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SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

I-34014 Trieste ITALY - Via Beirut 4 - Tel. [+]39-40-37871 - Telex:460269 SISSA I - Fax: [+]39-40-3787528

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Aspects of D-brane dynamics in superstring theory

Candidate
Claudio A. Scrucca

Supervisor
Prof. Roberto Iengo

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Contents

Introduction

One of the most fascinating and intriguing issues ever addressed in theoretical physics is the search for a consistent and unified quantum theory of fundamental interactions. The first major difficulty in this attempt is that there seems to be no consistent quantum field theory associated to Einstein's classical theory of general relativity [1], whereas all other fundamental interactions have instead been successfully formulated in this language. This suggests that perhaps quantum field theory is not the correct framework for the formulation of a so-called *theory of everything*. Moreover, the astonishing and appealing possibility of unifying gravity and gauge theories through the Kaluza-Klein mechanism of compactification [2, 3] has given strong support to the idea that actually our world might have more than four spacetime dimensions. All these arguments, together with supersymmetry [4, 5], have led to the formulation of supergravity [6, 7] and especially superstring theory [8, 9, 10, 11, 12], which is undoubtly the most promising candidate to this date for a unified theory of fundamental interactions.

Superstring theory describes one-dimensional extended strings, rather than point-like particles as does quantum field theory. The infinitely many vibrational modes of the string can be regarded as particle excitations with growing mass and spin, belonging to a so far unknown (and probably very complicated) field theory with an infinite tower of elementary fields. The tension $T = 1/(2\pi\alpha')$ of the string (energy per unit length) introduces a length scale $l_s = \sqrt{\alpha'}$ in the theory, so that the typical mass of the modes is $m_s = 1/\sqrt{\alpha'}$. For energies much below m_s, only the lowest lying massless modes are relevant, and their dynamics is encoded in a low-energy effective action (LEEA) obtained by integrating out all the massive modes. The effective field theories obtained in this way are various versions of super Yang-Mills (SYM) and supergravity (SUGRA) theories for open and closed strings respectively, α'-corrections appearing through higher dimensional effective operators.

The first quantized version of string theory is defined by assigning a conformally invariant world-sheet action, weighting the free propagation. More precisely, this action is in general a non-linear σ-model defining an embedding form the world-sheet Σ to a generic spacetime (or super spacetime for the Green-Schwarz formulation of the superstring) $\mathcal{M}$. The scalar fields appearing in the action are the spacetime coordinates of the string, whereas its spin is encoded in additional fermionic degrees of freedom. Free propagation of open and closed strings corresponds to world-sheets with the topology of a strip and a cylinder respectively. Interactions can instead be associated in a natural and geometric way to topologically more complex world-sheets representing the splitting and joining of strings. Thanks to the coupling of the dilaton background field ϕ to the world-sheet scalar curvature, whose integral gives (in two dimensions) the Euler characteristic χ_Σ of the world-sheet Σ, the amplitude corresponding to Σ is automatically weighted by its topology through a factor $e^{-\langle\phi\rangle\chi_\Sigma}$ involving the vacuum expectation value (VEV) of ϕ. It is therefore natural to identify the string coupling as $g_s = e^{-\langle\phi\rangle}$, so that Σ is weighted by $g_s^{-\chi_\Sigma}$.

The second quantized theory can then be built perturbatively, à la Feynman, assuming that the coupling constant g_s is small. Despite the powerful underlying mathematical structure inherited from conformal invariance [13], the lack of a truly second-quantized formulation of the theory is a severe limitation which is responsible for the difficulty of studying non-perturbative effects. Consistency at the quantum level requires D=10 spacetime dimensions, giving therefore a interesting prediction for the spacetime dimensionality. However, the theory is not unique, as one might have hoped. In fact, five apparently different consistent string theories are known:

- **Type I**
  N=1 SUSY, open strings with gauge group SO(32) and closed strings, unoriented.
  The LEEA is N=1 SUGRA coupled to SO(32) SYM.

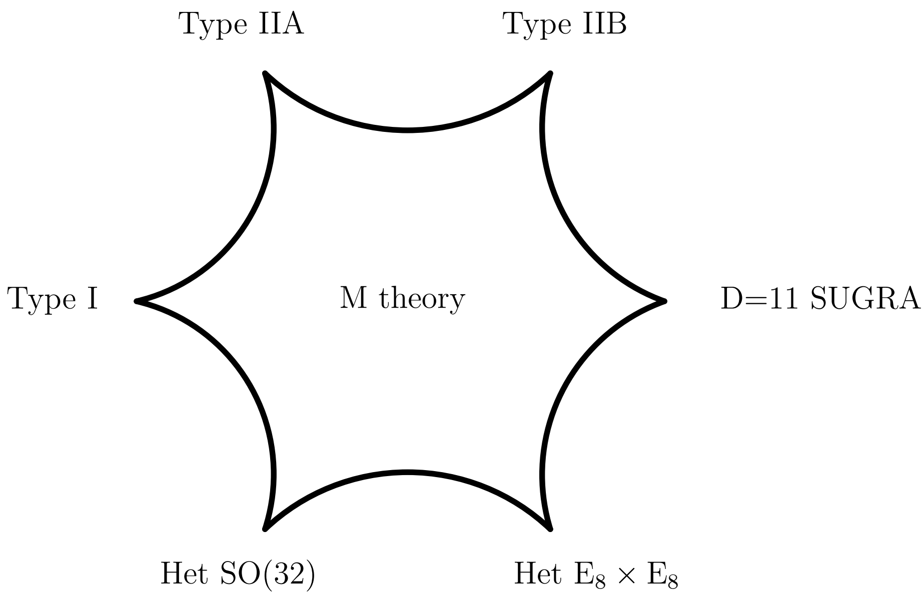
- **Type IIA,B**
  N=2 non-chiral (A) or chiral (B) SUSY, closed strings only.
  The LEEA is N=2A,B SUGRA.

- **Heterotic SO(32) and $\mathbf{E}_8 \times \mathbf{E}_8$**
  N=1 SUSY, closed strings with gauge groups SO(32) or $E_8 \times E_8$.
  The LEEA is N=1 SUGRA coupled to SO(32) or $E_8 \times E_8$ SYM.

Phenomenologically interesting models can be obtained upon compactification. More precisely, one makes a Kaluza-Klein ansatz of the form $\mathcal{M}_{10} = \mathbb{R}^{3,1} \otimes \mathcal{M}_6$ for the ten dimensional spacetime background, where $\mathbb{R}^{3,1}$ is four-dimensional flat Minkowski's space and $\mathcal{M}_6$ a compact manifold. The condition for this background to be an acceptable vacuum solution of the theory translates into the requirement that the non-linear σ-model describing string propagation be at a conformal fixed-point. For this to be true it is enough to chose $\mathcal{M}_6$ to be Ricci-flat. In order to preserve some of the original ten-dimensional supersymmetry in the four-dimensional effective theory, $\mathcal{M}_6$ has to be also a complex Kähler manifold. These two properties define a class of manifolds called Calabi-Yau manifolds, which turn out to be extremely important in string theory compactifications [14]. Another important class of compact spaces of great relevance in this context are the so-called orbifolds [15, 16]. They are defined by a manifold $\mathcal{M}_6$ modded out by a discrete equivalence group Γ, that is identifying points on $\mathcal{M}_6$ which are related by an element of Γ. This leads typically to isolated conical singularities on $\mathcal{M}_6/\Gamma$, where the Riemannian structure is lost. There exists nevertheless a well defined procedure, called *blow-up*, in which the singularities and their neighborhoods are substituted with a smooth space, yielding a regular manifold. In this sense, orbifolds represents singular limits of regular manifolds. Their importance lies essentially in the fact that even very simple versions of them (e.g. taking $\mathcal{M}_6$ to be flat so that $\mathcal{M}_6/\Gamma$ is also everywhere flat but at its singular points, where all the curvature is concentrated), can be topologically equivalent to some more complicated Calabi-Yau manifold. Correspondingly, the non-linear σ-model describing compactified string theory turns into a solvable orbifold CFT. A famous example is the $T^n/\mathbb{Z}_m$ orbifold, constructed by identifying points of a torus related by discrete rotations.

A major break-through in the understanding of string theory has been achieved by recognizing that its five different perturbative versions are actually related by various dualities [17, 18, 19, 20, 21], and are therefore not independent. Actually, most of these duality relations are conjectures that cannot be rigorously proven, since according to their non-perturbative character, this would mean to solve exactly the involved theories. Rather, evidence for a duality between two theories typically emerges from the matching of their BPS spectra, LEEA and supersymmetries. BPS states are invariant under a fraction of supersymmetry and have a mass m equal to the central charge $|z|$, saturating the BPS bound $m \geq |z|$. Since this property is related to the supersymmetry algebra only, these states are expected to be stable under any change in the free parameters of the theory, generically called moduli, and can therefore be used to infer duality relations even when the latter involve a map which changes dramatically the moduli. A first kind of duality are the so-called non-perturbative S-dualities, which are generalizations of electric-magnetic duality [22, 23, 24, 25]. They connect two theories whose couplings are related by a duality map, and typically weak coupling in one of the theory is mapped to strong coupling in the other. Important examples of the latter are the strong-weak coupling duality between Type I and SO(32) Heterotic theories in ten dimensions [26, 27, 28, 29], and between the Type IIA theory compactified on K_3 and the Heterotic theory compactified on T^4 [30, 31, 32, 33, 34]. There is also a conjectured SL(2,$\mathbb{Z}$) self-duality of the Type IIB theory in ten dimensions [30] which generalizes the corresponding symmetry of the Type IIB supergravity equations of motion [35, 36, 37, 38]. Another important class of duality are the so-called perturbative T-dualities [39]. In this case, the duality map inverts all the compactification radii ($R_i \to \alpha'/R_i$) and exchanges Kaluza-Klein and winding modes, but affects only in a multiplicative way the coupling ($g_s \to g_s\sqrt{\alpha'}/R$). These dualities can be proven to all orders in perturbation theory and relate for examples the two Type II theories or the two Heterotic theories when compactified on T^n. Finally, a more general kind of duality containing S and T-duality, called U-duality, has been proposed [30]. All string dualities correspond to the quantum realization of a discrete subgroup of some continuous global symmetry of the LEEA. In the SUGRA context, these symmetries have been known for a long time under the name of *hidden symmetries* and have been extensively studied [40, 41]. They correspond simply to the allowed isometries of the scalar manifold.

Another very important ingredient in defining a fundamental theory underlying the five perturbative string theories is the fact that D=10 Type IIA SUGRA can be obtained [40, 42, 43, 44, 45] from dimensional reduction of D=11 SUGRA [46]. Actually, upon this geometric compactification on a circle, one obtains also a tower of Kaluza-Klein modes which are BPS states with mass $m_n = |n|/R_{11}$ and charge $q_n = n/R_{11}$. The ten and eleven-dimensional couplings $\kappa_{(10)} \sim l_s^4$ and $\kappa_{(11)} \sim l_{11}^{9/2}$ are related through the compactification radius R_{11}: $\kappa_{(11)}/\kappa_{(10)} = \sqrt{2\pi R_{11}}$. Moreover, since the D=10 SUGRA action has an $e^{-2\phi}$ dependence on the dilaton ϕ, the effective ten-dimensional coupling is actually $g_s\kappa_{(10)}$, and a precise analysis of the compactification metric shows that the eleven-dimensional radius and length scale are given by $R_{11} = g_s l_s$ and $l_{11} = g_s^{1/3} l_s$ in terms of the string coupling g_s and length scale l_s [26]. This leads to the crucial observation that the strong coupling regime of D=10 Type IIA SUGRA is described by D=11 SUGRA. In fact, for $g_s \to 0$ all the infinite Kaluza-Klein modes become a continuum of massless states, signaling the opening of the eleven-th dimension. Lifting this statement from the LEEA up to the level of string theory, this means that the strong coupling of Type IIA superstring theory is described by some mysterious eleven-dimensional theory, called M theory, whose LEEA is D=11 SUGRA [47, 26, 48]. Using the various dualities relating the Type IIA theory to the other four perturbative string theories, it becomes clear that M theory can be thought as a fundamental and non-perturbative theory which reduces in various corners of its moduli space to D=11 SUGRA or one of the five perturbative superstring theories, as depicted in the now famous *hexagon* figure of dualities. An extremely important issue is to understand



the nature and the role of Kaluza-Klein modes arising from eleven dimensions, both in SUGRA and in string theory. It is worth to recall that their identification with massive string states has been ruled out long ago for a number of reason. In particular, they carry a non-vanishing Ramond-Ramond (RR) charge, whereas string states carry Neveu-Schwarz-Neveu-Schwarz (NSNS) charge but couple only non-minimally to the RR field-strengths. In the SUGRA context, there exist solitonic p-extended solutions called *p-branes* [49, 50, 51] (see also [52, 53]), both with RR and NSNS charges. They are black-hole-like BPS configurations with a tension proportional to $1/g_s^2$ for the NSNS ones which are ordinary solitons, and to $1/g_s$ only for the RR ones. In particular, RR p-branes have a charge μ_p with respect to the RR (p+1)-form $C_{(p+1)}$, and since the latter are related by Hodge-duality, ${}^*F_{(p)} = F_{(10-p)}$, low-dimensional p-brane can be considered as electrically charged with respect to $C_{(p+1)}$ and high-dimensional p-brane as magnetically charged with respect to $C_{(7-p)}$ if one chooses the low-dimensional forms as fundamental degrees of freedom. It is moreover convenient to define $\hat{\mu}_p$ as the charge in inverse units of $\sqrt{2}\kappa_{(10)}$, so that the true charge is $\mu_p = \hat{\mu}_p/(\sqrt{2}\kappa_{(10)})$. Dirac's quantization condition, appropriately generalized to extended objects [54, 55], then requires that $\hat{\mu}_p\hat{\mu}_{6-p} = 2\pi n$ for consistency of the theory at the quantum level, so that the allowed charges μ_p for p-branes are quantized. This allows the tantalizing identification of solitonic 0-branes with Kaluza-Klein modes [47, 26].

At the string level, it is natural to expect the appearance of solitonic states whose LEEA counterparts are p-branes. A major difficulty that one has then to face is to fit these non-perturbative states into the conformal field theory (CFT) defining perturbative string theory. This has been possible due to the extremely important observation [56] that p-branes correspond in string theory to topological hyper-plane defects on which string world-sheets can end through a boundary [57, 58]. Since this corresponds to choosing Dirichlet rather than traditional Neumann boundary conditions for the fields in the world-volume directions, these stringy generalizations of p-branes have been called *Dp-branes* (see [59, 60, 61, 62]). These objects exists in the Type I theory with p=5,9 and in the Type IIA and Type IIB theories for p even and odd respectively, where they couple to the corresponding RR (p+1)-forms. Dp-branes are BPS states preserving half of the supersymmetry, with a charge density μ_p which is equal to their tension. They carry the elementary quanta of RR charges $\hat{\mu}_p = \sqrt{2\pi}(2\pi\sqrt{\alpha'})^{3-p}$ in inverse units of $\sqrt{2}\kappa_{(10)}$, which satisfy the Dirac quantization condition with the minimum allowed integer $n = 1$. As before, the true charge is $\mu_p = \hat{\mu}_p/(\sqrt{2}\kappa_{(10)})$. Also, due to the $e^{-\phi}$ dependence of its effective action, the Dp-brane has an effective tension proportional to $1/g_s$ as for the RR p-brane solitons of supergravity. This gives further evidence for the identification of Dp-branes as the stringy version of p-branes. In particular, since the Kaluza-Klein modes of D=10 Type IIA SUGRA have been identified with solitonic 0-branes at the quantum level, it follows that the corresponding objects in string theory are D0-branes, which play indeed a very important role. More precisely, the n-th Kaluza-Klein mode carries n units of fundamental quantum charge and is therefore identified with the *threshold bound state* of n D0-branes. These bound-states of zero binding energy are BPS states with m=q and their existence as genuine quantum states [63, 64] is crucial for the identification to be possible.

The discovery of D-branes and a powerful and efficient CFT description of them has opened the possibility of studying non-perturbative aspects of string theory, like $\mathcal{O}(e^{-1/g_s})$ effects that where expected from large order in string perturbation theory [65] and recognized to be related to boundary effects [66]. Another extremely important theoretical issue is related to the black hole nature of D-branes. Since the first disappointments with the quantum version of Einstein's theory, it has been been accepted that general relativity should be the LEEA of some microscopic theory like string theory. This belief is also strongly sustained by the discovery that black holes, which can be considered in some sense as solitonic solutions of Einstein's theory, are actually thermodynamical objects with a non-vanishing temperature [67] and entropy [68, 69, 70]. The study of black holes and their thermodynamics is therefore of extreme importance and should allow to test significantly (and non-perturbatively) string theory, probably more than any possible future high-energy particle physics experiment. In fact, one of the most exciting and significant successes of string theory is the microscopic explanation of the entropy of *extremal black holes* as a statistical entropy associated to its microscopic stringy constituents (see [71]). In much the same way as singular p-brane solitons are described in string theory by D-branes, the string theory description of regular point-like black holes is generically given in terms of several D-branes wrapped in various way on the compact part of spacetime, possibly with massless open-string stretched between them. The statistical entropy of the resulting composite object is associated to the degeneracy of microscopic states yielding the same macroscopic properties like mass and charge, and correctly reproduces the Hawking-Beckenstein *area low* (even the correct numerical factor comes out).

M theory is strongly suspected [47] to be the theory of a supermembrane [72, 73], whose world-volume action is known to reproduce the Green-Schwarz action of the Type IIA superstring by double (world-sheet and spacetime) dimensional reduction [74]. Unfortunately, the quantization of the eleven-dimensional supermembrane presents some extremely subtle aspects and has not yet been accomplished. The key point seems to be the infinite dimensional symmetry group of area-preserving diffeomorphisms that is responsible for decoupling of ghost modes. This admits a finite dimensional SU(N) regularization which allows to formulate the theory in the light-cone gauge as the large N limit of U(N) maximally supersymmetric quantum mechanics (SQM) [75, 76], which can also be thought as the dimensional reduction of D=10 SYM theory to D=1 (see also [77, 78]). This is one of the arguments that have led to the *matrix model* conjecture [79], according to which M theory in the infinite momentum frame admits a parton description with a dynamics governed by U(N) SQM in the limit of infinite number N $\to \infty$ of partons (for a review see [80, 81, 82]). Furthermore, it has been established [83, 84] that this *matrix theory* has a meaning even for finite N and describes the discrete light-cone quantization of M theory compactified on a light-like circle R_l at fixed $p^- = \mathrm{N}/R_l$. An important point of the conjecture is the identification of the partons with D0-branes. This is suggested by the fact that the short-distance dynamics of a cluster of N of them is known to be governed precisely by U(N) SQM [85]. The appearance of the eleven-dimensional Plank scale in D0-brane quantum mechanics [86, 87, 88, 89] was indeed one of the most important ingredients in the formulation of matrix theory.

