathcal{A}^{(Cou.)} = \frac{1}{v} \frac{ee'}{2\pi} \frac{\vec{b}}{b^2} \;\Rightarrow\; \mathcal{A}^{(Cou.)} = \frac{1}{v} \frac{ee'}{2\pi} \ln \frac{b}{b_0}$$

For a Lorentz-like force, one has

$$\vec{F}^{(Lor.)} = \frac{eg'}{4\pi} \frac{\vec{v} \times \vec{r}}{r^3} \;\Rightarrow\; \vec{\nabla} \mathcal{A}^{(Lor.)} = \frac{eg'}{2\pi} \frac{{}^*\vec{b}}{b^2} \;\Rightarrow\; \mathcal{A}^{(Lor.)} = \frac{eg'}{2\pi} (\theta - \theta_0)$$

Defining $b = x + iy$, the relativistic result for charges a_i, m_i, e_i and g_i is

$$\mathcal{A}^{(f.t.)} = \frac{a_1 a_2 + \frac{m_1 m_2}{4} \mathrm{ch} 2\pi\epsilon - (e_1 e_2 + g_1 g_2) \mathrm{ch} \pi\epsilon}{\mathrm{sh} \pi\epsilon} \mathrm{Re} \Delta_{(2)}(b) + (e_1 g_2 - e_2 g_1) \mathrm{Im} \Delta_{(2)}(b)$$

There are subtleties in the cancellation between ghosts and long. modes.

The odd spin-structure encodes magnetic interaction between D-branes. It produces the correct velocity dependence and is non vanishing only for the dual system Dp-D(6-p) in D=10.

The magnetic coupling satisfies the Dirac quantization condition

$$e_1 g_2 + (-1)^{\frac{D}{2}-1} g_1 e_2 = 2\pi n$$

The partition function is

$$Z(l, \epsilon, \theta^a) = \frac{1}{N} \sum_{\{z^a\}} Z'(l, \epsilon, \theta^a + z^a)$$

with

$$Z'(l, \epsilon, \beta^a) = \sum_{\alpha} (-1)^{1+\alpha} \frac{\vartheta_\alpha(i\epsilon|2il)}{\vartheta_1(i\epsilon|2il)} \prod_a \frac{(2\sin\beta^a)\, \vartheta_\alpha(\frac{\beta^a}{\pi}|2il)}{\vartheta_1(\frac{\beta^a}{\pi}|2il)}$$

In the limit $l \to \infty$ this reduces to

$$Z'(l, \epsilon, \beta^a) \underset{l\to\infty}{\longrightarrow} \frac{2\sum_a \cos 2\beta^a + 2\mathrm{ch}2\pi\epsilon - 8\prod_a \cos\beta^a \mathrm{ch}\pi\epsilon}{\mathrm{sh}\pi\epsilon} + 8\prod_a \sin\beta^a$$

For $\mathbb{T}$, $\mathbb{Z}_2$, the averaging has no effect and the couplings depend on all θ^as:

$$\mathcal{A} \underset{b\to l_s}{\longrightarrow} M^2 \frac{\frac{1}{4}\sum_a \cos 2\theta^a + \frac{1}{4}\mathrm{ch}2\pi\epsilon - \prod_a \cos\theta^a \mathrm{ch}\pi\epsilon}{\mathrm{sh}\pi\epsilon} \mathrm{Re}\Delta_{(2)}(b) + M^2 \prod_a \sin\theta^a\, \mathrm{Im}\Delta_{(2)}(b)$$

For $\mathbb{Z}_3$, the averaging is non-trivial and the couplings depend only on $\sum_a \theta^a$:

$$\mathcal{A} \underset{b\to l_s}{\longrightarrow} M^2 \frac{\frac{1}{4}\mathrm{ch}2\pi\epsilon - \frac{1}{4}\cos\sum_a \theta^a \mathrm{ch}\pi\epsilon}{\mathrm{sh}\pi\epsilon} \mathrm{Re}\Delta_{(2)}(b) + \frac{M^2}{4} \sin\sum_a \theta^a\, \mathrm{Im}\Delta_{(2)}(b)$$

The Dirac condition is satisfied in $D = 10$ with $n = 1$ ($\mu_3^2 = 2\pi$). In $D = 4$ it is satisfied with $n \neq 1$ related to the winding number ($e_1 g_2 - g_1 e_2 = 2\pi n$). One can then determine how the $D = 4$ charges depend on the angles θ_i^a.