Because of these recent developments, it has become more and more clear that D-branes represent an important opportunity to learn about non-perturbative aspects of string theory. For instance, the distinction between Type I (containing also open strings) and Type II theories (containing only closed strings) is no longer really sensitive in a D-brane background, in which both types of theories can have open strings ending on D-branes. In the modern language, these theories are referred to as theories with D-branes, and ordinary Type I theory is simply a theory with D9-branes (i.e. open strings are free to end everywhere since a D9-brane has a world-volume occupying all ten-dimensional spacetime). In particular, the study of their string theory dynamics [90] has proven to be an extremely rich source of information. For instance, interesting and important relations between SUGRA and SYM effective actions at long and short distances have emerged in this context. A fundamental issue in the study of D-brane dynamics is the determination of their effective world-volume action in a generic SUGRA background, which encodes the couplings of D-branes as sources of massless fields of string theory. This is determined at leading order in the string coupling g_s by a string world-sheet with the topology of a disk attached to the D-brane. It represents the tree-level effective action induced by open string fluctuations, to be interpreted as the propagation of a virtual open string first appearing and then disappearing on the D-brane. Higher order corrections in g_s are instead associated with more complicated world-sheets, possibly with holes and all boundaries attached to the D-brane. Another important issue is the study of interactions between two D-branes, and the determination of the effective action governing their dynamics. This is given at leading order in the string coupling g_s by a string world-sheet with the topology of a cylinder connecting the two D-branes. In Euclidean signature, this world-sheet can be interpreted either as a loop of open strings stretched between the two D-branes or as a tree-level exchange of closed strings emitted by one of the D-branes and absorbed by the other. The corresponding interaction can therefore be considered equivalently as a one-loop effective action obtained by integrating out open string vacuum fluctuations or the tree-level effective interaction obtained by taking into account closed string exchange. Higher order corrections come from more complicated world-sheets with all boundaries lying on one of the two D-branes and possibly containing also holes. This open-closed string duality is not peculiar to the cylinder world-sheet but is in fact a particular example of the more general fact that a generic Euclidean world-sheet can be interpreted in different way by changing the time slicing. This was noted long ago in the context of Neumann boundaries [91, 92, 93], but it is really only for Dirichlet boundaries that this arguments acquires a physical significance, beside its mathematical convenience. In particular, a powerful formalism called *boundary state formalism* has been developed [94, 95, 96, 97] to treat boundaries, which are naturally associated with open strings, from a closed string point of view. The main idea is that the boundary itself can be regarded as a closed string coherent state, the boundary state, implementing the boundary conditions. This state is obtained essentially by a Bogoliubov transformation on the closed string Fock vacuum, and represents the couplings of the boundary to closed strings.

In this work, we study various aspects of D-brane dynamics in superstring theory, using mainly the boundary state formalism to analyze string amplitudes with one or two boundaries ending on D-branes. In Chapter 1 we recall some basic concepts of modern superstring theory, focusing on those aspects which are most relevant to D-branes and the understanding of their basic properties. In Chapter 2 we review in some detail the basic computations of the phase-shift for static and moving D-branes, focusing one the Dp-Dp and Dp-D(p+4) systems preserving 1/2 and 1/4 of the supersymmetries [56, 90, 98]. In Chapter 3, we give a brief introduction to the boundary state formalism, discussing in general its properties and its utility in studying D-branes. In Chapter 4, we study point-like D-brane configurations in D=4 Type II compactifications which are particularly interesting as potential microscopic descriptions of various D=4 black holes. We concentrate on T^6, $T^2 \times T^4/\mathbb{Z}_2$ and $T^6/\mathbb{Z}_6$ compactifications, whose LEEA is D=4 SUGRA with N=8, 4 and 2 supersymmetry. We first study the interactions between two of these point-like D-brane configurations in the boundary state formalism [99], focusing on the interesting cases of the dimensionally reduced D0-brane and the wrapped D3-brane. We then study the probability amplitude for the emission of a massless closed string state from two of these configurations in interaction [100], focusing on the four-dimensional axion, dilaton and graviton arising in the NSNS sector. The computation involves the evaluation of the one-point function of the corresponding vertex operator on the cylindrical world-sheet connecting the two D-branes, which we carry out again in the boundary state formalism. Both analysis show that the dimensionally reduced D0-brane represents a singular extremal dilatonic solution of the relevant low-energy SUGRA, charged under the various scalar and vector fields of the theory, with no horizon and vanishing entropy. Similarly, the D3-brane wrapped on T^6 and $T^2 \times T^4/\mathbb{Z}_2$ represents charged singular solutions of the corresponding N=8 and N=4 SUGRA in D=4. The D3-brane wrapped on $T^6/\mathbb{Z}_6$ does instead not couple to any scalar of the relevant D=4 N=2 SUGRA, and therefore represents a regular Reissner-Nordström black hole solution with a finite horizon and a non-vanishing entropy. In Chapter 5, we address more in detail the issue of identifying four-dimensional point-like configurations with black hole solutions of the appropriate D=4 SUGRA. We first study magnetic properties in order to determine the possible magnetic charges inherited by the point-like configurations. After recalling some basic concepts about the interactions of generic dyonic extended objects, we propose a precise way of computing electromagnetic interactions between magnetically dual Dp and D(6−p)-branes in string theory [101], and show that the electric and magnetic interactions are encoded in the even and odd RR spin-structures arising in the RR sector as a consequence of the GSO-projection. We then apply this general description to the D3-brane, both in ten dimensions and wrapped on T^6 and $T^6/\mathbb{Z}_6$. By studying the couplings appearing in the electric and magnetic interactions, we determine the D=4 electric and magnetic charges of the point-like configurations, which are related to the orientation of the D3-brane in the compact part of spacetime. On T^6 we find a four-parameter family of singular dyonic black holes of D=4 N=8 SUGRA, whereas on $T^6/\mathbb{Z}_3$ we find a one-parameter family of regular dyonic black holes of D=4 N=2 SUGRA. We then discuss the construction of an explicitly solution of D=10 SUGRA corresponding to a 3-brane wrapped on a 3-cycle of a generic CY threefold $\mathcal{M}_3^{CY}$ and, specializing to the limit $\mathcal{M}_3^{CY} = T^6/\mathbb{Z}_3$, we show that it represents indeed a regular R-N black hole from the four-dimensional point of view [102]. We also deduce the couplings to massless fields in the string theory description by computing the overlap of the corresponding closed string state with the boundary state describing the wrapped D3-brane, finding the same values for the four-dimensional electric and magnetic charges as those extracted from the computation of electromagnetic phase-shifts. We also give a geometric interpretation of the angle parameterizing the charges within the previously constructed SUGRA solution. Finally, in Chapter 6 we study the spin dependence of the interactions between two moving D-branes using the boundary state formalism in the Green-Schwarz formulation of superstring theory. We focus our attention on the leading terms for small velocities v, which are found to behave as v^{4-n}/r^{7-p+n} and v^{2-n}/r^{3-p+n} for the Dp-Dp and Dp-D(p+4) systems [103]. These leading interactions are completely determined by the fermionic zero modes, the contributions of massive non-BPS states canceling by supersymmetry. This implies the scale-invariance of these leading spin-effects, and supports the equivalence between the SYM and SUGRA descriptions of D-brane dynamics [104]. We compute also one-point functions of massless fields encoding all the non-minimal spin-dependent couplings, and give a detailed field theory interpretation of our results. We conclude by arguing that the matching between the SYM and SUGRA truncations for one-loop leading interactions is dictated by supersymmetry, which determines them completely without leaving any dynamical freedom.

Chapter 1

Superstrings and D-branes

In this chapter, we recall some generalities about superstrings and D-branes in the covariant formulation. In particular, we review how D-branes arises as as hyper-planes on which string world-sheets can end through a Dirichlet boundary. We also discuss T-duality and its important consequences in the context of D-branes.

1.1 Strings

In the Ramond-Neveu-Schwarz (RNS) covariant formulation, the superstring action in a flat Minkowski background and in the conformal gauge reads

$$S_0 = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu + i\alpha' \bar{\Psi}^\mu \partial\!\!\!/ \Psi_\mu\right) \ , \tag{1.1}$$

where Σ denotes the (Euclidean) two-dimensional world-sheet with coordinate σ_1 and σ_2 and μ runs from 0 to 9. The coordinates X^μ in ten-dimensional target-space are world-sheet scalars, whereas the internal spin degrees of freedom Ψ^μ_a are two-component Majorana world-sheet spinors which can be decomposed into one-dimensional Majorana-Weyl components as $\Psi^\mu_a = \binom{\psi^\mu}{\tilde{\psi}^\mu}$. The theory has N=1 or N=2 world-sheet superconformal symmetry depending on Σ. The equations of motion are obtained by setting the variation of this action with respect to X^μ and ψ^μ, $\tilde{\psi}^\mu$ to zero. The variation has a bulk and a boundary term:

$$\begin{aligned}\delta S_0 = &\frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma \left(\delta X_\mu \Box X^\mu + i\alpha' \delta\Psi_\mu \partial\!\!\!/ \Psi^\mu\right) \\ &+\frac{1}{2\pi\alpha'}\oint_{\partial\Sigma} d\sigma_\alpha \epsilon^{\alpha\beta}\left(\delta X_\mu \partial_\beta X^\mu + i\alpha' \delta\psi_\mu \rho_\beta \Psi^\mu\right) \ .\end{aligned} \tag{1.2}$$

In order make this to vanish without ruining locality, the integrands of the bulk term and each of the boundary terms have to be separately zero.

The bulk equations of motion are the usual Laplace and Dirac equations on Σ

$$\Box X^\mu = 0 \ , \tag{1.3}$$

$$\partial\!\!\!/ \Psi^\mu = 0 \ . \tag{1.4}$$

The boundary equations of motion can be solved in two different ways. For the bosons, one can set either the normal derivative $\partial_n X^\mu$ or the variation δX^μ to zero on the boundary.

This two choices are referred to as Neumann (N) and Dirichlet (D) boundary conditions (b.c.) respectively. The former amount to set the momentum flowing out of the boundary to zero, whereas the latter corresponds to fix the end-points at some fixed value Y^μ for the coordinate, which is also equivalent to requiring that the tangential derivative $\partial_t X^\mu$ be zero on the boundary. Summarizing,

$$\partial_n X^\mu \delta X^\mu|_{\partial\Sigma} = 0 \;\Rightarrow\; \begin{cases} \partial_n X^\mu|_{\partial\Sigma} = 0 & \text{N} \\ \partial_t X^\mu|_{\partial\Sigma} = 0 & \text{D} \end{cases} . \tag{1.5}$$

For the fermions, one has to identify the two chiral components up to a sign. The two possibilities correspond therefore to equal (+) and opposite (−) sign b.c.

$$\psi_\mu \delta\psi^\mu - \tilde\psi_\mu \delta\tilde\psi^\mu\Big|_{\partial\Sigma} = 0 \;\Rightarrow\; \begin{cases} \psi^\mu = \tilde\psi^\mu\Big|_{\partial\Sigma} & + \\ \psi^\mu = -\tilde\psi^\mu\Big|_{\partial\Sigma} & - \end{cases} . \tag{1.6}$$

As we will see, an important features shared both by N and D b.c. for the bosons and + or − b.c. for fermions, is that they identify two otherwise independent components of the corresponding fields with a ± sign. For a number of reasons, it is natural to associate the N and D bosonic b.c. respectively with the + and − fermionic b.c.. In this way, choosing say the first p+1 coordinates X^α and fermions ψ^α to be N and +, and the remaining 9−p coordinates X^i and fermions ψ^i to be D and −, the original SO(9,1) Lorentz invariance of the theory is broken to SO(p,1) × SO(9−p), corresponding to a flat (p+1)-dimensional topological defect: a Dp-brane. A crucial feature emerging from this setting is that end-points of strings can move only in the (p+1)-hyperplane corresponding to the Dp-brane.

1.1.1 Open strings

Consider for instance a world-sheet like in Fig. 1.1 with the topology of a strip, with coordinates τ running from $-\infty$ to ∞ and σ from 0 to π, representing the propagation of an open string. Using the notation $z, \bar z = \tau \pm i\sigma$, and correspondingly $\partial, \bar\partial = 1/2(\partial_\tau \pm i\partial_\sigma)$,

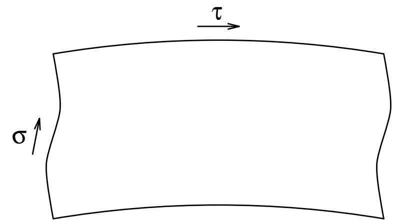


Figure 1.1: The propagation of an open string.

the bulk equation of motion imply the usual splitting of the fields in left and right movers

$$\partial\bar\partial X^\mu = 0 \quad\Rightarrow\quad X^\mu = X^\mu(z) + \tilde X^\mu(\bar z) \,, \tag{1.7}$$
$$\partial\psi^\mu = \bar\partial\tilde\psi^\mu = 0 \quad\Rightarrow\quad \psi = \psi(z) \,,\;\; \tilde\psi = \tilde\psi(\bar z) \,. \tag{1.8}$$

At each of the two boundaries, one can then choose N or D b.c. for the bosons and + or − b.c. for the fermions. The N,D and ± b.c. can be rewritten as

$$\left\{ \begin{array}{ll} \partial X^\mu = \bar\partial \bar X^\mu|_{\partial\Sigma} & \text{N} \\ \partial X^\mu = -\bar\partial \bar X^\mu|_{\partial\Sigma} & \text{D} \end{array} \right. \quad , \quad \left\{ \begin{array}{ll} \psi^\mu = \tilde\psi^\mu\Big|_{\partial\Sigma} & + \\ \psi^\mu = -\tilde\psi^\mu\Big|_{\partial\Sigma} & - \end{array} \right. . \tag{1.9}$$

Therefore, the b.c. identify left and right movers up to a sign, independently on each connected component of the boundary $\partial\Sigma$.

For generality, suppose that the two boundaries at $\sigma = 0, \pi$ end on a Dp and a Dq-brane respectively. Correspondingly, various combinations of b.c. arise both for the bosons and the fermions. The bosonic coordinates satisfy NN, DD, ND or DN b.c. depending on if they belong or not to the world-volumes of the Dp and the Dq-branes. The mode expansion for X^μ in these four cases is given by the following expressions

$$X^\mu = \left\{ \begin{array}{ll} x^\mu - 2i\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \frac{\alpha_n^\mu}{n} \left(e^{-nz} + e^{-n\bar z}\right) & , \ \text{NN} \\ Y_0^\mu + \frac{Y_0^\mu - Y_\pi^\mu}{\pi}\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \frac{\alpha_n^\mu}{n} \left(e^{-nz} - e^{-n\bar z}\right) & , \ \text{DD} \\ Y_{0,\pi}^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\pm\frac{1}{2}} \frac{\alpha_n^\mu}{n} \left(e^{-nz} + e^{-n\bar z}\right) & , \ \text{DN,ND} \end{array} \right. . \tag{1.10}$$

x^μ and p^μ are the center of mass position and momentum operators and satisfy the canonical commutation relation $[x^\mu, p^\mu] = i\eta^{\mu\nu}$, whereas the modes α_n^μ satisfy $[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}$. For the fermions, the b.c. can be either $++$, $--$, $+-$ or $-+$. The overall sign between left and right mover is a matter of definition since it can be changed by a field redefinition, so that only the relative sign between the two boundaries is relevant. Let us therefore choose as starting convention to associate the ± fermionic b.c. to N,D bosonic b.c. respectively, in agreement with superconformal and broken Lorentz invariance. There is then still the freedom of changing the relative sign between the two boundaries. We shall refer with R and NS to the sectors respectively with and without an additional flip in the relative sign. With these conventions, ψ^μ has integer moding for NN and DD directions and half-integer moding for ND and DN directions, in the R sector, and vice versa in the NS sector. Notice that in this way the moding of the fermions in the R and NS sectors is always respectively equal and opposite to that of the bosons. The fermion mode expansion is generically

$$\psi^\mu = \sqrt{\alpha'} \sum_n \psi_n^\mu e^{-nz} \ , \tag{1.11}$$

$$\tilde\psi^\mu = \sqrt{\alpha'} \sum_n \psi_n^\mu e^{-n\bar z} \ . \tag{1.12}$$

with appropriate moding and $\{\psi_m^\mu, \psi_n^\nu\} = \delta_{m+n}\eta^{\mu\nu}$.

The Fock space is constructed by acting with negative frequency modes on a vacuum $|0\rangle$ annihilated by all the positive frequency modes. Whenever fermions have integer moding, there are fermionic zero modes ψ_0^μ satisfying the Clifford algebra $\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}$. The vacuum $|0\rangle$ then becomes a 32-dimensional SO(9,1) spinor representation of this algebra, with the fermionic zero modes acting as Γ-matrices, $\psi_0^\mu = \Gamma^\mu/\sqrt{2}$. In the sectors where fermions have half-integer moding, the vacuum $|0\rangle$ is instead a scalar. The world-sheet Hamiltonian can be written as the sum of a zero mode and an oscillator parts

$$H = H_0 + H_{osc} \ . \tag{1.13}$$

The zero mode part depends on the b.c.

$$H_0 = \begin{cases} \dfrac{\alpha'}{2}p^2 & , \ \text{NN} \\ \dfrac{\alpha'}{2}\left(\dfrac{\Delta Y}{\pi\alpha'}\right)^2 & , \ \text{DD} \\ 0 & , \ \text{ND,DN} \end{cases} \ . \tag{1.14}$$

In the NN case, it represents the kinetic energy due to the center of mass motion, whereas in the DD case, it accounts for the potential energy due to the stretching form Y_0^μ to Y_π^μ, proportional to the distance $\Delta Y = |Y_0^\mu - Y_\pi^\mu|$. The oscillator part can be written in a universal way as

$$H_{osc} = N - a \ , \tag{1.15}$$

where

$$N = \sum_{n>0} (\alpha_{-n}\cdot\alpha_n + n\psi_{-n}\cdot\psi_n) \tag{1.16}$$

is the excitation level, with appropriate moding, and a is the total normal-ordering zero-point energy. The contributions to a from a single physical boson and fermion is $-1/24$ and $1/24$ for integer moding, and $1/48$ and $-1/48$ for half-integer moding.