Consider the $\mathbb{Z}_3$ case. Calling $\alpha_i = \Sigma_a \theta_i^a$ and $\alpha = \alpha_1 - \alpha_2$, one reads

$$m_1 m_2 = M^2 \ , \quad e_1 e_2 + g_1 g_2 = \frac{M^2}{4}\cos\alpha \ , \quad e_1 g_2 - e_2 g_1 = \frac{M^2}{4}\sin\alpha$$

We conclude that

$$m_i = M \ , \quad e_i = \frac{M}{2}\cos\alpha_i \ , \quad g_i = \frac{M}{2}\sin\alpha_i$$

For $\mathbb{T}$, $\mathbb{Z}_2$, one can do the same thing, and one finds more charges.

One-Point functions

The wrapped D3-brane couples to the D=4 fields arising from the D=10 graviton $h^{\mu\nu}$ in the NSNS sector and to the 4-form $C^{\mu\nu\rho\sigma}$ in the RR sector. The couplings are encoded in the one-point functions

$$\langle\Psi\rangle = \langle\Psi|P_N|B,\theta^a\rangle = \frac{1}{N}\sum_{\{z^a\}}\langle\Psi|B,\theta^a + z^a\rangle$$

For $\mathbb{T}$, $\mathbb{Z}_2$, one finds:

$$\begin{aligned}\langle h\rangle = M\Big\{ & h^{00}+h^{11}+h^{22}+h^{33} \\ & - \sum_a \Big[\cos 2\theta^a\left(h^{aa}-h^{a+1a+1}\right) - 2\sin 2\theta^a\, h^{aa+1}\Big]\Big\} \\ \langle C\rangle = 2M\Big\{ & \prod_a \cos\theta^a C^{0468} + \Big[\cos\theta^4\cos\theta^6\sin\theta^8 C^{0469} + \text{perm.}\Big] \\ & + \Big[\cos\theta^4\sin\theta^6\sin\theta^8 C^{0479} + \text{perm.}\Big] + \prod_a \sin\theta^a A^{0579}\Big\}\end{aligned}$$

For $\mathbb{Z}_3$, one finds instead:

$$\begin{aligned}\langle h\rangle &= M\left\{h^{00}+h^{11}+h^{22}+h^{33}\right\} \\ \langle C\rangle &= \frac{M}{2}\Big\{\cos\sum_a\theta^a\left(C^{0468}-C^{0479}-C^{0569}-C^{0578}\right) \\ &\qquad + \sin\sum_a\theta^a\left(C^{0579}-C^{0568}-C^{0478}-C^{0469}\right)\Big\}\end{aligned}$$

Specialize again to $\mathbb{Z}_3$. We see that the only couplings are:

$$h^{\mu\nu} \;\Rightarrow\; m = M$$

$$A^\mu = C^{\mu 468} - C^{\mu 479} - C^{\mu 569} - C^{\mu 578} \;\Rightarrow\; e_A = \cos\alpha M$$

$$B^\mu = C^{\mu 579} - C^{\mu 568} - C^{\mu 478} - C^{\mu 469} \;\Rightarrow\; e_B = \sin\alpha M$$

It follows from $F_C = {}^*F_C$ in D=10 that $F_B = {}^*F_A$ in D=4. Therefore, $e_B \Leftrightarrow g_A$ and/or $e_A \Leftrightarrow g_B$. Using e.g. A^μ and eliminating B^μ one recovers

$$m = M\ ,\quad e = \frac{M}{2}\cos\alpha\ ,\quad g = \frac{M}{2}\sin\alpha$$

For $\mathbb{T}$, $\mathbb{Z}_2$ one gets more fields.