Consider in particular the purely Neumann standard open string theory, that is open strings whose end-points live on a D9-brane. In this case, world-sheet fermions have integer and half-integer moding in the R and NS sectors respectively. The R ground state is a spacetime spinor, whereas the NS one is a scalar, so that R and NS states are spacetime fermions and bosons respectively. The total normal ordering constant a is equal to 0 and $-1/2$, so that the lowest lying NS mode is tachyonic. The GSO projection is implemented through the projection $P = 1/2(1 + (-1)^F)$. It achieves spacetime supersymmetry and projects out the tachyon by keeping only the states with even world-sheet fermion number. The lowest lying modes of each sector are massless and fill the following irreducible representations of the SO(8) little group:

Type I

$$\begin{array}{lcccc} \text{NS} & : & \mathbf{8_v} & \Leftrightarrow & A_\mu \\ \text{R} & : & \mathbf{8_s} & \Leftrightarrow & \psi_\alpha \end{array} \tag{1.17}$$

In total, one has therefore a massless vector multiplet. The LEEA for this light mode is N=1 D=10 SYM, and is completely determined by supersymmetry

$$S = \frac{1}{g_{YM}^{(10)2}} \int d^{10}x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi\right) \ , \tag{1.18}$$

with

$$g_{YM}^{(10)} = \sqrt{g_s}(2\pi)^{\frac{7}{2}}\alpha'^{\frac{3}{2}} \sim \sqrt{g_s} l_s^3 \; . \tag{1.19}$$

Actually, SYM theory is not renormalizable in ten dimensions, so that it cannot be promoted to a consistent microscopic theory. Rather, it can be used consistently only up to some cut-off energy scale of the order of the string mass $m_s = 1/\sqrt{\alpha'}$. At higher energies, α'-corrections to the LEEA coming from integrating out virtual massive string modes and entering through higher-dimensional operators involving the scale α', become important. The determination of these α'-corrections is a long-standing problem, which can be faced in a number of different way. One can for example reconstruct order by order the effective action by requiring it to reproduce the tree-level (disk) n-photons correlation functions computed in string theory. In principle, one could also compute the generating functional for such correlation functions by directly evaluating the Polyakov path-integral on a disk with a generic electromagnetic background coupling to the end-points of open strings, which amounts to exponentiate the photon vertex-operator. This can be done exactly in the constant field approximation, which is the lowest order approximation of a derivative expansion and corresponds to resum all the α'-corrections with at most second derivatives. One obtains a non-linear Born-Infeld generalization of the SYM theory [105, 106]

$$S = -\frac{T_9}{g_s}\int d^{10}x \sqrt{-\det\left(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right)} + \text{ferm.} \; , \tag{1.20}$$

with

$$T_9 = \sqrt{\alpha'}^{-1}\left(2\pi\sqrt{\alpha'}\right)^{-9} \sim l_s^{-10} \; . \tag{1.21}$$

It is worth mentioning that there exists an very interesting connection between the open string LEEA and dissipative quantum mechanics (DQM) [96, 107, 108] (see also [109, 110]). DQM can be introduced by coupling a particle with ordinary dynamics to a bath of infinite harmonic oscillators with a linearly growing frequency spectrum, which once integrated out leave an effective dissipation term [111]. It is quite obvious that the same is happening the the end-point of the string. Since the electromagnetic background only couples to the end-points of the open string, one can evaluate the path-integral by first integrating out the free bulk oscillations. One is then left with a path-integral over the end-points with a dissipative dynamics on all of the ten coordinates, which then yields immediately the Born-Infeld Lagrangian. Yet another way to determine the exact LEEA is to require the interacting σ-model for a generic electromagnetic background to be at a conformal fixed-point. This can be implemented at lowest order by require the vanishing of the β-function, obtaining the same Born-Infeld Lagrangian [112].

When some of the ten directions are Dirichlet, say the last 9−p so that the end-points of the open strings now live on a Dp-branes, all the discussion goes through essentially in the same way. As already explained, ten-dimensional Lorentz invariance SO(9,1) is broken to SO(p,1) × SO(9−p) so that one can imagine the theory as effectively living in the (p+1)-dimensional world-volumes of the Dp-branes, with and SO(p,1) Lorentz invariance and an SO(9−p) global R-symmetry inherited from the invariance of the theory under transverse rotations. It is a straightforward exercise to determine how the lowest lying massless modes of each sector transform under the Lorentz and R-symmetry groups. One obtains precisely the content corresponding to the dimensional reduction of a D=10 vector multiplet to D=p+1 dimensions. In particular, the ten-dimensional vector field living on the D9-brane of the standard Type I theory splits into a (p+1)-dimensional vector A_μ living on the Dp-brane and 9−p scalar fields q^i. The LEEA for these light modes is therefore N=1 SYM reduced from D=10 to D=p+1, and is again completely determined by supersymmetry

$$S = \frac{1}{g_{YM}^{(p+1)2}} \int d^{p+1}x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2(2\pi\alpha')^2}\partial_\mu q^i \partial^\mu q^i\right) + \text{ferm.} \,, \tag{1.22}$$

with

$$g_{YM}^{(p+1)} = \sqrt{g_s}(2\pi)^{\frac{p-2}{2}}\alpha'^{\frac{p-3}{2}} \sim \sqrt{g_s} l_s^{\frac{p-3}{2}} \,. \tag{1.23}$$

As before, there are α'-prime corrections to this LEEA, which in the constant field approximation yield again a non-linear Born-Infeld generalization of the SYM theory

$$S = -\frac{T_p}{g_s}\int d^{p+1}x \sqrt{-\det\left(\eta_{\mu\nu} + \partial_\mu q^i \partial_\nu q^i + 2\pi\alpha' F_{\mu\nu}\right)} + \text{ferm.} \,, \tag{1.24}$$

with

$$T_p = \sqrt{\alpha'}^{-1}\left(2\pi\sqrt{\alpha'}\right)^{-p} \sim l_s^{-(p+1)} \,. \tag{1.25}$$

Finally, some of the ten directions can have mixed ND or DN b.c.. This happens for example when for open strings stretched between a Dp and a Dq-brane. In this case the analysis is some what more involve and one finds typically hypermultiplets in an N=1 SYM theory reduced from D<10 to D=p+1 (see [113]).

1.1.2 Closed strings

Before going on, it is worth recalling the basic properties of closed strings in absence of branes. Since in this case there are no boundaries on the world-sheet, left and right moving degrees of freedom remain completely independent. Consider in particular a world-sheet with the topology of a cylinder like in Fig. 1.2, with τ running from $-\infty$ to ∞ and σ from 0 to 2π. Using as before the notation $z, \bar{z} = \tau \pm i\sigma$, the equations of motion again imply

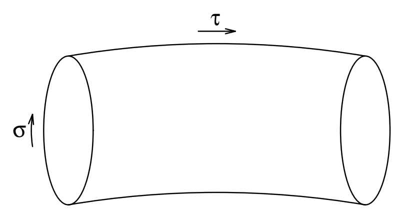


Figure 1.2: The propagation of a closed string.

the splitting of the fields in left and right movers

$$\partial\bar{\partial}X^\mu = 0 \quad \Rightarrow \quad X^\mu = X^\mu(z) + \tilde{X}^\mu(\bar{z}) \,, \tag{1.26}$$

$$\partial\psi^\mu = \bar{\partial}\tilde{\psi}^\mu = 0 \quad \Rightarrow \quad \psi = \psi(z) \,, \;\; \tilde{\psi} = \tilde{\psi}(\bar{z}) \,. \tag{1.27}$$

The bosonic coordinates must be periodic in σ and have therefore integer moding

$$X^\mu = x^\mu - 2i\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n}\left(\alpha_n^\mu e^{-nz} + \tilde{\alpha}_n^\mu e^{-n\bar{z}}\right) \,. \tag{1.28}$$

As before $[x^\mu, p^\mu] = i\eta^{\mu\nu}$ and the modes α_n^μ and $\tilde{\alpha}_n^\mu$ satisfy the usual commutation relation $[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}$. Each chiral component of the fermions can instead be either periodic (R) with integer moding or antiperiodic (NS) with half-integer moding. There are therefore four sectors, RR, NSNS, RNS and NSR, corresponding all the possible choices. The mode expansion is generically

$$\psi^\mu = \sqrt{\alpha'} \sum_n \psi_n^\mu e^{-nz} \ , \tag{1.29}$$

$$\tilde{\psi}^\mu = \sqrt{\alpha'} \sum_n \tilde{\psi}_n^\mu e^{-n\bar{z}} \ , \tag{1.30}$$

with appropriate moding. The commutation relations are $\{\psi_m^\mu, \psi_n^\nu\} = \{\tilde{\psi}_m^\mu, \tilde{\psi}_n^\nu\} = \delta_{m+n}\eta^{\mu\nu}$.

The closed string Fock space is essentially the tensor product of two open string Fock spaces for the left and right-moving sectors. Again, it is constructed by acting with negative frequency modes on a vacuum $|0\rangle \otimes |\tilde{0}\rangle$ annihilated by all the positive frequency modes. Depending on the sector, $|0\rangle$ and $|\tilde{0}\rangle$ are either SO(9,1) spinors or scalars, and as before, fermionic zero modes act as gamma matrices, $\psi_0^\mu = \Gamma^\mu/\sqrt{2}$, $\tilde{\psi}_0^\mu = \tilde{\Gamma}^\mu/\sqrt{2}$. The world sheet Hamiltonian can again be written as the sum of a zero mode and an oscillator parts,

$$H = H_0 + H_{osc} \ , \tag{1.31}$$

with

$$H_0 = \frac{\alpha'}{2} p^2 \tag{1.32}$$

and

$$H_{osc} = N - a + \tilde{N} - \tilde{a} \ . \tag{1.33}$$

Here

$$N = \sum_{n>0} \left(\alpha_{-n} \cdot \alpha_n + n\psi_{-n} \cdot \psi_n\right) \ , \tag{1.34}$$

$$\tilde{N} = \sum_{n>0} \left(\tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + n\tilde{\psi}_{-n} \cdot \tilde{\psi}_n\right) \ , \tag{1.35}$$

are, with appropriate moding, the left and right excitation levels subject to the level-matching condition $N = \tilde{N}$, and a and $\tilde{a}$ represent the total left and right normal-ordering zero-point energies.

The left and right R ground states are spacetime spinors, whereas the NS ones are scalars, so that RR and NSNS states are spacetime bosons, whereas RNS and NSR states are spacetime fermions. The total normal-ordering constants a and $\tilde{a}$ are equal to 0 and $-1/2$ for R and NS b.c., so that the lowest lying RNS, NSR and NSNS modes are tachyonic. The GSO projection is implemented independently in the left and right sectors as for the open string, $P = 1/2(1 + (-1)^F)$, $\tilde{P} = 1/2(1 + (-1)^{\tilde{F}})$. Again, it achieves spacetime supersymmetry and projects out the tachyons by keeping only the states with even left and right world-sheet fermion number. Since the R spacetime chirality choice for the P and $\tilde{P}$ projections is arbitrary, there are two distinct consistent theories, with negative and positive left-right relative chirality. These theories are called Type IIA and Type IIB, and are globally non-chiral and chiral respectively. The irreducible representations of the little group SO(8) filled by the lowest lying massless modes of the two versions of the theory can be deduced by tensoring those found for Type I open strings, Eq. (1.17). One finds the following content

$$\begin{array}{lcll} \textbf{Type IIA} & & & \\ \text{NSNS} & : & \mathbf{8_v} \otimes \mathbf{8_v} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v} & \Leftrightarrow \ \phi\ ,\ b_{\mu\nu}\ ,\ g_{\mu\nu} \\ \text{RR} & : & \mathbf{8_s} \otimes \mathbf{8_c} = \mathbf{8_v} \oplus \mathbf{56_v} & \Leftrightarrow \ C_\mu\ ,\ C_{\mu\nu\rho} \\ \text{NSR} & : & \mathbf{8_v} \otimes \mathbf{8_c} = \mathbf{8_s} \oplus \mathbf{56_c} & \Leftrightarrow \ \lambda^1_\alpha\ ,\ \psi^{1\mu}_\alpha \\ \text{RNS} & : & \mathbf{8_s} \otimes \mathbf{8_v} = \mathbf{8_c} \oplus \mathbf{56_s} & \Leftrightarrow \ \tilde{\lambda}^2_\alpha\ ,\ \tilde{\psi}^{2\mu}_\alpha \end{array} \tag{1.36}$$

and

$$\begin{array}{lcll} \textbf{Type IIB} & & & \\ \text{NSNS} & : & \mathbf{8_v} \otimes \mathbf{8_v} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v} & \Leftrightarrow \ \phi\ ,\ b_{\mu\nu}\ ,\ g_{\mu\nu} \\ \text{RR} & : & \mathbf{8_s} \otimes \mathbf{8_s} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_t} & \Leftrightarrow \ C\ ,\ C_{\mu\nu}\ ,\ C^+_{\mu\nu\rho\sigma} \\ \text{NSR} & : & \mathbf{8_v} \otimes \mathbf{8_s} = \mathbf{8_c} \oplus \mathbf{56_s} & \Leftrightarrow \ \lambda^1_\alpha\ ,\ \psi^{1\mu}_\alpha \\ \text{RNS} & : & \mathbf{8_s} \otimes \mathbf{8_v} = \mathbf{8_c} \oplus \mathbf{56_s} & \Leftrightarrow \ \lambda^2_\alpha\ ,\ \psi^{2\mu}_\alpha \end{array} \tag{1.37}$$

In total, one has therefore the non-chiral and chiral massless gravitational multiplets. The LEEA for this light mode is N=2A,B D=10 SUGRA, and is again completely determined by supersymmetry. Its generic form is

$$S = \frac{1}{2\kappa^2_{(10)}} \int d^{10}x \sqrt{g} e^{-2\phi} \left[\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) - \sum_{n=0}^{4} \frac{1}{2n!} F^{(n)}_{\mu_1\cdots\mu_n} F^{\mu_1\cdots\mu_n}_{(n)} \right] + \text{ferm.}\ , \tag{1.38}$$

where the rank n of the RR field strengths $F_{(n)}$ is even or odd for Type IIA and Type IIB and

$$\kappa_{(10)} = \frac{1}{\sqrt{2}} (2\pi)^{\frac{7}{2}} \alpha'^2 \sim l_s^4\ . \tag{1.39}$$

Actually, there is a subtlety for the self-dual 4-form of the Type IIB theory. In fact, the self-duality constraint makes the usual action to vanish, so that there is, strictly speaking, no simple action reproducing the constrained equations of motion. One can nevertheless decide to use for simplicity the conventional unconstrained kinetic term and impose the self-duality condition as a true constraint. As for open strings, the LEEA for massless modes can be reconstructed in various way. The action Eq. (1.38) can in this case be determined only to leading order in α'. This can be done most easily by computing tree-level (sphere) n-supergraviton correlation functions in string theory [114], where by supergravitons we mean here and in the following any of the massless particles of SUGRA. Equivalently, Eq. (1.38) can be deduced, at least for NSNS part, by requiring conformal invariance through the vanishing of the β-function of the non-linear σ-model describing string propagation in a general curved spacetime [115, 116]. One can also face the problem in its whole generality by trying to compute directly the Polyakov path-integral on the sphere yielding the generating functional at leading order in the string coupling constant g_s [117]. However, due to the non-linear coupling to the gravitational background, it is not possible in this case to obtain and α'-exact result similar to the Born-Infeld Eq. (1.20) action for open strings, and the best one can do is a heat-kernel expansion in α'. The effective action Eq. (1.38) contains a factor $e^{-2\phi}$ corresponding to the sphere topology. Since $g_s = \langle e^{\phi} \rangle$, the effective coupling is therefore the product $g_s \kappa_{(10)}$. In the RR sector, the dilatonic factor has been reabsorbed into the fields, $C_{(p+1)} \to e^{\phi} C_{(p+1)}$ in order to get the usual Maxwell equations and Bianchi identities even for non-trivial dilaton backgrounds. In the NSNS sector instead, this is not necessary. These field definitions correspond to the so-called *string frame*. For practical calculation, it is convenient to reabsorb the dilatonic factor in the Einstein term, in order to avoid the mixing between dilaton and graviton porpagators. This is achieved in the so-called *Einstein frame* by rescaling the metric, $g^E_{\mu\nu} = e^{-\phi/2} g^S_{\mu\nu}$. The effective action in the Einstein frame is

$$S = \frac{1}{2\kappa^2_{(10)}} \int d^{10}x \sqrt{g} \left[\left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} e^{-\phi} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) - \sum_{n=0}^{4} \frac{1}{2n!} e^{\frac{5-n}{2}\phi} F^{(n)}_{\mu_1 \dots \mu_n} F^{\mu_1 \dots \mu_n}_{(n)} \right] \\ + \text{ferm.} \, . \tag{1.40}$$

An important characteristic of this action is the presence of non-minimal exponential couplings of the dilaton to the RR gauge forms and the NSNS Kalb-Ramond antisymmetric tensor. A crucial consequence is that the latter act as sources for the dilaton, and charged solitonic solutions have in general a non-trivial dependence on the dilaton. The only exception is the RR 4-form, whose self-dual 5-form field-strength do not couple to the dilaton.

1.1.3 T-duality

Historically D-branes where discovered by studying the behavior of open string compactified on a small circle of radius $R \to 0$ [57]. In particular, the T-duality symmetry of closed strings which reverses the relative sign between left and right movers, has proven to be extremely important also in the open string context.

Closed strings

Consider for instance closed strings with one of the spacetime coordinates, say X^9, compactified on a circle of radius R. The zero mode part of the mode expansion will be modified. The momentum $p^9 = n/R$ is quantized in units of $1/R$, for the wave function $\exp\{ip \cdot X\}$ to be well-defined under the shift $X^9 \to X^9 + 2\pi R$, and a winding $w^9 = mR$, quantized in units of R, can appear since X^9 is allowed to change by a integer multiple of $2\pi R$ when going around the string, $\sigma \to \sigma + 2\pi$. Therefore, the mode expansion is

$$X^9 = x^\mu - 2i\alpha' \frac{n}{R} \tau + mR\sigma + \text{osc.} \, . \tag{1.41}$$

This can be written as $X^9 = X^9 + \tilde{X}^9$ where

$$X^9(z) = \frac{x^\mu}{2} - i\sqrt{2\alpha'} \alpha^9_0 z + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \frac{\alpha^9_n}{n} e^{-nz} \, , \tag{1.42}$$

$$\bar{X}^9(\bar{z}) = \frac{x^\mu}{2} - i\sqrt{2\alpha'} \tilde{\alpha}^9_0 \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \frac{\tilde{\alpha}^9_n}{n} e^{-n\bar{z}} \, , \tag{1.43}$$

with

$$\alpha_0^9 = \sqrt{\frac{\alpha'}{2}} \left(\frac{n}{R} + \frac{mR}{\alpha'} \right) , \tag{1.44}$$

$$\tilde{\alpha}_0^9 = \sqrt{\frac{\alpha'}{2}} \left(\frac{n}{R} - \frac{mR}{\alpha'} \right) . \tag{1.45}$$

Correspondingly, the mass operator in the remaining non-compact directions becomes

$$M^2 = \frac{1}{\alpha'} \left[\left(\frac{n}{R} \right)^2 + \left(\frac{mR}{\alpha'} \right)^2 + \text{osc.} \right] , \tag{1.46}$$

which can be written more precisely as

$$M^2 = \frac{1}{\alpha'} \left[(\alpha_0^9)^2 + (\tilde{\alpha}_0^9)^2 + 2 \left(N - a + \tilde{N} - \tilde{a} \right) \right] . \tag{1.47}$$

Sending R into α'/R and exchanging winding and Kaluza-Klein modes $n \leftrightarrow m$ results in the transformation $\alpha_0^9, \tilde{\alpha}_0^9 \to \alpha_0^9, -\tilde{\alpha}_0^9$, which leaves the mass spectrum invariant [118, 119]. By superconformal symmetry, one has also to transform the fermionic zero modes in the 9 direction, $\psi_0^9, \tilde{\psi}_0^9 \to \psi_0^9, -\tilde{\psi}_0^9$. The interactions are identical as well [120]. This symmetry is best formalized by generalizing it to reverse the sign of the whole right-moving fields (changing the sign to the oscillator modes is trivial). In this way, the T-duality transformation looks like a one-side parity transformation and reads

$$\begin{array}{rcl} R & \to & R' = \alpha'/R \\ m, n & \to & n, m \\ X^9, \bar{X}^9 & \to & X^9, -\bar{X}^9 \\ \psi^9, \tilde{\psi}^9 & \to & \psi^9, -\tilde{\psi}^9 \end{array} . \tag{1.48}$$

In particular, whereas the original theory was written in terms of the usual coordinate $X^9 = X^9 + \bar{X}^9$, the dual theory is written in terms of the dual one, $X'^9 = X^9 - \bar{X}^9$. Notice also that $\partial_{\tau,\sigma} X^9 \sim \partial X^9 \pm \bar{\partial} \bar{X}^9$ goes into $\partial_{\sigma,\tau} X'^9 \sim \partial X^9 \mp \bar{\partial} \bar{X}^9$.

An important feature of the T-duality transformation is that, due to the sign change in $\tilde{\psi}^9$ it reverse the spacetime chirality of the RR vacuum. As a result, T-duality actually exchange the two versions of the theory, Type IIA and Type IIB. This statement can be translated at the level of spin-fields [121] $S^\alpha, \tilde{S}^\alpha$, which are the chiral spinors entering the construction of the RR vertex-operator. In fact, the transformation $\psi^9, \tilde{\psi}^9 \to \psi^9, -\tilde{\psi}^9$ implies

$$S^\alpha, \tilde{S}^\alpha \to S^\alpha, (\Gamma^9 \Gamma^{11})^{\alpha\beta} \tilde{S}_\alpha . \tag{1.49}$$

The Γ^{11} gives just a chirality-dependent sign. To understand the effect of this transformation, recall that the RR vertex-operator is given, in the $(-1/2,-1/2)$ picture, by

$$V_{RR} = \int d^2 z \, e^{-\frac{\phi}{2}} e^{-\frac{\tilde{\phi}}{2}} S C F \tilde{S} e^{ip\cdot X} . \tag{1.50}$$

Here ϕ and $\tilde{\phi}$ are the bosonization of the left and right superghosts, C is the charge-conjugation matrix and $F_{\alpha\beta}$ is the RR chiral bi-spinor field-strength, which can be decomposed into antisymmetric tensors $F^{(n)}$ with even and odd rank n in the Type IIA and Type IIB theories respectively

$$F_{\alpha\beta} = \sum_n \frac{1}{n!} F^{(n)}_{\mu_1 \ldots \mu_n} \Gamma^{\mu_1 \ldots \mu_n}_{\alpha\beta} \ . \tag{1.51}$$

These RR field-strengths are related by Hodge-duality, ${}^*F_{(k)} = \pm F_{(10-k)}$, so that they are actually not all independent, and lead to a total of 256 components both in the Type IIA and Type IIB theories, corresponding to 64 on-shell degrees of freedom for the corresponding potentials $C_{(n)}$, as displayed in Eqs. (1.36) and (1.37). It is now straightforward to see that the effect of the matrix $\Gamma^9\Gamma^{11}$ in Eq. (1.49) is to add a 9 index to $F_{(n)}$ if none is present, and to remove it if one is present. This is precisely the map relating the RR sector of the Type IIA and Type IIB theories when compactified on a circle.

Finally, notice that T-duality acts non-trivially on the string coupling g_s. In fact, the effective coupling of the compactified theory is $g_s/\sqrt{R}$. T-duality requires this to be equal to the corresponding effective coupling $g_s'/\sqrt{R'}$ of the dual theory at radius $R' = \alpha'/R$. This yields

$$g_s \to g_s' = \frac{\sqrt{\alpha'}}{R} g_s \ . \tag{1.52}$$

Open strings

Consider now purely Neumann open strings, with as before one of the spacetime coordinates, say X^9, compactified on a circle of radius R. The zero mode part of the mode expansion will be modified in this case too. The momentum $p^9 = n/R$ is again quantized in units of $1/R$, but there is no analog of the winding in this case. Therefore,

$$X^9 = x^\mu - 2i\alpha' \frac{n}{R}\tau + \text{osc.} \ . \tag{1.53}$$

This can be written as $X^9 = X^9 + \tilde{X}^9$ where

$$X^9(z) = \frac{x^9 + Y^9}{2} - i\sqrt{2\alpha'}\alpha_0^9 z + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \frac{\alpha_n^9}{n} e^{-nz} \ , \tag{1.54}$$

$$\bar{X}^9(\bar{z}) = \frac{x^9 - Y^9}{2} - i\sqrt{2\alpha'}\alpha_0^9 \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \frac{\alpha_n^9}{n} e^{-n\bar{z}} \ , \tag{1.55}$$

with

$$\alpha_0^9 = \sqrt{\frac{\alpha'}{2}} \frac{n}{R} \ . \tag{1.56}$$

In this case, there is no manifest left-right symmetry. Nevertheless, its is natural to study the theory for $R \to 0$ in terms of the dual variable $X'^9 = X^9 - \bar{X}^9$. One finds

$$X'^9 = Y^9 + 2nR'\sigma + \text{osc.} \ . \tag{1.57}$$

The osc. part vanishes now at both ends of the string, so that in the new variable one gets a Dirichlet string with its two end-points fixed on a D-brane hyper-plane located at Y^9, which is periodically identified with $Y^9 + 2\pi n R'$

$$X'^9\Big|_{\sigma=0} = Y^9 \ , \quad X'^9\Big|_{\sigma=\pi} = Y^9 + 2\pi n R' \ . \tag{1.58}$$

The integer n labeling the momentum quantum number in the original theory becomes a winding quantum number in the dual theory, and represents the number of times the open string, starting on the D-brane, winds around the compactification circle before ending again on the D-brane. Thus, T-duality changes Neumann into Dirichlet b.c. and vice versa. This could have been anticipated from the fact that

$$\begin{array}{ccc} \partial_{\tau,\sigma} X^9 & \leftrightarrow & \partial_{\sigma,\tau} X'^9 \\ \psi^9 \pm \tilde{\psi}^9 & \leftrightarrow & \psi^9 \mp \tilde{\psi}^9 \end{array} \ . \tag{1.59}$$

Summarizing, T-duality is a symmetry of closed strings compactified on a circle. It relates the two versions of the theory, Type IIA and Type IIB, which contain respectively odd and even RR forms. For open strings compactified on a circle, T-duality relates two versions of the theory with different b.c. along the compact directions, exchanging N and D b.c.. In the general case of a theory in a Dp-brane background, with both open and closed strings, there is an important consistency condition that the theory has to fulfill due to the fact that D-branes couple to fundamental strings. In fact, compactifying on a circle and performing a T-duality transformation, the closed string spectrum will change from Type IIA to Type IIB or vice versa, and Dp-brane will be turned into a D(p±1)-brane. The coupling between closed strings and D-branes has therefore to be consistent with this transformation. Indeed, we will see that Dp-branes couple minimally to RR (p+1)-forms, and T-duality consistently relates theories with even branes and odd RR forms to theories with odd branes and even RR forms.

1.2 D-branes

Having introduced D-branes and the important notion of T-duality, it is possible to analyze in somewhat more detail their fundamental characteristics. In particular, the fact they do couple to fundamental strings implies that their are not rigid and inert objects, but rather dynamical ones. In particular, it is natural to expect that they will play the role of some special background in the framework of string theory LEEA.

1.2.1 Supersymmetry

The first important property of D-branes is that they are BPS states. More precisely, they are backgrounds of theory preserving half of the supersymmetry, which are trivially realized. This can be seen by recalling the expression for the left and right supersymmetry charges, given by

$$Q^\alpha = \oint dz\, e^{-\frac{\phi}{2}} S^\alpha(\alpha) \ , \tag{1.60}$$

$$\tilde{Q}^\alpha = \oint d\bar{z}\, e^{-\frac{\tilde{\phi}}{2}} \tilde{S}^\alpha(\alpha) \ . \tag{1.61}$$

The presence of a boundary on the world-sheet Σ identify left and right movers, so that at most a linear combination of these two supersymmetries can be expected to survive. In ordinary Type I theory, that is a D9-brane, the surviving supersymmetry is generated simply by the sum of the left and right supercharges, $Q_+^\alpha = (Q^\alpha + \tilde{Q}^\alpha)/\sqrt{2}$. This is easily understood recalling that Type I theory can be regarded as Type IIB theory witch gauged world-sheet parity Ω, open strings emerging in Ω-twisted sectors of the hilbert space as strings which are closed only up to an Ω parity transformation. The result for a more general Dp-brane can be obtained simply by T-duality. As already shown, a T-duality transformation along some direction X^i has the effect of multiplying the right-moving spin-field $\tilde{S}^\alpha$ by the matrix $\Gamma^i\Gamma^{11}$, as in Eq. (1.49), so that according to the definitions Eqs. (1.60) and (1.61), the unbroken combination of supersymmetry is in this case $Q_+^\alpha = (Q^\alpha + \prod_{i=p+1}^9 \left(\Gamma^i\Gamma^{11}\right)^{\alpha\beta} \tilde{Q}_\alpha)/\sqrt{2}$. Summarizing, in the presence of a Dp-brane, the two Type II left and right supercharges (1.60) and (1.61) split into an unbroken (Q_+^α) and a broken (Q_-^α) combinations given by

$$Q_\pm^\alpha = \frac{1}{\sqrt{2}}\left(Q^\alpha \pm M_p^{\alpha\beta}\tilde{Q}_\beta\right) , \tag{1.62}$$

where

$$M_p^{\alpha\beta} = \prod_{i=p+1}^{9} \left(\Gamma^i\Gamma^{11}\right)^{\alpha\beta} . \tag{1.63}$$

Dp-branes are therefore BPS string theory backgrounds preserving half of the supersymmetry.

When more than one D-brane is present, that is when the world-sheet Σ has more than one boundary, the combination of supersymmetry left over is the intersection of those left over by each of the branes. More precisely, a Dp-brane and a Dq-brane (suppose p<q) preserve two generically different combinations of supersymmetries involving the matrices

$$M_p^{\alpha\beta} = \prod_{i=p+1}^{9} \left(\Gamma^i\Gamma^{11}\right)^{\alpha\beta} , \tag{1.64}$$

$$M_q^{\alpha\beta} = \prod_{i=q+1}^{9} \left(\Gamma^i\Gamma^{11}\right)^{\alpha\beta} . \tag{1.65}$$

The number of supersymmetries preserved by the theory with both of these D-branes as background is equal to the dimension of the eigenspace common to both of the two matrices M_p and M_q. This is given by the number of +1 eigenvalues of the matrix

$$N_{pq}^{\alpha\beta} = \left(M_p M_q^{-1}\right)^{\alpha\beta} = \prod_{i=p+1}^{q} \left(\Gamma^i\Gamma^{11}\right)^{\alpha\beta} . \tag{1.66}$$

Notice that q−p has to be even since in any case there can be only even (Type IIA) or odd (Type IIB) branes together in a consistent theory. It is easy to check that for q−p = 2 or 6, $N_{pq}^2 = -\mathbb{1}$ so that all its eigenvalues are imaginary. For q−p=4 or 8, N_{pq} is traceless and $N_{pq}^2 = \mathbb{1}$ so that its eigenvalues are ±1 in equal number. Finally, if q−p = 0 obviously $N_{pq} = \mathbb{1}$ and all the eigenvalues are +1. Therefore, since N_{pq} acts actually on a

16-dimensional chiral subspace, one finds that

$$\# \text{ of SUSY} = \begin{cases} 16 & , \quad q-p=0 \\ 8 & , \quad q-p=4,8 \\ 0 & , \quad q-p=2,6 \end{cases} \quad . \tag{1.67}$$

This means that beside single Dp-branes preserving 1/2 of the 32 supersymmetries, there exists BPS configurations formed by two of them, also preserving a fraction of these supersymmetries. For example, two parallel Dp-branes form a BPS state preserving 1/2 of the supersymmetry. Similarly, a Dp-brane together with a parallel D(p+4)-brane or a D(p+8)-brane form a BPS state preserving 1/4 of the supersymmetry. This discussion can be generalized to the more general case in which there are more D-branes at arbitrary angles and possibly with some ND or DN directions. For instance, one finds that any of the former BPS configurations can be generalized by replicating an arbitrary number of times each of its constituents. The question of whether or not these composite configurations can be considered as threshold bound-states representing genuine elementary quantum states and not merely as superpositions is a subtle issue.

1.2.2 Effective action

D-branes are genuine dynamical excitations of superstring theory, since they couple to fundamental strings. Furthermore, this coupling is completely encoded in Polchinski's b.c. prescription, which gives in principle an exact σ-model description of fundamental strings in presence of D-branes. We have already seen that the bosonic massless degree of freedom describing the Dp-brane split into a (p+1)-dimensional vector A_μ living on its world-volume and a set of 9−p scalar q^i related to its position. More precisely, the gauge field A_μ describes internal excitations on the world-volume, and the VEV of its field-strength is related to the electromagnetic flux that the Dp-brane carries. The scalar fields q^i describe instead the transverse fluctuations of the Dp-brane, and their VEV give its position $\langle q^i \rangle = Y^i$. The vertex-operators corresponding to these excitations is obtained in a straightforward way by T-duality from the vertex-operator for a Type I photon:

$$V_{A_\mu} = \oint_{\partial\Sigma} d\sigma_\alpha A_\mu \left(\partial^\alpha X^\mu + i\alpha' p \cdot \bar{\Psi}\rho^\alpha \Psi^\mu\right) e^{ip\cdot X} \; , \tag{1.68}$$

$$V_{q^i} = \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} d\sigma_\alpha \epsilon^{\alpha\beta} q^i \left(\partial_\beta X^i + i\alpha' p \cdot \bar{\Psi}\rho_\beta \Psi^i\right) e^{ip\cdot X} \; . \tag{1.69}$$

The momentum p entering these vertex-operators corresponds to the dependence on the coordinates X of the fields $A_\mu(X)$ and $q^i(X)$. For simplicity, we shall take a dependence only on the world-volume N directions X^μ, corresponding to a non-vanishing N momentum p^μ. The path-integral representation of the corresponding generating functionals is obtained by adding to the free string action S_0, Eq. (1.1), the deformations S_1 and/or S_2 obtained by exponentiating and Fourier transforming these vertex-operators

$$S_1 = \oint_{\partial\Sigma} d\sigma_\alpha \left[A_\mu \partial^\alpha X^\mu + \alpha' F_{\mu\nu} \bar{\Psi}^\mu \rho^\alpha \Psi^\nu\right] \; , \tag{1.70}$$

$$S_1 = \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} d\sigma_\alpha \epsilon^{\alpha\beta} \left[q^i \partial_\beta X^i + \alpha' \partial_\mu q^i \bar{\Psi}^\mu \rho_\beta \Psi^i\right] \; . \tag{1.71}$$

In this way, one obtains a complete definition of the string partition function as a function of the background fields A_μ and q^i specifying the Dp-brane. The LEEA can then be found with the usual methods, either by directly computing the generating functional or requiring conformal invariance through the vanishing of the β-function. At leading order in the string coupling constant g_s, the relevant world-sheet has the topology of a disk, and the result is the Dirac-Born-Infeld action [58] augmented by a Wess-Zumino term [56, 122, 123, 124, 125]

$$S = -T_p \int_{W_{p+1}} d^{p+1}\xi e^{-\phi} \sqrt{-\det\left(\hat{g}_{\mu\nu} + \mathcal{F}_{\mu\nu}\right)} - \mu_p \int_{W_{p+1}} \left(C \wedge e^{\mathcal{F}} \wedge \sqrt{\mathcal{A}}\right)_{(p+1)} + \text{ferm.} \,. \quad (1.72)$$

Here $\hat{g}_{\mu\nu} = g_{ij}\partial_\mu q^i \partial_\nu q^j$ represents the induced metric on the world-volume W_{p+1}. The pull-back of the antisymmetric tensor $\hat{b}_{\mu\nu} = b_{ij}\partial_\mu q^i \partial_\nu q^j$ and the world-volume field-strength $F_{\mu\nu}$ appear only in the gauge-invariant combination

$$\mathcal{F}_{\mu\nu} = 2\pi\alpha' F_{\mu\nu} - \hat{b}_{\mu\nu} \,. \quad (1.73)$$