Geometric interpretation and SUGRA solution

In complex notation ($z^a = \frac{1}{\sqrt{2}}(x^a + i x^{a+1})$), $T^6/\mathbb{Z}_3$ has 9 $(1,1)$ forms, 0 $(2,1)$-forms and 1 $(3,0)$-form which are harmonic. It is the limit of a Calabi-Yau manifold with $h^{(1,1)} = 9$ and $h^{(1,2)} = 0$.

The effective theory in D=4 is N=2 SUGRA with $n_V = 0$ and $n_H = 9 + 1$. The only gauge field is the graviphoton, described by A^μ, B^μ with $F_B = {}^*F_A$. A^μ and B^μ arise by decomposing $C^{\mu\nu\rho\sigma}$ on ReΩ and ImΩ, where

$$\Omega = dz^4 \wedge dz^6 \wedge dz^8$$

The D3-brane wrapped with angles θ^a corresponds to

$$\Omega_B = \mathrm{Re}[e^{-i\theta^4} dz^4 \wedge e^{-i\theta^6} dz^6 \wedge e^{-i\theta^8} dz^8] = \mathrm{Re}[e^{-i\alpha}\Omega]$$

$$= \cos\alpha\, \mathrm{Re}\Omega + \sin\alpha\, \mathrm{Im}\Omega$$

One can construct a SUGRA solution corresponding to a D3-brane wrapped on a 3-cycle of a Calabi-Yau. The solution is a dyonic RN $\times$ CY:

$$d^2s = g^{(RN)}_{\mu\nu} dx^\mu dx^\nu + g^{(CY)}_{ab} dx^a dx^b\ ,\quad F_{(5)} = F^{(RN)}_{(2)} \wedge \Omega_B\ ,\quad \phi = 0$$

SPIN EFFECTS IN D-BRANE DYNAMICS

D-branes are BPS states preserving 16 of the 32 SUSYs of Type II theories. They fill short multiplets with $2^8 = 256$ components with different spins realizing the 16 broken SUSYs.

The cylinder amplitude gives only the universal spin-independent part of the interaction between two Dp-branes.

Performing SUSY transformations, one can generate all the other spin-dependent leading interactions. This program can be carried out in the Green-Schwarz formalism, finding the scale-invariant potential

$$V \sim \sum_{k=0}^{4} \frac{v^{4-k}}{r^{7-p+k}}$$

Boundary state and SUSY.

Consider the Type II theories in the light-cone gauge. $X^+ = x^+ + p^+\tau$ whereas X^- is completely determined and after fixing the κ-symmetry, we are left with two spinors S^a e $\tilde{S}^a$ in the $\mathbf{8_s}$ of $SO(8)$.

The Fock space is constructed on a vacuum representing the algebra of S_0^a e $\tilde{S}_0^a$. The representation is $\mathbf{8_v} \oplus \mathbf{8_c}$ both for the left and the right parts, and

$$S_0^a|i\rangle = \frac{1}{\sqrt{2}}\gamma^i_{a\dot{a}}|\dot{a}\rangle \quad , \quad S_0^a|\dot{a}\rangle = \frac{1}{\sqrt{2}}\gamma^i_{a\dot{a}}|i\rangle$$

$$\tilde{S}_0^a|\tilde{i}\rangle = \frac{1}{\sqrt{2}}\gamma^i_{a\dot{a}}|\tilde{\dot{a}}\rangle \quad , \quad \tilde{S}_0^a|\tilde{\dot{a}}\rangle = \frac{1}{\sqrt{2}}\gamma^i_{a\dot{a}}|\tilde{i}\rangle$$

The light-cone coordinates $X^\pm$ automatically satisfy Dirichlet b.c., whereas the b.c. of the transverse coordinates X^i, $i = 1, 2, ..., 8$ can be chosen freely.

It is possible to define a Dp-brane-like configuration by choosing Neumann b.c. for $\mu = 1, 2, ..., p+1$ and Dirichlet b.c. for $I = p+2, ..., 8-p$. The usual description is recovered through an analytic continuation.

The boundary conditions are

$$\partial X^i = M_{ij} \bar{\partial} \bar{X}^j \ , \quad S^a = i M_{ab} \tilde{S}^b$$

where

$$M_{ij} = \begin{pmatrix} -1\!\!1_{p+1} & 0 \\ 0 & 1\!\!1_{7-p} \end{pmatrix} \ , \quad M_{ab} = (\gamma^1 \gamma^2 ... \gamma^{p+1})_{ab}$$

The solution for the boundary state is

$$|B\rangle = \exp \sum_{n>0} \left(\frac{1}{n} M_{ij} \alpha^i_{-n} \tilde{\alpha}^j_{-n} - i M_{ab} S^a_{-n} \tilde{S}^b_{-n} \right) |B_0\rangle$$

with the zero mode part

$$|B_0\rangle = M_{ij} |i\rangle |\tilde{j}\rangle - i M_{\dot{a}\dot{b}} |\dot{a}\rangle |\tilde{\dot{b}}\rangle$$

The boundary state in configuration space is $|B, \vec{Y}\rangle = |B\rangle \otimes |\vec{Y}\rangle$ with

$$|\vec{Y}\rangle = \delta^{(9-p)}(\vec{x}_0 - \vec{Y}) |\vec{0}\rangle = \int \frac{d^{9-p} q}{(2\pi)^{9-p}} \, e^{i \vec{q} \cdot \vec{Y}} \, |\vec{q}\rangle$$

Consider the combinations of supercharges

$$Q^a_{\pm} = \frac{1}{\sqrt{2}} \left(Q^a \pm i M_{ab} \tilde{Q}^b \right) \ , \quad Q^{\dot{a}}_{\pm} = \frac{1}{\sqrt{2}} \left(Q^{\dot{a}} \pm i M_{\dot{a}\dot{b}} \tilde{Q}^{\dot{b}} \right)$$

satisfying the algebra

$$\{ Q^a_+, Q^b_- \} = 2 p^+ \delta^{ab} \ , \quad \{ Q^{\dot{a}}_+, Q^{\dot{b}}_- \} = P^- \delta^{\dot{a}\dot{b}}$$
$$\{ Q^a_+, Q^{\dot{a}}_- \} = \frac{1}{\sqrt{2}} \gamma^i_{a\dot{a}} \left(p^i + M_{ij} \tilde{p}^j \right)$$

The boundary state satisfies the BPS conditions

$$Q^a_+ |B\rangle = 0 \quad , \quad Q^{\dot{a}}_+ |B\rangle = 0 \ \Rightarrow \ Q^a_+, Q^{\dot{a}}_+ \ \text{preserved}$$

$$Q^a_- |B\rangle \neq 0 \quad , \quad Q^{\dot{a}}_- |B\rangle \neq 0 \ \Rightarrow \ Q^a_-, Q^{\dot{a}}_- \ \text{broken}$$

Multipole expansion

Performing an arbitrary broken SUSY transformation on $|B\rangle$, one obtains informations on the couplings of any component of the multiplet. The state

$$|B,\eta\rangle = e^{\eta Q^-}|B\rangle = \sum_{m=0}^{16} \frac{1}{m!}(\eta Q^-)^m |B\rangle$$

encodes the couplings to closed string states of a semi-classical current formed by an "in" and an "out" Dp-branes ($\eta = (\eta_a, \tilde{\eta}_{\dot{a}})$ and $Q^- = (Q_a^-, Q_{\dot{a}}^-)$). The sum corresponds to a multipole expansion, and terms with m even and odd are relevant for bosonic and fermionic currents.