In fact, at the world-sheet level, the antisymmetric tensor gauge invariance is violated by a surface term on the D-brane boundary in the variation $\delta b_{\mu\nu} = \partial_\mu \chi_\nu - \partial_\nu \chi_\mu$, which has to be compensated with a gauge transformation of the gauge field living on the world-volume, $\delta A_\mu = \chi_\mu$. The quantity C indicates the somewhat formal sum of all the RR n-form potentials $C_{(n)} = C^{(n)}_{\mu_1 \ldots \mu_n} d\xi^{\mu_1} \wedge \ldots \wedge d\xi^{\mu_n}$, $\mathcal{F} = \mathcal{F}_{\mu\nu} dX^\mu \wedge dX^\nu$ is the gauge-invariant two-form constructed out of Eq. (1.73) and $\hat{\mathcal{A}}(\mathcal{R})$ is the *roof genus* constructed out of the (pulled-back) curvature two-form $\mathcal{R} = \mathcal{R}_{\mu\nu} d\xi^\mu \wedge d\xi^\nu$. In this notation $C \wedge e^F \wedge \sqrt{\mathcal{A}}$ is therefore a sum of forms, and it is implicitly understood in the notation that one has to pick-up the part of it which is a (p+1)-form and can therefore be integrated over the world-volume W_{p+1}. The tension T_p and charge density μ_p are equal, indicating BPS saturation of the Dp-brane, and are given by

$$T_p = \mu_p = \sqrt{\alpha'}^{-1} \left(2\pi\sqrt{\alpha'}\right)^{-p} \,. \quad (1.74)$$

The world-volume action Eq. (1.72) is written in the string frame. The factor $e^{-\phi}$ corresponds to the disk topology and therefore the effective tension is T_p/g_s. As in the closed string effective action Eq. (1.38), the dilatonic prefactor appears only in the NSNS part and not in the RR one, because the RR gauge forms have been rescaled. Eq. (1.72) encodes all the interactions of the Dp-branes with the massless modes of open and closed fundamental strings. The complete LEEA in the string frame is therefore that of these massless modes, Eq. (1.38) augmented with Eq. (1.72) as a source. In the Einstein frame, Eq. (1.38) becomes Eq. (1.40). Correspondingly, Eq. (1.72) becomes

$$S = -T_p \int_{W_{p+1}} d^{p+1}\xi e^{-\frac{3-p}{4}\phi} \sqrt{-\det\left(\hat{g}_{\mu\nu} + e^{-\frac{\phi}{2}}\mathcal{F}_{\mu\nu}\right)} - \mu_p \int_{W_{p+1}} \left(C \wedge e^{\mathcal{F}} \wedge \sqrt{\mathcal{A}}\right)_{(p+1)} + \text{ferm.} \quad (1.75)$$

The couplings to the massless fields can be obtained by expanding this action around flat spacetime. The gravitational mass $\hat{T}_p$, p-form charge $\hat{\mu}_p$ and dilaton coupling $\hat{a}_p$ in units of the coupling $\sqrt{2}\kappa_{(10)}$, are found to be

$$\hat{T}_p = \hat{\mu}_p = \sqrt{2\pi}(2\pi\sqrt{\alpha'})^{3-p} \,, \quad \hat{a}_p = \frac{p-3}{4}\sqrt{2\pi}(2\pi\sqrt{\alpha'})^{3-p} \,. \quad (1.76)$$

It is then easy to compute the asymptotic fields generated by the Dp-brane. Some of the details are reported in Appendix C, together with the results Eqs. (C.11) and (C.12) in the Einstein and string frames.

1.2.3 Charge quantization

A flat Dp-brane with vanishing gauge field couples minimally to the RR (p+1)-form $C_{(p+1)}$ with the charge (1.74). This charge is electric and the Dp-brane is a true source for $C_{(p+1)}$. However, not all the RR-forms are independent degrees of freedom. Rather, they are related by Hodge duality through their field strength. In the string frame, one has simply $F_{(p+2)} = {}^*F_{(8-p)}$, whereas in the Einstein frame this becomes $F_{(p+2)} = e^{(p-3)/2\phi *}F_{(8-p)}$. One could keep considering all the Dp-branes as electrically charged with respect to the corresponding (p+1)-form, and impose the Hodge duality as a constraint. Another possibility is two eliminate high forms and keep only those with p≤4 as propagating degree of freedom. Doing so, Dp-branes with p≤3 are still electrically charged with respect to $C_{(p+1)}$, whereas those with p≥3 become magnetically charged with respect to $C_{(7-p)}$, which is dual to $C_{(p+1)}$ that has been eliminated. In any case, a Dp-brane and a D(6−p)-brane are magnetically dual and experience an electric-magnetic interaction. In fact, working for example in the string frame, the $C_{(p+1)}$ form has a field-strength $F_{(p+2)}$ which is identified with ${}^*F_{(8-p)}$, and locally $F_{(8-p)}$ admits the potential $C_{(7-p)}$. Therefore, the potential $C_{(p+1)}$ generated by a Dp-brane can be described in terms of the potential $C_{(7-p)}$ to which a D(6−p)-brane couples, everywhere but on a Dirac hyper-string, where the potential is singular. One way of obtaining Dirac's quantization condition is then to require that this singularity not be observable. In particular, the Aharonov-Bohm effect in transporting a D(6−p)-brane around the Dirac string attached to a Dp-brane, and therefore describing a S_{7-p} sphere as world-sheet, results in a shift $\delta\Phi$ in the phase of the wave-function given by the interaction action. One finds (in the string frame)

$$\delta\Phi = \mu_{6-p} \int_{S_{7-p}} C_{(7-p)} \; . \tag{1.77}$$

Using Gauss' law and Hodge duality one finds that

$$\begin{aligned} \delta\Phi &= \mu_{6-p} \int_{S_{7-p}} C_{(7-p)} = \mu_{6-p} \int_{S_{8-p}} F_{(8-p)} = \mu_{6-p} \int_{S_{p+2}} {}^*F_{(p+2)} \\ &= 2\kappa_{(10)}^2 \mu_p \mu_{6-p} \; . \end{aligned} \tag{1.78}$$

In order the singularity not to be observable, this phase has to be an irrelevant multiple of 2π, yielding therefore the condition

$$2\kappa_{(10)}^2 \mu_p \mu_{6-p} = 2\pi n \; , \tag{1.79}$$

which is satisfied with n=1 by Eq. (1.74). In order to avoid the annoying appearance of the coupling $2\kappa_{(10)}^2$, it is convenient to work as before in Eqs. (1.76) with the rescaled charge $\hat{\mu}_p = \sqrt{2\pi}(2\pi\sqrt{\alpha'})^{3-p}$ defined such that $\mu_p = \hat{\mu}_p/(\sqrt{2}\kappa_{(10)})$, which satisfy

$$\hat{\mu}_p \hat{\mu}_{6-p} = 2\pi \; . \tag{1.80}$$

1.2.4 Low energy interpretation

One of the most important characteristics of Dp-branes is that they correspond at low energy to charged solitonic solutions of the LEEA called p-branes, that we shall now briefly describe. The string frame effective action Eq. (1.38) has the following exact solutions

$$\begin{cases} g_{\mu\nu} = H_p^{-\frac{1}{2}} \bar{\eta}_{\mu\nu} \ , \quad g_{ij} = H_p^{\frac{1}{2}} \delta_{ij} \\ C_{\mu_1 \ldots \mu_{p+1}} = \bar{\epsilon}_{\mu_1 \ldots \mu_{p+1}} \left(H_p^{-1} - 1 \right) \\ \phi = \dfrac{3-p}{4} \ln H_p \end{cases} . \tag{1.81}$$

Here Greek indices correspond to the p+1 world-volume directions, whereas Latin indices span the 9−p transverse directions, and $\bar{\eta}^{\mu\nu}$ and $\bar{\epsilon}^{\mu_1 \cdots \mu_p+1}$ are the Minkowski and Levi-Civita tensors on the Dp-brane world-volume, with indices running from 0 to p. H_p is a harmonic function of the transverse distance r and can be parameterized as

$$H_p(r) = 1 + 2\kappa_{(10)}^2 \tilde{T}_p \Delta_{(9-p)}(r) \tag{1.82}$$

in terms of the transverse Green function $\Delta_{(9-p)}$ and a so far arbitrary coupling $\tilde{T}_p$. The corresponding solution of Eq. (1.40) in the Einstein frame is

$$\begin{cases} g_{\mu\nu} = H_p^{\frac{p-7}{8}} \bar{\eta}_{\mu\nu} \ , \quad g_{ij} = H_p^{\frac{p+1}{8}} \delta_{ij} \\ C_{\mu_1 \ldots \mu_p+1} = \bar{\epsilon}_{\mu_1 \ldots \mu_{p+1}} \left(H_P^{-1} - 1 \right) \\ \phi = \dfrac{3-p}{4} \ln H_p \end{cases} . \tag{1.83}$$

This p-brane solution has a tension $\tilde{T}_p$ and dilaton coupling $\tilde{a}_p = (3-p)/4\tilde{T}_p$. It is electrically charged with respect to the the RR (p+1)-form, with a charge

$$\tilde{\mu}_p = \frac{1}{2\kappa_{(10)}^2} \int_{S_{8-p}} *F_{(p+2)} = \tilde{T}_p \ . \tag{1.84}$$

By Hodge duality, this can also be interpreted as a magnetic charge with respect to the RR (7−p)-form. Consistency at the quantum level leads as before to the Dirac quantization condition $2\kappa_{(10)}^2 \tilde{\mu}_p \tilde{\mu}_{6-p} = 2\pi n$. On dimensional grounds, this fixes $\tilde{\mu}_p$ to be an integer multiple of the fundamental charge μ_p. This means $\tilde{T}_p = nT_p$ and suggests that p-branes are related to the low energy description of Dp-branes.

The p-brane solution is BPS for every $\tilde{T}_p$. This can be verified by computing the gravitino and dilatino variations in the background of the solution, which are found to be proportional to the projection $1/2(1-M)\eta$ of the supersymmetry parameter η, with $M = \prod_{n=p+1}^{9} (\Gamma^i \Gamma^{11})$ as in Eq. (1.63). Since $P_\pm = 1/2(1 \pm M)$ are orthogonal projection operators, the initially arbitrary supersymmetry parameter η splits into the two components $\eta_\pm = P_\pm \eta$. η_+ appears in the supersymmetry variations and corresponds therefore to broken supersymmetries, whereas η_- does never appear and corresponds therefore to trivially realized preserved supersymmetries. In a more technical language, η_+ is Killing spinor of the solution, whereas

η_- are fermionic zero modes. By applying the broken supersymmetries to the fundamental p-brane solution, one obtaines other solutions carrying non-vanishing spin. These can be grouped in a supermultiplet representing the fermionic degeneracy related to the fermionic zero modes on which the broken supersymmetry is realized. The p-brane solution is therefore a BPS background of the LEEA preserving 1/2 of the 32 supersymmetries. Moreover, the unbroken and broken combinations $\eta_\pm$ correspond precisely to the unbroken and broken supercharges Eq. (1.62) of Type II superstrings in a Dp-brane background. This gives strong evidence that p-branes are the low energy description of Dp-branes. Further evidence for this identification is obtained by analyzing the couplings to massless fields. It is straightforward to verify that the asymptotic fields of the p-brane solution obtained in the weak field limit $\kappa_{(10)} \to 0$ from Eqs. (1.81) or (1.83) match with $\tilde{T}_p$ instead of T_p those computed in Appendix C, Eqs. (C.11) or (C.12), starting from the knowledge of the Dp-brane couplings Eqs. (1.76). This demonstrate that the elementary p-brane with $\tilde{T}_p = T_p$ can be identified with the Dp-brane at low energy. Since, as we will see in detail, parallel Dp-branes do not exert any force on each other due to their BPS character, the multiply charged p-brane with $\tilde{T}_p = T_p$ can be interpreted as the superposition of n Dp-branes at the same position.

Chapter 2

D-brane dynamics

In this chapter, we describe in some detail the one-loop amplitude encoding the interaction between D-branes, focusing on the systems constituted by two parallel Dp-branes and by a Dp-brane and a D(p+4)-brane. We also discuss the amplitude in the closed string channel, by performing a modular transformation.

2.1 Static D-branes

An important and extremely interesting issue in modern superstring theory is the study of the interactions between D-branes. This opens a new domain of investigation in the theory which proves to be extremely rich and generous of information.

Consider then two parallel D-branes, say a Dp-brane and a Dq-brane with p<q. The world-sheet encoding their interaction at leading order in the string coupling g_s has the topology of a cylinder whose boundaries end on the two D-branes. Using the usual time slicing, this world-sheet represents a loop of open strings stretched between the two D-branes. The resulting amplitude has to be interpreted, once integrated over the real modulus parameterizing all the inequivalent cylinders, as a one-loop effective action written in Schwinger's proper time parameterization. Physically, this represents the Casimir energy that the two D-branes experience due to open string vacuum fluctuations in the space in-between them.

To compute the amplitude, it is convenient to parameterize the cylinder with a fixed length π and circumference t. The amplitude is given as usual by

$$\mathcal{A} = \int_0^\infty \frac{dt}{t} Z(t) \; . \tag{2.1}$$

Here $Z(t)$ indicates the GSO-projected open string partition function

$$Z(t) = \mathrm{STr}[P e^{-\frac{\pi}{2} t H}] \; , \tag{2.2}$$

where $P = 1/2(1 + (-1)^F)$ is the GSO operator projecting onto states with even world-sheet fermion number. The supertrace STr runs over the two sectors, R and NS, of the open string spectrum, and counts spacetime bosons with a $+$ sign and spacetime fermions with a $-$ sign. Decomposing the projector P, the partition function Eq. (2.2) splits into four distinct contributions

$$Z(t) = \frac{1}{2}\Big(\mathrm{Tr}_{NS}[e^{-\frac{\pi}{2}tH}] + \mathrm{Tr}_{NS}[(-1)^F e^{-\frac{\pi}{2}tH}] - \mathrm{Tr}_R[e^{-\frac{\pi}{2}tH}] - \mathrm{Tr}_R[(-1)^F e^{-\frac{\pi}{2}tH}]\Big)$$
$$= \frac{1}{2}\Big(Z^{(NS+)}(t) - Z^{(NS-)}(t) - Z^{(R+)}(t) + Z^{(R-)}(t)\Big)\ . \tag{2.3}$$

The two sectors without $(-1)^F$ insertion have world-sheet fermions which are antiperiodic around the loop, whereas the two sectors with an $(-1)^F$ insertion have world-sheet fermions which are instead periodic around the loop. Moreover, integer moding corresponds to periodicity and half-integer moding to antiperiodicity in the cycle of the covering torus. The four sectors R$\pm$ and NS$\pm$ correspond therefore to all the possible periodicities of the fermions on the covering torus, and are referred to as *spin-structures*. For reasons that will become clear in the following, the spin-structures which have antiperiodic b.c. around at least one of the two cycles of the covering torus are called *even*, whereas the spin-structure which has periodic b.c. along both of the cycles of the covering torus is called odd. Recall that fermions have integer and half-integer moding in the R and NS sectors for a NN or DD directions and vice versa for a ND or DN directions. The partition functions $Z(t)$ can be split in each of the four sectors into the product of a bosonic and a fermionic contributions, Z^B and Z^F. Each of these can be further decomposed into a product of zero mode and oscillator parts, Z_0 and Z_{osc}. It is convenient to analyze these four parts separately for a single field corresponding to a given direction.

Consider first the bosons. The contribution of the bosonic zero mode depends crucially on the b.c.. In the DD case, the contribution is trivial since the z.m. part H_0 of the Hamiltonian, Eq. (1.14), is in this case a number. In the NN case, the trace becomes an integral over momentum, with H_0 given by Eq. (1.14) and there is an infinite degeneracy proportional to the volume V of the direction under analysis, due to translational invariance. Finally, in the ND or DN cases there are no zero modes, and we can therefore assign them conventionally a partition function equal to 1. Summarizing one finds

$$Z_0^B(t) = \begin{cases} V(4\pi^2\alpha' t)^{-\frac{1}{2}} & ,\ \mathrm{NN} \\ e^{-\frac{\Delta Y^2}{4\pi\alpha'}t} & ,\ \mathrm{DD} \\ 1 & ,\ \mathrm{ND,DN} \end{cases} . \tag{2.4}$$

The bosonic oscillators have integer moding for a NN or DD direction, and half-integer moding for a ND or DN direction. One finds

$$Z_{osc}^B(t) = \begin{cases} q^{-\frac{1}{12}} \prod_{n=1}^{\infty} \Big(1 - q^{2n}\Big)^{-1} & ,\ \mathrm{NN,DD} \\ q^{\frac{1}{24}} \prod_{n=1}^{\infty} \Big(1 - q^{2n-1}\Big)^{-1} & ,\ \mathrm{ND,DN} \end{cases} . \tag{2.5}$$

where $q = e^{-\pi t}$.

Consider next the fermions. As already pointed out, they have integer and half-integer moding in the R and NS sectors for a NN or DD direction, and vice versa for a ND or DN direction. Let us call temporarily P and A the sectors with integer and half-integer moding for the direction under analysis. In the P sector there is a fermionic z.m. which anticommutes with $(-1)^F$ and does not contribute to the energy. At each level one has therefore a two-fold degeneracy corresponding to the freedom of inserting or not this z.m., yielding an equal number of states with even and odd fermion number and equal energy. Correspondingly, the P$-$ spin-structure partition function, which has a $(-1)^F$ inserted, vanishes. On can therefore assign to the z.m. a vanishing partition function in the P$-$ spin-structure. In the P+ spin-structure, with no $(-1)^F$ inserted, states with even and odd fermion number are counted with the same sign and therefore do not cancel but sum. Taking properly into account the multiplicity of the states for each direction, one can attribute a partition function equal to $\sqrt{2}$ to the fermionic z.m. of the P+ spin-structure. Finally, in the A sector there are no fermionic z.m. at all, so that one can conventionally assign them a partition function equal to 1. Summarizing

$$Z_0^{F(P+)}(t) = \sqrt{2}\ , \quad Z_0^{F(P-)}(t) = 0\ , \tag{2.6}$$

$$Z_0^{F(A\pm)}(t) = 1\ . \tag{2.7}$$

The fact that the P$-$ fermionic z.m. give a vanishing result reflects the fact that they correspond to true z.m. on the covering torus which give a vanishing result in the Polyakov path-integral representation of the partition function. On the contrary, the P+ fermionic z.m. are fake. They are z.m. only with respect to the open string Fourier decomposition, since in this sector the fields are antiperiodic around the loop. For the oscillator modes, one finds

$$Z_{osc}^{F(P\pm)}(t) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} \left(1 \pm q^{2n}\right)\ , \tag{2.8}$$

$$Z_{osc}^{F(A\pm)}(t) = q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 \pm q^{2n-1}\right)\ . \tag{2.9}$$

Recall that

$$\mathrm{P} = \begin{cases} R & ,\ \mathrm{NN,DD} \\ NS & ,\ \mathrm{ND,DN} \end{cases}\ ,\quad \mathrm{A} = \begin{cases} NS & ,\ \mathrm{NN,DD} \\ R & ,\ \mathrm{ND,DN} \end{cases}\ . \tag{2.10}$$

Making use of the formulæ reported in Appendix A, it is straightforward to write these partition functions in terms of ϑ-functions. One can summarize by quoting the results for a boson and a fermion with periodicities P_1 and P_2 along the two cycles of the torus, indicated with the symbol

$$ {}_{P_2}\square_{P_1} \tag{2.11}$$

P_1 is the periodicity along the σ-cycle and is P for integer moding and A for half-integer moding. P_2 is the periodicity along the τ-cycle and is respectively P and A with ($-$ spin structure) and without (+ spin-structure) $(-1)^F$ insertion, for the fermions, and always P for the bosons. The results are reported in Appendix B, Eqs. (B.2)-(B.7). Using these results, it is straightforward to compute the one-loop amplitude giving the leading interaction energy between a Dp and a Dq-brane. To carry out a complete and precise computation within the covariant formalism, one should explicitly consider also the contribution of b,c diffeomorphisms ghosts and β,γ superdiffeomorphisms superghosts running in the loop.