For elastic scatterings, it suffice to consider even powers of Q^-. Moreover, in each $(\eta Q^-)^2 = (\eta_a Q_a^- + \tilde{\eta}_{\dot{a}} Q_{\dot{a}}^-)^2$ it is enough to consider the SO(8) part

$$V_\eta = \eta_a Q_a^- \tilde{\eta}_{\dot{a}} Q_{\dot{a}}^-$$

In this way, the boundary state $|B,\eta\rangle$ for a generic static D-brane current is

$$|B,\eta\rangle = \sum_{n=0}^{8} \frac{V_\eta^n}{(n!)^2} |B\rangle$$

The generalization to non-zero velocity is obtained through a boost and

$$|B,\eta,\epsilon\rangle = e^{-i\pi\epsilon_i J^{1i}}|B,\eta\rangle = \sum_{m=0}^{\infty} \frac{1}{m!}(-i\pi\epsilon_i J^{1i})^n |B,\eta\rangle$$

It corresponds to insertions of the operator

$$V_\epsilon = -i\pi\epsilon_i J^{1i}$$

Finally, the boundary state $|B,\eta,\epsilon\rangle$ for a generic moving D-brane current is

$$|B,\eta,\epsilon\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{8} \frac{V_\epsilon^m V_\eta^n}{m!(n!)^2}|B\rangle$$

Therefore, Lorentz and SUSY transformations correspond to insert zero momentum bosonic and fermionic vertices.

Zero modes and one-point functions

Using the b.c. implemented by $|B\rangle$, the zero mode parts of V_η and V_ϵ can be written in terms of the left-moving $SO(8)$ generators

$$R_0^{ij} = \frac{1}{4} S_0^a \gamma^{ij}_{ab} S_0^b$$

Both $V_{\eta 0}$ and $V_{\epsilon 0}$ are linear in R_0^{ij} and one finds

$$V_{\epsilon 0} = -2\pi \epsilon_i R_0^{1i}$$

$$V_{\eta 0}^n = q_{i_1} ... q_{i_n}\, \omega^{i_1 ... i_n}_{j_1 ... j_{2n}}(\eta) R_0^{j_1 j_2} ... R_0^{j_{2n-1} j_{2n}} \ , \quad n \leq 4$$

where the tensor

$$\omega^{i_1 ... i_n}_{j_1 ... j_{2n}}(\eta) = \frac{1}{2^n} \left[\eta_{[a_1} (\tilde\eta \gamma^{i_1})_{a_2} ... \eta_{a_{2n-1}} (\tilde\eta \gamma^{i_n})_{a_{2n}]} \right] \gamma^{j_1 j_2}_{a_1 a_2} ... \gamma^{j_{2n-1} j_{2n}}_{a_{2n-1} a_{2n}}$$

encodes the dependence on the SUSY parameter.

Using the action of R_0^{ij} in the $\mathbf{8_v}$ and $\mathbf{8_c}$ representations, one can compute

$$|B_0, \eta, \epsilon\rangle = M_{ij}(\eta,\epsilon) |i\rangle |\tilde{j}\rangle - i M_{\dot a \dot b}(\eta,\epsilon) |\dot a\rangle |\tilde{\dot b}\rangle$$

The couplings of a generic Dp-brane to massless closed strings states $|\Psi\rangle$ are encoded in $\langle \Psi \rangle = \langle \Psi | B_0, \eta \rangle$ as a multipole expansion. The n-pole term is $\langle \Psi \rangle_{(n)} = \langle \Psi | V_{\eta 0}^n | B_0 \rangle$ and for the bosonic states

$$|\xi\rangle = \xi_{mn} |m\rangle |\tilde{n}\rangle \ , \quad |C\rangle = C_{\dot a \dot b} |\dot a\rangle |\tilde{\dot b}\rangle$$

one finds

$$\langle \xi \rangle_{(n)} = T_p^2\, q_{i_1} ... q_{i_n}\, \xi^{ij} \omega^{i_1 ... i_n}_{i k_1 k_1 ... k_{n-1} k_{n-1} k_n}(\eta) M_{k_n j}$$

$$\langle C \rangle_{(n)} = T_p^2\, q_{i_1} ... q_{i_n}\, \omega^{i_1 ... i_n}_{j_1 ... j_{2n}}(\eta) \mathrm{Tr}[C \gamma^{j_1 j_2} ... \gamma^{j_{2n-1} j_{2n}} M]$$

Inserting a closed string propagator and Fourier transforming, one finds the asymptotic fields generated by a static Dp-brane.