Nevertheless, it is easy to show that, a part from the z.m., their contribution is exactly the inverse of that of a pair of bosonic and fermionic fields in two NN or DD directions. Canceling a priori the contributions of the ghosts and superghosts with an appropriate pair of bosons and fermions amounts to recover a light-cone gauge treatment in which only physical states propagate. Actually, there are some subtleties concerning fermionic and superghost z.m. in P$-$ odd spin-structure on which we shall return further on. For simplicity and concreteness, we shall concentrate in the following on two simple and illustrative cases: the system of two parallel Dp-branes, preserving 1/2 of the supersymmetry, and the system of a Dp and a parallel D(p+4)-brane, preserving 1/4 of the supersymmetry.

2.1.1 Dp-Dp interaction

For the Dp-Dp system, the first p+1 directions are NN, whereas the last 9$-$p are DD. Only the three even spin-structure NS$\pm$ and R+ contribute, whereas the R$-$ odd spin-structure gives a vanishing result because of the fermionic zero modes. Considering also the ghost and superghost contributions, the result for the one-loop amplitude is [56]

$$\mathcal{A} = \frac{V_{p+1}}{(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dt}{t^{\frac{p+3}{2}}} e^{-\frac{r^2}{4\pi\alpha'}t} Z_{open}(t) \ , \tag{2.12}$$

where $\vec{r} = \Delta\vec{Y}$ and

$$Z_{open}(t) = \frac{1}{2}\frac{\vartheta_3^4(0|\frac{it}{2}) - \vartheta_4^4(0|\frac{it}{2}) - \vartheta_2^4(0|\frac{it}{2})}{\eta^{12}(\frac{it}{2})} \ . \tag{2.13}$$

The combination of ϑ-functions appearing in the numerator of the partition function sum up to zero by means of Jacobi's *æquatio identico satis abstrusa*, which is a particular case of Eq. (A.24). Using Eq. (A.24) one can formally rewrite $Z_{open}(t)$ as

$$Z_{open}(t) = \frac{\vartheta_1^4(0|\frac{it}{2})}{\eta^{12}(\frac{it}{2})} = 0 \ . \tag{2.14}$$

The vanishing of the amplitude is a consequence the 1/2 of spacetime supersymmetry left unbroken by the BPS system of two parallel Dp-branes. This is expected from the fact that the vacuum of a theory with some unbroken supersymmetry must have strictly zero energy. A cancellation occurs as usual between loops of spacetime bosons and spacetime fermions contributing with opposite signs to the vacuum energy, which holds level by level within each supermultiplet with growing mass and spin.

It is extremely interesting to analyze the amplitude from the closed string channel point of view. To do so, it is enough to perform a modular transformation turning the open string modulus $t/2$ into the closed string modulus $2l$ and rewrite the amplitude in terms of $l = 1/t$. In the open string parameterization, the cylinder has fixed length π equal to the "length" of open strings π and a variable circumference t corresponding to the loop proper time, whereas in the closed string parameterization the cylinder has fixed circumference 2π equal to the "length" of closed strings and variable length l corresponding to the propagation proper time. Using Eqs. (A.16)-(A.19), the amplitude Eq. (2.12) can be rewritten as

$$\mathcal{A} = \frac{V_{p+1}}{2^4(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dl}{l^{\frac{9-p}{2}}} e^{-\frac{r^2}{4\pi\alpha' l}} Z_{closed}(l) \ , \tag{2.15}$$

where

$$Z_{closed}(l) = \frac{1}{2}\frac{\vartheta_3^4(0|2il) - \vartheta_2^4(0|2il) - \vartheta_4^4(0|2il)}{\eta^{12}(2il)} \ . \tag{2.16}$$

Again, the combination of ϑ-functions appearing in the numerator of the partition function sum up to zero by means of Jacobi's identity and using Eq. (A.24) one can rewrite $Z_{closed}(l)$ as

$$Z_{closed}(l) = \frac{\vartheta_1^4(0|2il)}{\eta^{12}(2il)} = 0 \ . \tag{2.17}$$

The amplitude is now interpreted as the interaction energy coming from the exchange of all closed string states between the two Dp-branes. The vanishing of the amplitude reflects the no-force condition holding for the interaction between a BPS combination of states. Only spacetime bosons are exchanged, and a level by level cancellation occurs between the attractive exchange of NSNS bosons and the repulsive exchange of RR bosons within each supermultiplet with growing mass and spin.

In both channels, the amplitude receives contributions both from the lowest lying states and from the infinite tower of higher mass states. Once one sums up all the contributions, obtaining complete modular functions for the partition function, the open and closed string channel descriptions are completely equivalent. Nevertheless, it is interesting to understand which modes contributes at large and short distances in the two descriptions.

Consider first the short distance limit, in which the distance between the two Dp-branes is small with respect to the string scale, $r \ll l_s$. In Eq. (2.12), only very bride world-sheets with $t \to \infty$ contribute, corresponding to loops of almost massless open string modes. Loops of higher mass open string modes give contributions which are exponentially suppressed. Mathematically, the fact that only the lowest lying mode contributes reflects into the fact that the limit $t \to \infty$ selects a single factor in the modular functions entering the partition function. One finds

$$Z_{open}(t) \underset{t\to\infty}{\longrightarrow} (8-8) \ , \tag{2.18}$$

and the amplitude (2.12) becomes

$$\mathcal{A} \underset{r\ll l_s}{\longrightarrow} \frac{V_{p+1}}{(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dt}{t^{\frac{p+3}{2}}} e^{-\frac{r^2}{4\pi\alpha'}t} (8-8) \ . \tag{2.19}$$

Vice versa, in Eq. (2.15), only very short world-sheets with $l \to 0$ contribute, corresponding to the exchange of closed strings propagating for a very short distance. For this reason, beside the massless closed string modes, also all the tower of massive closed string modes are significant since they are suppressed only by an exponential factor with a vanishing exponent, and their contribution has to be resummed, yielding a power law behavior. Mathematically, the fact that all the modes contributes reflects into the fact that the limit $l \to 0$ does not select any factor in the modular functions entering the partition function. Rather, one has to perform a Poisson resummation to compute its behavior. One finds in this case

$$Z_{closed}(l) \underset{l\to 0}{\longrightarrow} (8-8)(2il)^4 \ , \tag{2.20}$$

and the amplitude (2.15) reduces to

$$\mathcal{A} \underset{r\ll l_s}{\longrightarrow} \frac{V_{p+1}}{(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dl}{l^{\frac{1-p}{2}}} e^{-\frac{r^2}{4\pi\alpha' l}} (8-8) \ . \tag{2.21}$$

Eqs. (2.19) and (2.21) are identical and yield the short distance behavior of the amplitude in the open and closed string parameterization. It clearly emerges that that the most natural description of the short distance interaction of D-branes is obtained in the open string channel, where a simple truncation to the massless mode is sufficient to give a good approximation. The result can be rewritten more conveniently as

$$\mathcal{A} \underset{r \ll l_s}{\longrightarrow} \frac{V_{p+1}}{(4\pi)^{\frac{p+1}{2}}} \int_0^\infty \frac{dt}{t^{\frac{p+3}{2}}} e^{-(\frac{r}{2\pi\alpha'})^2 t} (8-8) \ . \tag{2.22}$$

One recognizes Schwinger's proper time parameterization of the effective action for a supermultiplet of particles with mass $m = r/(2\pi\alpha')$ in $p+1$ spacetime dimensions, corresponding to the lowest lying modes of the open strings stretched between the two Dp-branes.

Consider now the large distance limit, in which the distance between the two Dp-branes is large with respect to the string scale, $r \gg l_s$. In Eq. (2.12), only very small world-sheets with $t \to 0$ now contribute, corresponding to loops of all the open string modes. Since open string modes with high masses are not sufficiently suppressed, their contributions have to be resummed. The behavior of the partition function is

$$Z_{open}(t) \underset{t \to 0}{\longrightarrow} (8-8) \left(\frac{it}{2}\right)^4 \ , \tag{2.23}$$

and the amplitude (2.12) becomes

$$\mathcal{A} \underset{r \gg l_s}{\longrightarrow} \frac{V_{p+1}}{2(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dt}{t^{\frac{p-5}{2}}} e^{-\frac{r^2}{4\pi\alpha'}t} (1-1) \ . \tag{2.24}$$

Vice versa, in Eq. (2.15), only very long world-sheets with $l \to \infty$ now contribute, corresponding to the exchange of closed string propagating for a very long distance. Closed string modes with high masses give exponentially suppressed contributions, so that the dominant interaction comes from the exchange of massless closed string modes. The behavior of the partition function is in this case

$$Z_{closed}(l) \underset{l \to \infty}{\longrightarrow} (8-8) \ , \tag{2.25}$$

and the amplitude (2.15) reduces to

$$\mathcal{A} \underset{r \gg l_s}{\longrightarrow} \frac{V_{p+1}}{2(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dl}{l^{\frac{9-p}{2}}} e^{-\frac{r^2}{4\pi\alpha' l}} (1-1) \ . \tag{2.26}$$

Eqs. (2.24) and (2.26) are identical and yield the large distance behavior of the amplitude in the open and closed string parameterizations. Furthermore, the most natural description of the large distance interaction of D-branes is now obtained in the closed string channel, where a simple truncation the massless mode is sufficient to give a good approximation. The result can finally be rewritten as

$$\mathcal{A} \underset{r \gg l_s}{\longrightarrow} V_{p+1} \hat{T}_p^2 (1-1) \Delta_{(9-p)}(r) \ , \tag{2.27}$$

where $\hat{T}_p = \sqrt{2\pi}(2\pi\sqrt{\alpha'})^{3-p}$ is the tension of the Dp-brane in inverse units of the effective coupling $\sqrt{2}\kappa_{(10)}$ and $\Delta_{(d)}(r)$ is the green function for a scalar massless particle in d space dimensions

$$\Delta_{(d)}(r) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip \cdot r}}{p^2} = \frac{1}{4\pi^{d/2}} \Gamma\left(\frac{d-2}{2}\right) \frac{1}{r^{d-2}} \ . \tag{2.28}$$

2.1.2 Dp-D(p+4) interaction

For the Dp-D(p+4) system, the first p+1 directions are NN, the last 5−q DD and the four ranging from p+1 to p+4 are ND. This time, only the two + spin structures NS+ and R+ contributes, whereas the − spin structures NS− and R− have both four z.m. and do not contribute. Considering also the ghost and superghost contributions, the result for the one-loop amplitude is [98]

$$\mathcal{A} = \frac{V_{p+1}}{(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dt}{t^{\frac{p+3}{2}}} e^{-\frac{r^2}{4\pi\alpha'}t} Z_{open}(t) \ , \tag{2.29}$$

where $\vec{r} = \Delta\vec{Y}$ as before and

$$Z_{open}(t) = \frac{1}{2}\frac{\vartheta_3^2(0|\frac{it}{2})\vartheta_2^2(0|\frac{it}{2}) - \vartheta_2^4(0|\frac{it}{2})\vartheta_3^2(0|\frac{it}{2})}{\eta^6(\frac{it}{2})\vartheta_4^2(0|\frac{it}{2})} \ . \tag{2.30}$$

The combination of ϑ-functions in the numerator vanishes identically. Notice nevertheless that using Eqs. (A.25), one can formally rewrite $Z_{open}(t)$ as

$$Z_{open}(t) = \frac{\vartheta_1^2(0|\frac{it}{2})}{\eta^6(\frac{it}{2})} = 0 \ . \tag{2.31}$$

As before, the vanishing of the amplitude is a consequence the 1/4 of spacetime supersymmetry preserved by the BPS system of two branes, and the cancellation occurs level by level within all supermultiplets with growing mass and spin.

As before, the closed string channel interpretation is obtained by performing a modular transformation turning the open string modulus $t/2$ into the closed string modulus $2l$ and rewriting the amplitude in terms of $l = 1/t$. Using Eqs. (A.16)-(A.19), one finds

$$\mathcal{A} = \frac{V_{p+1}}{2^2(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dl}{l^{\frac{5-p}{2}}} e^{-\frac{r^2}{4\pi\alpha' l}} Z_{closed}(l) \ , \tag{2.32}$$

where

$$Z_{closed}(l) = \frac{1}{2}\frac{\vartheta_3^2(0|2il)\vartheta_4^2(0|2il) - \vartheta_4^2(0|2il)\vartheta_3^2(0|2il)}{\eta^6(2il)\vartheta_2^2(0|2il)} \ , \tag{2.33}$$

or using Eq. (A.26)

$$Z_{closed}(l) = \frac{\vartheta_1^2(0|2il)}{\eta^6(\frac{it}{2})} = 0 \ . \tag{2.34}$$

The vanishing of the total amplitude again reflects the no-force condition holding for the interaction between a BPS combination of states, the cancellation occurring level by level between the attractive and repulsive exchange of various NSNS bosons within each massive supermultiplet.

In the short distance limit, $r \ll l_s$, only very bride world-sheets with $t \to \infty$ contribute in Eq. (2.29), corresponding to loops of almost massless open string modes. Loops of higher mass open string modes again give contributions which are exponentially suppressed. One finds

$$Z_{open}(t) \underset{t\to\infty}{\longrightarrow} (2-2) \ , \tag{2.35}$$

and Eq. (2.29) becomes

$$\mathcal{A} \underset{r \ll l_s}{\longrightarrow} \frac{V_{p+1}}{(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dt}{t^{\frac{p+3}{2}}} e^{-\frac{r^2}{4\pi\alpha'}t} (2-2) \; . \tag{2.36}$$

Vice versa, in Eq. (2.32), only very short world-sheets with $l \to 0$ contribute, corresponding to the exchange of closed string propagating for a very short distance. All the closed string modes modes contribute significantly and their contributions have to be resummed. One finds in this case

$$Z_{closed}(l) \underset{l \to 0}{\longrightarrow} (2-2)(2il)^2 \; , \tag{2.37}$$

and Eq. (2.32) reduces to

$$\mathcal{A} \underset{r \ll l_s}{\longrightarrow} \frac{V_{p+1}}{(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dl}{l^{\frac{1-p}{2}}} e^{-\frac{r^2}{4\pi\alpha' l}} (2-2) \; . \tag{2.38}$$

Eqs. (2.36) and (2.38) are identical and yield the short distance behavior of the amplitude in the open and closed string parameterization. As before, the most natural description of the short distance interaction is obtained in terms of the lowest lying open string modes. Finally, the result can be rewritten more conveniently as

$$\mathcal{A} \underset{r \ll l_s}{\longrightarrow} \frac{V_{p+1}}{(4\pi)^{\frac{p+1}{2}}} \int_0^\infty \frac{dt}{t^{\frac{p+3}{2}}} e^{-(\frac{r}{2\pi\alpha'})^2 t} (2-2) \; . \tag{2.39}$$

One recognizes Schwinger's proper time parameterization of the effective action for an supermultiplet of particles with mass $m = r/(2\pi\alpha')$ in $p+1$ spacetime dimensions, corresponding to the lowest lying modes of the open strings stretched between the Dp and the D(p+4)-brane.

In the large distance limit, $r \gg l_s$, only very small world-sheets with $t \to 0$ now contribute in Eq. (2.12). All open string modes loops contribute significantly. The behavior of the partition function is

$$Z_{open}(t) \underset{t \to 0}{\longrightarrow} (2-2) \left(\frac{it}{2}\right)^2 \; , \tag{2.40}$$

and Eq. (2.29) becomes

$$\mathcal{A} \underset{r \gg l_s}{\longrightarrow} \frac{V_{p+1}}{2(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dt}{t^{\frac{p-1}{2}}} e^{-\frac{r^2}{4\pi\alpha'}t} (1-1) \; . \tag{2.41}$$

Vice versa, in Eq. (2.32), only very long world-sheets with $l \to \infty$ now contribute, corresponding to the exchange of closed strings propagating for a very long distance. Closed string modes with high masses give again exponentially suppressed contributions, and the dominant interaction comes from the exchange of massless closed string modes. The behavior of the partition function is in this case

$$Z_{closed}(l) \underset{l \to \infty}{\longrightarrow} (2-2) \; , \tag{2.42}$$

and Eq. (2.32) reduces to

$$\mathcal{A} \underset{r \gg l_s}{\longrightarrow} \frac{V_{p+1}}{2(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dl}{l^{\frac{5-p}{2}}} e^{-\frac{r^2}{4\pi\alpha' l}} (1-1) \; . \tag{2.43}$$

Eqs. (2.41) and (2.43) are identical and yield the large distance behavior of the amplitude in the open and closed string parameterizations. The most natural description of the large distance interaction of D-branes is obtained in terms of the massless closed string modes. The result can finally be rewritten as

$$\mathcal{A} \underset{r \gg l_s}{\longrightarrow} V_{p+1} \hat{T}_p \hat{T}_{p+4} (1-1) \Delta_{(5-p)}(r) \ . \tag{2.44}$$

It is possible to study extremely important generalizations of the one-loop amplitude discussed above, related to the vertex operators Eqs. (1.68) and (1.69). These vertex-operators correspond geometrically to perturbations parallel and orthogonal to the Dp-brane world-volume. The path-integral representation of the corresponding generating functionals is obtained by adding to the free string action S_0, Eq. (1.1), the deformations S_1 and/or S_2 in Eqs. (1.70) and (1.71) obtained by exponentiating and Fourier transforming these vertex-operators.

2.2 Rotated and boosted D-branes

Consider the case of two D-branes tiled with arbitrary angles $\pi\alpha_{1,2}$ in the plane of two space-like NN and DD directions x^i and x^{i+1}. The rotations are implemented by giving an VEV to the N derivative ∂_i of the D position described by the scalar field q^{i+1}, precisely $\langle \partial_i q^{i+1} \rangle = \tan \pi\alpha_{1,2}$. In other words, the VEVs of the scalar field itself are given by $\langle q^{i+1} \rangle = Y^{i+1} + \tan \pi\alpha_{1,2} X^i$. The free N and D b.c. get rotated and read

$$\partial_\sigma \left(\cos \pi\alpha_{1,2} X^i + \sin \pi\alpha_{1,2} X^{i+1} \right) = 0 \Big|_{\sigma=0,\pi} \ , \tag{2.45}$$

$$\partial_\tau \left(\cos \pi\alpha_{1,2} X^{i+1} - \sin \pi\alpha_{1,2} X^{i} \right) = 0 \Big|_{\sigma=0,\pi} \ , \tag{2.46}$$

or equivalently

$$\partial X^i - \bar{\partial} \bar{X}^i = \tan \pi\alpha_{1,2} \left(\partial X^{i+1} - \bar{\partial} \bar{X}^{i+1} \right) \Big|_{\sigma=0,\pi} \ , \tag{2.47}$$

$$\partial X^{i+1} + \bar{\partial} \bar{X}^{i+1} = - \tan \pi\alpha_{1,2} \left(\partial X^{i} + \bar{\partial} \bar{X}^{i} \right) \Big|_{\sigma=0,\pi} \ . \tag{2.48}$$

Similarly, the $\pm$ fermionic b.c. become

$$\psi^i \mp \tilde{\psi}^i = \tan \pi\alpha_{1,2} \left(\psi^{i+1} \mp \tilde{\psi}^{i+1} \right) \Big|_{\sigma=0,\pi} \ , \tag{2.49}$$

$$\psi^{i+1} \pm \tilde{\psi}^{i+1} = - \tan \pi\alpha_{1,2} \left(\psi^{i} \pm \tilde{\psi}^{i} \right) \Big|_{\sigma=0,\pi} \ . \tag{2.50}$$

In terms of the complex combinations $X^i_\pm = (X^i \pm i X^{i+1})/\sqrt{2}$ and $\psi^i_\pm = (\psi^i \pm i \psi^{i+1})/\sqrt{2}$, Eqs. (2.47), (2.48) and (2.49), (2.50) can be rewritten as

$$\partial X^\pm = e^{\mp 2\pi\alpha_{1,2} i} \bar{\partial} \bar{X}^\mp \Big|_{\sigma=0,\pi} \ , \tag{2.51}$$

$$\psi^\pm = \pm \, e^{\mp 2\pi\alpha_{1,2} i} \tilde{\psi}^\mp \Big|_{\sigma=0,\pi} \ . \tag{2.52}$$

It straightforward to write the mode expansions that follow from these twisted b.c.. The integer or half-integer modes of the free case are shifted by the relative angle $\alpha = \alpha_2 - \alpha_1$.

As a consequence, there are no longer zero modes. The two bosonic coordinates $X^{\pm}$ with the b.c. (2.51) have the following mode expansion

$$X^{\pm} = i\sqrt{\frac{\alpha'}{2}}\sum_{n}\left(\frac{\alpha_n^{\pm}}{n\pm\alpha}e^{-(n\pm\alpha)z\mp i\pi\alpha_1} + \frac{\alpha_n^{\mp}}{n\mp\alpha}e^{-(n\mp\alpha)\bar{z}\pm i\pi\alpha_1}\right) , \tag{2.53}$$

where $[\alpha_m^{\pm}, \alpha_n^{\mp}] = (m\pm\alpha)\delta_{m+n}$. Notice that n is integer in this case. Taking n to be half-integer would correspond to the case in which the two directions to be tilted are ND-DN (initially orthogonal D-branes) instead of NN-DD (initially parallel D-branes). Similarly, the mode expansion for the two fermions $\psi^{\pm}$ and $\tilde{\psi}^{\pm}$ satisfying the b.c. (2.52) is

$$\psi^{\pm} = \sqrt{\alpha'}\sum_{n}\psi_n^{\pm}e^{-(n\pm\alpha)z\mp i\pi\alpha_1} , \tag{2.54}$$

$$\tilde{\psi}^{\pm} = \sqrt{\alpha'}\sum_{n}\psi_n^{\mp}e^{-(n\mp\alpha)\bar{z}\pm i\pi\alpha_1} , \tag{2.55}$$

with n integer or half-integer depending on the sector and $\{\psi_m^{\pm}, \psi_n^{\mp}\} = (m\pm\alpha)\delta_{m+n}$. Notice that from the mode expansions Eq. (2.53), (2.54) and (2.55) it follows that the derivative (or equivalently the oscillator part) of $\pm$ bosons, as well as the $\pm$ fermions, have a definite monodromy under the transformation $\sigma \to \sigma + 2\pi$

$$\partial X^{\pm}(\tau, \sigma + 2\pi) = e^{\mp 2\pi\alpha i}\partial X^{\pm}(\tau, \sigma) , \tag{2.56}$$

$$\psi^{\pm}(\tau, \sigma + 2\pi) = \pm e^{\mp 2\pi\alpha i}\psi^{\pm}(\tau, \sigma) . \tag{2.57}$$

These relation also follow directly from the b.c. Eqs. (2.51) and (2.52). In fact, the b.c. at $\sigma = 0$ can be automatically implemented by identifying left and right movers with the right phase through the involution $z \to \bar{z} + 2\pi$ which allows to obtain a cylinder of length π from a torus with one of the periods equal to 2π. More precisely, one identifies

$$\bar{\partial}\bar{X}^{\pm}(\bar{z}) = e^{\mp 2\pi\alpha_1 i}\partial X^{\pm}(z) , \quad z = \bar{z} , \tag{2.58}$$

$$\tilde{\psi}^{\pm}(\bar{z}) = \pm e^{\mp 2\pi\alpha_1 i}\psi^{\pm}(z) , \quad z = \bar{z} , \tag{2.59}$$

at the first boundary $\sigma = 0$. Substituting these expressions into the b.c. Eqs. (2.51) and (2.52) after having used then the equivalence under 2π shifts along the cycle of the torus, one finds indeed Eqs. (2.56) and (2.57).

The contribution of the $\pm$ fields to the Hamiltonian is

$$H^{(+,-)} = N^{(+,-)} - a^{(+,-)} , \tag{2.60}$$

where

$$N^{(+,-)} = \sum_{n\geq 0}\left[\alpha_{-n}^{-}\alpha_n^{+} + (n+\alpha)\psi_{-n}^{-}\psi_n^{+}\right] + \sum_{n>0}\left[\alpha_{-n}^{+}\alpha_n^{-} + (n-\alpha)\psi_{-n}^{+}\psi_n^{-}\right] \tag{2.61}$$

and $a^{(+,-)}$ is the total normal ordering zero-point energy. This can be easily computed using the generalized ζ-function regularization

$$\sum_{n=0}^{\infty}(n+q) = \zeta(-1, q) = -\frac{1}{2}B_2(q) = -\frac{1}{2}\left[\frac{1}{6} + q(q-1)\right] . \tag{2.62}$$

The contribution of the bosonic and fermionic pairs with integer or half-integer is

$$a_{B,F}^{(+,-)} = \mp \begin{cases} \dfrac{1}{2}\left[-\dfrac{1}{6}+\alpha(1-\alpha)\right] & , \text{ P} \\ \dfrac{1}{2}\left(\dfrac{1}{12}-\alpha^2\right) & , \text{ A} \end{cases} \quad . \tag{2.63}$$

Consider now the case of two D-branes moving with constant velocities v_1 and v_2, in some transverse D direction x^i. The velocities correspond to non-zero VEVs for the time derivative ∂_0 of the scalar field q^i specifying the position in the D direction, $\langle \partial_0 q^i \rangle = v_{1,2}$. Correspondingly, the VEVs of the scalar field itself are $\langle q^i \rangle = v_{1,2} X^0 + Y^i$. It is well known that these constant velocities can be considered as imaginary rotations in the (x^0, x^i) plane of Minkowski space. The angles are π times the *rapidities* $\epsilon_{1,2}$ defined such that $v_{1,2} = \tanh \pi\epsilon_{1,2}$. The boundary interaction term associated to the velocities rotate the free N and D b.c. of the X^0 and X^i coordinates

$$\partial_\sigma \left(\cosh \pi\epsilon_{1,2} X^0 - \sinh \pi\epsilon_{1,2} X^i\right) = 0\Big|_{\sigma=0,\pi} \ , \tag{2.64}$$

$$\partial_\tau \left(\cosh \pi\epsilon_{1,2} X^i - \sinh \pi\epsilon_{1,2} X^0\right) = 0\Big|_{\sigma=0,\pi} \ , \tag{2.65}$$

or equivalently

$$\partial X^0 - \bar\partial \bar X^0 = \tan \pi\epsilon_{1,2} \left(\partial X^i - \bar\partial \bar X^i\right)\Big|_{\sigma=0,\pi} \ , \tag{2.66}$$

$$\partial X^i + \bar\partial \bar X^i = \tan \pi\epsilon_{1,2} \left(\partial X^0 + \bar\partial \bar X^0\right)\Big|_{\sigma=0,\pi} \ . \tag{2.67}$$

Similarly, the $\pm$ fermionic b.c. become

$$\psi^0 \mp \tilde\psi^0 = \tan \pi\epsilon_{1,2} \left(\psi^i \mp \tilde\psi^i\right)\Big|_{\sigma=0,\pi} \ , \tag{2.68}$$

$$\psi^i \pm \tilde\psi^i = \tan \pi\epsilon_{1,2} \left(\psi^0 \pm \tilde\psi^0\right)\Big|_{\sigma=0,\pi} \ . \tag{2.69}$$

In terms of the light-cone combinations $X^\pm = (X^0 \pm X^i)/\sqrt{2}$ and $\psi^\pm = (\psi^0 \pm \psi^i)/\sqrt{2}$ the b.c. Eqs. (2.66), (2.67) and (2.68), (2.69) become

$$\partial X^\pm = e^{\pm 2\pi\epsilon_{1,2}} \bar\partial \bar X^\mp\Big|_{\sigma=0,\pi} \ , \tag{2.70}$$

$$\psi^\pm = \pm\, e^{\pm 2\pi\epsilon_{1,2}} \tilde\psi^\mp\Big|_{\sigma=0,\pi} \ . \tag{2.71}$$

Therefore, the only difference with respect to the case of real rotations is that the twists are imaginary rather than real, $\alpha_{1,2} = i\epsilon_{1,2}$.

Because of their monodromy properties, the pairs of $\pm$ fields can be considered as two fields twisted with two opposite angles $\pm 2\pi\gamma$ around the σ-cycle of the covering torus. $\gamma = \alpha$ for tilted D-branes and $\gamma = i\epsilon$ for boosted D-branes. The contribution to the partition function of a boson or fermion with generic periodicities and an additional twist $\pm\gamma$ is indicated with the symbol

$$ {}^{P_2}\square_{P_{1\pm\gamma}} \tag{2.72}$$

Skipping the details, the results for contribution to the partition function of these twisted pairs in the various sectors can be summarized by quoting the results for a pair of twisted bosons and fermions with all possible periodicities on the covering torus. The results are reported in Appendix B, Eqs. (B.9)-(B.14). Using these results, it is straightforward at this point to generalize the computation of the partition functions entering in the one-loop amplitude for the interaction energy between two Dp-branes and between a Dp-brane and a D(p+4)-brane. In particular, we discuss now the case of constant velocities. The case of real angles is similar.

2.2.1 Dp-Dp dynamics

The only change with respect to the static case is that the light-cone pairs of fields get twisted. Eq. (2.12) becomes [90]

$$\mathcal{A} = \frac{V_p}{(2\pi\sqrt{\alpha'})^p} \int_0^\infty \frac{dt}{t^{\frac{p+2}{2}}} e^{-\frac{b^2}{4\pi\alpha'}t} Z_{open}(t,\epsilon) \;, \tag{2.73}$$

where $\vec{b} = \Delta\vec{Y}$ is the impact parameter and

$$Z_{open}(t,\epsilon) = \frac{1}{2}\frac{\vartheta_3(\frac{\epsilon t}{2}|\frac{it}{2})\vartheta_3^3(0|\frac{it}{2}) - \vartheta_4(\frac{\epsilon t}{2}|\frac{it}{2})\vartheta_4^3(0|\frac{it}{2}) + \vartheta_2(\frac{\epsilon t}{2}|\frac{it}{2})\vartheta_2^3(0|\frac{it}{2})}{\vartheta_1(\frac{\epsilon t}{2}|\frac{it}{2})\eta^9(\frac{it}{2})} \;. \tag{2.74}$$

Using Eq. (A.24), this can be rewritten as

$$Z_{open}(t,\epsilon) = \frac{\vartheta_1^4(\frac{\epsilon t}{4}|\frac{it}{2})}{\vartheta_1(\frac{\epsilon t}{2}|\frac{it}{2})\eta^9(\frac{it}{2})} \;. \tag{2.75}$$

Supersymmetry is broken and the amplitude vanishes only in the limit $\epsilon \to 0$ in which supersymmetry is restored.

The short distance limit $b \ll l_s$ is conveniently analyzed in the open string channel. Only very bride world-sheets with $t \to \infty$ contribute, corresponding to loops of massless open strings. In this limit, one finds simply

$$Z_{open}(t,\epsilon) \underset{t\to\infty}{\longrightarrow} \frac{6 + 2\cos 2\frac{\pi\epsilon t}{2} - 8\cos\frac{\pi\epsilon t}{2}}{2\sin\frac{\pi\epsilon t}{2}} = \frac{\left(2\sin\frac{\pi\epsilon t}{4}\right)^4}{2\sin\frac{\pi\epsilon t}{2}} \;. \tag{2.76}$$

Rescaling $t \to \pi\alpha' t$, the amplitude reduces to

$$\begin{aligned}\mathcal{A} &\underset{b\ll l_s}{\longrightarrow} \frac{V_p}{2(4\pi)^{\frac{p}{2}}} \int_0^\infty \frac{dt}{t^{\frac{p+2}{2}}} e^{-(\frac{b}{2\pi\alpha'})^2 t}\, \frac{6 + 2\cos 2\frac{\pi\epsilon}{2\pi\alpha'}t - 8\cos\frac{\pi\epsilon}{2\pi\alpha'}t}{\sin\frac{\pi\epsilon}{2\pi\alpha'}t} \\ &\underset{b\ll l_s}{\longrightarrow} \frac{V_p}{2(4\pi)^{\frac{p}{2}}} \int_0^\infty \frac{dt}{t^{\frac{p+2}{2}}} e^{-(\frac{b}{2\pi\alpha'})^2 t}\, \frac{\left(2\sin\frac{\pi\epsilon}{4\pi\alpha'}t\right)^4}{\sin\frac{\pi\epsilon}{2\pi\alpha'}t} \;.\end{aligned} \tag{2.77}$$

This can be interpreted as a one-loop effective action for the U(2) SYM theory reduced from D=10 to D=p+1, describing the massless strings living on the world-volumes of the two Dp-branes when these coincide. For $b \neq 0$ and $v \neq 0$, the transverse and longitudinal scalar fields acquire VEVs equal to $\vec{b}$ and vt respectively, and the theory is in the Coulomb phase. The strings starting and ending on the same Dp-brane remain massless, whereas those starting end ending on two different Dp-branes become massive, and the gauge symmetry is spontaneously broken to U(1)⊗U(1). One can factorize this two-fold symmetry into a U(1) related to the overall center of mass motion and a U(1) related to the relative motion of the two Dp-branes. By T-duality, the relative velocity corresponds to an effective relative U(1) electric field equal to $E = \pi\epsilon$. The fact the effective electric field is $\pi\epsilon$ rather than $\tanh \pi\epsilon$ is due to the non-minimal electromagnetic coupling of open strings. The particles running in the loop are the lightest modes of the open strings stretched between the two Dp-branes. They have a mass $m = b/(2\pi\alpha')$ and a charge $q = 1/(2\pi\alpha')$ under the relative motion U(1). Therefore, Eq. (2.77) can be interpreted as the effective action for a supermultiplet of mass $m = b/(2\pi\alpha')$ and charge $q = 1/(2\pi\alpha')$ in an effective electric field $E = \pi\epsilon$. The normalization differs from the usual one in the Euler-Heisenberg effective action because of the different overall degeneracy of the energy levels. The universal denominator come from the usual harmonic oscillator like spectrum of a particle in a constant electromagnetic field. The numerator only depends on the spin. It is associated to helicity supertraces coming from from the gyromagnetic coupling to the external field in the Hamiltonian. In Appendix C we report some relevant cases, whose dependence on the rapidity allows two disentangle unambiguously the contributions of various representation. In the present case, this allows to recognize that the numerator of Eq. (2.77) exactly corresponds to the content of a vector-like multiplet, with spin 0, 1/2 and 1 particles.