Spin-dependent dynamics

The phase-shift for two // Dp-branes with parameters η_i and ϵ_i is

$$\mathcal{A} = \frac{(4\pi^2\alpha')^{4-p}}{16} \int_0^\infty dl \, \langle B, \eta_1, \epsilon_1, \vec{Y}_1 | e^{-2\pi\alpha' l p^+ (P^- - p^-)} | B, \eta_2, \epsilon_2, \vec{Y}_2 \rangle$$

and can be rewritten as

$$\mathcal{A} = \frac{V_p \, (4\pi^2\alpha')^{4-p}}{16 \mathrm{sh}\pi |\epsilon_1 - \epsilon_2|} \int_0^\infty dl \int \frac{d^{8-p}q}{(2\pi)^{8-p}} \, e^{i\vec{q}\cdot\vec{b}} \, e^{-\pi\alpha' l \vec{q}^2} Z_0(\eta_i, \epsilon_i) Z_{osc}(l, \eta_i, \epsilon_i)$$

with

$$Z_0(\eta_i, \epsilon_i) = \langle B_0, \eta_1, \epsilon_1 | B_0, \eta_2, \epsilon_2 \rangle$$

$$Z_{osc}(l, \eta_i, \epsilon_i) = \langle B_{osc}, \eta_1, \epsilon_1 | e^{-2\pi\alpha' l p^+ P^-} | B_{osc}, \eta_2, \epsilon_2 \rangle$$

Consider for simplicity $\eta_1 = \epsilon_1 = 0$. Expanding the Lorentz and the SUSY transformations in powers of $\epsilon_2 = \epsilon$ and $\eta_2 = \eta$, the partition functions can be rewritten as

$$Z_0(\eta, \epsilon) = \sum_{m=0}^{\infty} \sum_{n=0}^{4} \frac{1}{m! n!^2} \langle B_0 | V_{\epsilon 0}^m V_{\eta 0}^n | B_0 \rangle$$

$$Z_{osc}(l, \eta, \epsilon) = \sum_{q=0}^{\infty} \sum_{p=0}^{8} \frac{1}{q! p!^2} \langle B_{osc} | e^{-2\pi\alpha' l p^+ P^-} \, V_\epsilon^q V_\eta^p | B_{osc} \rangle$$

For $\eta = 0$ and $\epsilon = 0$, the configuration is still BPS and

$$Z_0(0,0) = 8 - 8 = 0$$

$$Z_{osc}(l, 0, 0) = \prod_{n=1}^{\infty} \frac{(1 - e^{-2\pi l n})^8}{(1 - e^{-2\pi l n})^8} = 1$$

For $\eta \neq 0$ and/or $\epsilon \neq 0$, the configuration is no longer BPS and

$$Z_0(\eta, \epsilon) \neq 0$$

$$Z_{osc}(l, \eta, \epsilon) \neq 1$$

The leading terms for $\eta, \epsilon \to 0$ in the total $Z(l, \eta, \epsilon)$ come from $m, n \neq 0$ ($Z_0(\eta, \epsilon) \sim (\eta\tilde{\eta})^n v^m$) but $p, q = 0$ ($Z_{osc}(l, \eta, \epsilon) \sim 1$).

The zero mode correlation contains only left-moving S_0^as and can be written as a trace over fermionic zero modes in Type I theory. Indeed, it is the analog of the integration over the fermionic zero modes in the path-integral representation of the open string vacuum amplitude.

The trace is 0 unless at least 8 zero modes S_0^a are inserted. The first $\neq 0$ is

$$\begin{aligned} t_8^{i_1\ldots i_8} &= \mathrm{Tr}_{S_0}[R_0^{i_1i_2}R_0^{i_3i_4}R_0^{i_5i_6}R_0^{i_7i_8}] \\ &= -\frac{1}{2}\epsilon^{i_1\ldots i_8} - \frac{1}{2}\left[\delta^{i_1i_4}\delta^{i_2i_3}\delta^{i_5i_8}\delta^{i_6i_7} + \text{perm.}\right] \\ &\quad +\frac{1}{2}\left[\delta^{i_2i_3}\delta^{i_4i_5}\delta^{i_6i_7}\delta^{i_8i_1} + \text{perm.}\right] \end{aligned}$$

Therefore, the leading terms in the amplitude receive contributions only from the massless states associated to the fermionic zero modes, and

$$\begin{aligned} Z_{osc}(l,\eta,\epsilon) &\underset{\eta,\epsilon\to 0}{\longrightarrow} 1 \\ Z_0(\eta,\epsilon) &\underset{\eta,\epsilon\to 0}{\longrightarrow} \sum_{m=0}^{4}\frac{1}{(4-m)!m!^2}\mathrm{Tr}_{S_0}[V_{\epsilon 0}^{4-m}V_{\eta 0}^{m}] \end{aligned}$$

Schematically ($\pi\epsilon \sim v$):

$$Z(l,\eta,\epsilon) \underset{\eta,\epsilon\to 0}{\longrightarrow} \sum_{k=0}^{4} c_k\, q^k\, t_8\, v^{4-k}\,\eta^{2k}\ , \quad \text{indep. of } \ l$$

From the open string point of view, this is an index to which only ultra short multiplets contribute.

The corresponding leading potential is

$$\begin{aligned} V &\underset{\eta,\epsilon\to 0}{\longrightarrow} T_p^2\sum_{k=0}^{4} c_k\, v_{m_k}\ldots v_{m_{4-k}}\, t_8^{1m_1\ldots 1m_{4-k}i_1\ldots i_{2k}}\,\omega^{j_1\ldots j_k}_{i_1\ldots i_{2k}}(\eta)\,\partial_{j_1}\ldots\partial_{j_k}\Delta_{(9-p)} \\ &\sim \sum_{k=0}^{4}\frac{v^{4-k}\eta^{2k}}{r^{7-p+k}} \end{aligned}$$

It is exact in α', that is scale-invariant.

D0-branes

In $SO(9)$ notation, $\theta = \binom{\eta_a}{\tilde{\eta}_{\dot{a}}}$, the result for elastic D0-brane scattering is

$$V = \frac{1}{8}\,\Big[v^4 + 2i\,v^2 v_m(\theta\gamma^{mn}\theta)\,\partial_n - 2v_p\,v_q(\theta\gamma^{pm}\theta)(\theta\gamma^{qn}\theta)\,\partial_m\partial_n \\ -\frac{4i}{9}v_i(\theta\gamma^{im}\theta)(\theta\gamma^{nl}\theta)(\theta\gamma^{pl}\theta)\,\partial_m\partial_n\partial_p \\ +\frac{2}{63}(\theta\gamma^{ml}\theta)(\theta\gamma^{nl}\theta)(\theta\gamma^{pk}\theta)(\theta\gamma^{qk}\theta)\,\partial_m\partial_n\partial_p\partial_q\Big]\,\Delta_{(9)}$$

Since this is scale-invariant, it has to be reproduced both in the SUGRA and SYM limits. Several explicit checks have been done:

All terms in SUGRA : Plefka, Serone and Waldron

1^{st} term in SYM : Douglas, Kabat, Pouliot and Shenker

2^{nd} term in SYM : Kraus

3^{rd} term in SYM : McArthur

5^{th} term in SYM : Barrio, Helling and Polhemus

It has also been shown in some detail that the leading part of the effective action is completely determined by SUSY (Paban,Sethi,Stern).

The BPS nature of D0-branes and the cancellation of leading orders in interactions imply specific ratios of the couplings at each multipole order, generalizing the usual BPS relation. In particular, $g = 1$, as appropriated for KK states, rather than $g = 2$.

These results are in agreement with the identification of D0-branes with KK states of M-theory compactified on $R_{11} = g_s l_s$ (Townsend,Witten). It has also been conjectured that in the infinite momentum frame, D0-brane are the partons of M-theory (Banks,Fischler,Shenker,Susskind).