Again, one can understand the amplitude from the closed string channel point of view by performing a modular transformation. One finds

$$\mathcal{A} = \frac{V_p}{2^3(2\pi\sqrt{\alpha'})^p} \int_0^\infty \frac{dl}{l^{\frac{8-p}{2}}} e^{-\frac{b^2}{4\pi\alpha' l}} Z_{closed}(t,\epsilon) \; , \tag{2.78}$$

where

$$Z_{closed}(l,\epsilon) = \frac{1}{2}\frac{\vartheta_3(i\epsilon|2il)\vartheta_3^3(0|2il) - \vartheta_2(i\epsilon|2il)\vartheta_2^3(0|2il) - \vartheta_4(i\epsilon|2il)\vartheta_4^3(0|2il)}{\vartheta_1(i\epsilon|2il)\eta^9(2il)} \; , \tag{2.79}$$

or using Eq. (A.24)

$$Z_{closed}(l,\epsilon) = \frac{\vartheta_1^4(\frac{i\epsilon}{2}|2il)}{\vartheta_1(i\epsilon|2il)\eta^9(2il)} \; . \tag{2.80}$$

The large distance limit, $b \gg l_s$, is conveniently analyzed in this channel. In fact, only very long world-sheets with $l \to \infty$ contribute, corresponding to the exchange of massless closed strings. In this limit, the partition function becomes

$$Z_{closed}(l,\epsilon) \underset{l\to\infty}{\longrightarrow} \frac{6 + 2\cosh 2\pi\epsilon - 8\cosh \pi\epsilon}{2\sinh \pi\epsilon} = \frac{(2\sinh\frac{\pi\epsilon}{2})^4}{2\sinh \pi\epsilon} \; . \tag{2.81}$$

The remaining integration over the modulus l produces essentially a transverse propagator $\Delta_{(8-p)}(b)$, which combines with the $\sinh \pi\epsilon$ in the denominator to reconstruct the complete propagator integrated over the time τ parameterizing the path. Indeed

$$\frac{1}{\sinh \pi\epsilon}\Delta_{(d-1)}(b) = \int_{-\infty}^{+\infty} d\tau \Delta_{(d)}(r(\tau)) \; , \tag{2.82}$$

where $r(\tau) = \sqrt{b^2 + \sinh^2 \pi\epsilon\, \tau^2}$ is the true distance between the Dp-branes at time τ. Finally, the amplitude reduces to

$$\mathcal{A} \underset{b\gg l_s}{\longrightarrow} V_p \hat{T}_p^2 \left(\frac{3}{4} + \frac{1}{4}\cosh 2\pi\epsilon - \cosh \pi\epsilon\right) \int_{-\infty}^{\infty} d\tau \Delta_{(9-p)}(r(\tau))$$
$$\underset{b\gg l_s}{\longrightarrow} 2 V_p \hat{T}_p^2 \sinh^4 \frac{\pi\epsilon}{2} \int_{-\infty}^{\infty} d\tau \Delta_{(9-p)}(r(\tau)) \,. \tag{2.83}$$

This is interpreted as the phase-shift between two Dp-branes in SUGRA. The interaction comes from the exchange of the gravitational multiplet. The dependence on the rapidity allows to recognize the contribution of the different massless particles which are exchanged between the two Dp-branes. Using the results reported in Appendix C, one immediately recognizes the bosonic representation content of the gravitational multiplet.

Notice finally that, although generically different, the short distance and large distance behaviors Eqs. (2.73) and (2.78) become equal in the non-relativistic limit $v \simeq \pi\epsilon \to 0$. Yet more impressively, the exact amplitude given equivalently by Eqs. (2.73) or (2.78) in the open and closed string channels no longer depends on the string scale in the non-relativistic limit. Indeed, using Eq. (A.15), one finds

$$Z_{open}(t,\epsilon) \underset{\epsilon\to 0}{\longrightarrow} \frac{(\pi\epsilon t)^3}{16} \,, \tag{2.84}$$
$$Z_{closed}(l,\epsilon) \underset{\epsilon\to 0}{\longrightarrow} \frac{(\pi\epsilon)^3}{2} \,. \tag{2.85}$$

and Eqs. (2.73) and (2.78) become

$$\mathcal{A} \underset{v\to 0}{\longrightarrow} \frac{v^3}{8} V_p \hat{T}_p^2 \Delta_{(8-p)}(b) \,. \tag{2.86}$$

2.2.2 Dp-D(p+4) dynamics

In the Dp-D(p+4) case, Eq. (2.29) becomes [98]

$$\mathcal{A} = \frac{V_p}{(2\pi\sqrt{\alpha'})^p} \int_0^{\infty} \frac{dt}{t^{\frac{p+2}{2}}} e^{-\frac{b^2}{4\pi\alpha'}t} Z_{open}(t,\epsilon) \,. \tag{2.87}$$

where again $\vec{b} = \Delta\vec{Y}$ is the impact parameter and

$$Z_{open}(t,\epsilon) = \frac{1}{2} \frac{\vartheta_3(\frac{\epsilon t}{2}|\frac{it}{2})\vartheta_3(0|\frac{it}{2})\vartheta_2^2(0|\frac{it}{2}) - \vartheta_2(\frac{\epsilon t}{2}|\frac{it}{2})\vartheta_2(0|\frac{it}{2})\vartheta_3^2(0|\frac{it}{2})}{\vartheta_1(\frac{\epsilon t}{2}|\frac{it}{2})\eta^3(\frac{it}{2})\vartheta_4^2(0|\frac{it}{2})} \,. \tag{2.88}$$

Using Eq. (A.24), this can be rewritten as

$$Z_{open}(t,\epsilon) = \frac{\vartheta_1^2(\frac{\epsilon t}{4}|\frac{it}{2})\vartheta_4^2(\frac{\epsilon t}{4}|\frac{it}{2})}{\vartheta_1(\frac{\epsilon t}{2}|\frac{it}{2})\eta^3(\frac{it}{2})\vartheta_4^2(0|\frac{it}{2})} \,. \tag{2.89}$$

Supersymmetry is broken and the amplitude vanishes only in the limit $\epsilon \to 0$ in which supersymmetry is restored.

In the short distance limit $b \ll l_s$, only world-sheets with $t \to \infty$ contribute. In this limit, one finds

$$Z_{open}(t,\epsilon) \underset{t\to\infty}{\longrightarrow} \frac{2-2\cos\frac{\pi\epsilon t}{2}}{2\sin\frac{\pi\epsilon t}{2}} = \frac{\left(2\sin\frac{\pi\epsilon t}{4}\right)^2}{2\sin\frac{\pi\epsilon t}{2}} \ . \tag{2.90}$$

Rescaling $t \to \pi\alpha' t$, the amplitude reduces to

$$\begin{aligned}\mathcal{A} &\underset{b\ll l_s}{\longrightarrow} \frac{V_p}{2(4\pi)^{\frac{p}{2}}}\int_0^\infty \frac{dt}{t^{\frac{p+2}{2}}} e^{-(\frac{b}{2\pi\alpha'})^2 t} \frac{2-2\cos\frac{\pi\epsilon}{2\pi\alpha'}t}{\sin\frac{\pi\epsilon}{2\pi\alpha'}t} \\ &\underset{b\ll l_s}{\longrightarrow} \frac{V_p}{2(4\pi)^{\frac{p}{2}}}\int_0^\infty \frac{dt}{t^{\frac{p+2}{2}}} e^{-(\frac{b}{2\pi\alpha'})^2 t} \frac{\left(2\sin\frac{\pi\epsilon}{4\pi\alpha'}t\right)^2}{\sin\frac{\pi\epsilon}{2\pi\alpha'}t} \ .\end{aligned} \tag{2.91}$$

This can be interpreted as a one-loop effective action for the SYM theory reduced from D=6 to D=p+1, describing the massless strings living on the world-volumes of the Dp and D(p+4)-branes. For $b \neq 0$ and $v \neq 0$, the transverse and longitudinal scalar fields acquire a VEV equal to $\vec{b}$ and vt respectively, and again the gauge symmetry related to the relative motion is U(1). A before, by T-duality the relative velocity corresponds to an electric field equal to $E = \pi\epsilon$. The particles running in the loop are the lightest modes of the open strings stretched between the Dp-brane and the D(p+4)-brane. They have a mass $m = b/(2\pi\alpha')$ and fill a massive supermultiplet with a charge $q = 1/(2\pi\alpha')$ under U(1). Therefore, Eq. (2.91) can be interpreted as the effective action for a supermultiplet of mass $m = b/(2\pi\alpha')$ and charge $q = 1/(2\pi\alpha')$ in an effective electric field $E = \pi\epsilon$. Using the results of Appendix C, one recognizes that the numerator of Eq. (2.91) corresponds to the content of an hyper-like multiplet, with spin 0 and 1/2 particles.

Again, one can understand the amplitude from the closed string channel point of view by performing a modular transformation. One finds

$$\mathcal{A} = \frac{V_p}{2(2\pi\sqrt{\alpha'})^p}\int_0^\infty \frac{dl}{l^{\frac{4-p}{2}}} e^{-\frac{b^2}{4\pi\alpha' l}} Z_{closed}(t,\epsilon) \ , \tag{2.92}$$

where

$$Z_{closed}(l,\epsilon) = \frac{1}{2}\frac{\vartheta_3(i\epsilon|2il)\vartheta_3(0|2il)\vartheta_4^2(0|2il) - \vartheta_4(i\epsilon|2il)\vartheta_4(0|2il)\vartheta_3^2(0|2il)}{\vartheta_1(i\epsilon|2il)\eta^3(2il)\vartheta_2^2(0|2il)} \ , \tag{2.93}$$

or using Eq. (A.24)

$$Z_{closed}(l,\epsilon) = \frac{\vartheta_1^2(\frac{i\epsilon}{2}|2il)\vartheta_2^2(\frac{i\epsilon}{2}|2il)}{\vartheta_1(i\epsilon|2il)\eta^3(2il)\vartheta_2^2(0|2il)} \ . \tag{2.94}$$

In the large distance limit $b \gg l_s$, only world-sheets with $l \to \infty$ contribute. In this limit, the partition function becomes

$$Z_{closed}(l,\epsilon) \underset{l\to\infty}{\longrightarrow} \frac{-2+2\cosh 2\pi\epsilon}{8\sinh\pi\epsilon} = \frac{(2\sinh\pi\epsilon)^2}{8\sinh\pi\epsilon} \ . \tag{2.95}$$

As before,the remaining integration over the modulus l produces the transverse propagator $\Delta_{(4-p)}(b)$, which combines with the $\sinh\pi\epsilon$ in the denominator to reconstruct the complete propagator integrated over the time τ parameterizing the path, according to Eq. (2.82). Finally, the amplitude becomes

$$\mathcal{A} \underset{b\gg l_s}{\longrightarrow} V_p\hat{T}_p\hat{T}_{p+4}\left(-\frac{1}{4}+\frac{1}{4}\cosh 2\pi\epsilon\right)\int_{-\infty}^{\infty} d\tau\Delta_{(5-p)}(r(\tau))$$
$$\underset{b\gg l_s}{\longrightarrow} \frac{1}{2}V_p\hat{T}_p\hat{T}_{p+1}\sinh^2\pi\epsilon\int_{-\infty}^{\infty} d\tau\Delta_{(5-p)}(r(\tau)) \ . \quad (2.96)$$

This is interpreted as the phase-shift between a Dp and a D(p+4)-branes in SUGRA. The interaction comes from the exchange of part of the gravitational multiplet to which both the Dp-brane and the D(p+4)-brane couple. The dependence on the rapidity allows to recognize the contribution coming from the exchange of the different massless particles. Using the results of Appendix C, one recognizes in particular that there is no contribution from the RR gauge fields, and only the NSNS fields of the gravitational multiplet contribute. This is due to the fact that the Dp and the D(p+4)-branes are charged under different RR forms.

Notice finally that the to short distance and large distance behaviors Eqs. (2.87) and (2.92) match in the non-relativistic limit $v \simeq \pi\epsilon \to 0$. Actually, the exact amplitude given equivalently by Eqs. (2.87) or (2.92) in the open and closed string channels no longer depends on the string scale in the non-relativistic limit. Indeed, using Eq. (A.15) one finds

$$Z_{open}(t,\epsilon) \underset{\epsilon\to 0}{\longrightarrow} \frac{\pi\epsilon t}{4} \ , \quad (2.97)$$
$$Z_{closed}(l,\epsilon) \underset{\epsilon\to 0}{\longrightarrow} \frac{\pi\epsilon}{2} \ . \quad (2.98)$$

and Eqs. (2.87) and (2.92) become

$$\mathcal{A} \underset{v\to 0}{\longrightarrow} \frac{v}{2}V_p\hat{T}_p\hat{T}_{p+4}\Delta_{(4-p)}(b) \ . \quad (2.99)$$

2.3 D-branes with electromagnetic fluxes

The case of two constant electromagnetic fields $F^1_{\mu\nu}$ and $F^2_{\mu\nu}$ on the world-volumes of the two D-branes on which the cylindrical string world-sheets ends is perfectly similar. The boundary interaction term modifies the free N bosonic b.c. to

$$\partial_\sigma X^\mu = i2\pi\alpha'(F_{1,2})^\mu{}_\nu\partial_\tau X^\nu\Big|_{\sigma=0,\pi} \ , \quad (2.100)$$

or

$$\partial X^\mu - \bar\partial\bar X^\mu = 2\pi\alpha'(F_{1,2})^\mu_\nu\left(\partial X^\nu + \bar\partial\bar X^\nu\right)\Big|_{\sigma=0,\pi} \ . \quad (2.101)$$

Similarly, the $\pm$ fermionic b.c. become

$$\psi^\mu \mp \tilde\psi^\mu = 2\pi\alpha'(F_{1,2})^\mu_\nu\left(\psi^\mu \pm \tilde\psi^\mu\right)\Big|_{\sigma=0,\pi} \ . \quad (2.102)$$

Let us start by considering for simplicity non-zero magnetic fields $F^{1,2}_{ii+1} = B^{1,2}$ only in two space-like N directions, say x^i and x^{i+1}. It is convenient to parameterize this constant fields by two angles $\alpha_{1,2}$ defined such that $2\pi\alpha' B_{1,2} = \tan\pi\alpha_{1,2}$ and to form the complex combinations $X^i_\pm = (X^i \pm iX^{i+1})/\sqrt{2}$ and $\psi^i_\pm = (\psi^i \pm i\psi^{i+1})/\sqrt{2}$. By doing so, it becomes clear that the fields $F^{1,2}_{\mu\nu}$ induce a relative rotations between left and right movers. Eqs. (2.101) and (2.102) can in fact be rewritten as

$$\partial X^\pm = e^{\mp 2\pi i\alpha_{1,2}}\bar\partial \bar X^\pm\Big|_{\sigma=0,\pi} \ , \tag{2.103}$$

$$\psi^\pm = \pm e^{\mp 2\pi i\alpha_{1,2}}\tilde\psi^\pm\Big|_{\sigma=0,\pi} \ . \tag{2.104}$$

It straightforward to write the mode expansion that follow from these twisted b.c.. The integer or half-integer modes of the free case are shifted by $\alpha = \alpha_2 - \alpha_1$. As a consequence, in general there are no longer zero modes. The two bosonic coordinates $X^\pm$ with the b.c. (2.103) have the following mode expansion

$$X^\pm = x^\pm + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{\alpha^\pm_n}{n\pm\alpha}\left(e^{-(n\pm\alpha)z\mp i\pi\alpha_1} + e^{-(n\mp\alpha)\bar z\pm i\pi\alpha_1}\right) \ , \tag{2.105}$$

where $[\alpha^\pm_m, \alpha^\mp_n] = (m\pm\alpha)\delta_{m+n}$ and $[x^+, x^-] = \pi/\tan\pi\alpha$. Similarly, the mode expansion for the fermions $\psi^\pm$ and $\tilde\psi^\pm$ satisfying the b.c. (2.104) is

$$\psi^\pm = \sqrt{\alpha'}\sum_n \psi^\pm_n e^{-(n\pm\alpha)z\mp i\pi\alpha_1} \ , \tag{2.106}$$

$$\tilde\psi^\pm = \sqrt{\alpha'}\sum_n \psi^\pm_n e^{-(n\pm\alpha)\bar z\pm i\pi\alpha_1} \ , \tag{2.107}$$

with n integer or half-integer in the R and NS sectors and $\{\psi^\pm_m, \psi^\mp_n\} = (m\pm\alpha)\delta_{m+n}$. Notice that from the mode expansions Eq. (2.105), (2.106) and (2.107) it follow that the derivative (or equivalently the oscillator part) of the $\pm$ bosons, as well as the $\pm$ fermions, pick up a definite phase under the transformation $\sigma \to \sigma + 2\pi$

$$\partial X^\pm(\tau, \sigma + 2\pi) = e^{\mp 2\pi\alpha i}\partial X^\pm(\tau,\sigma) \ , \tag{2.108}$$

$$\psi^\pm(\tau, \sigma + 2\pi) = \pm e^{\mp 2\pi\alpha i}\psi^\pm(\tau,\sigma) \ . \tag{2.109}$$

These relation also follows directly from the b.c. Eqs. (2.103) and (2.104). In fact, the b.c. at $\sigma = 0$ can be automatically implemented by identifying left and right movers with the right phase through the involution $z \to \bar z + 2\pi$ which allows to obtain a cylinder of length π from a torus with one of the periods equal to 2π. More precisely, one identifies

$$\bar\partial \bar X^\pm(\bar z) = e^{\mp 2\pi\alpha_1 i}\partial X^\pm(z) \ , \quad z = \bar z \ , \tag{2.110}$$

$$\tilde\psi^\pm(\bar z) = \pm e^{\mp 2\pi\alpha_1 i}\psi^\pm(z) \ , \quad z = \bar z \ , \tag{2.111}$$

at the first boundary $\sigma = 0$. Substituting these expressions in the b.c. Eqs. (2.103) and (2.104) after having used then the equivalence under 2π shifts along the cycle of the torus, one finds indeed Eqs. (2.108) and (2.109).

The contribution of the $\pm$ fields to the Hamiltonian is

$$H^{(+,-)} = N^{(+,-)} - a^{(+,-)} \ , \tag{2.112}$$

where

$$N^{(+,-)} = \sum_{n\geq 0}\left[\alpha^-_{-n}\alpha^+_n + (n+\alpha)\psi^-_{-n}\psi^+_n\right] + \sum_{n>0}\left[\alpha^+_{-n}\alpha^-_n + (n-\alpha)\psi^+_{-n}\psi^-_n\right] \ . \tag{2.113}$$

The contribution to the normal-ordering zero-point energy $a^{(+,-)}$ for the bosonic and fermionic pairs with integer or half-integer is

$$a_{B,F}^{(+,-)} = \mp \begin{cases} \frac{1}{2}\left[-\frac{1}{6} + \alpha(1-\alpha)\right] & , \text{ P} \\ \frac{1}{2}\left(\frac{1}{12} - \alpha^2\right) & , \text{ A} \end{cases} . \quad (2.114)$$

The case of electric fields $F_{0i}^{1,2} = E^{1,2}$ along the x^i direction is similar. Again, it is convenient to parameterize these constant fields by two angles $\epsilon_{1,2}$ defined such that $2\pi\alpha' E_{1,2} = \tanh \pi\epsilon_{1,2}$ and to form the light-cone combinations $X^\pm = (X^0 \pm X^i)/\sqrt{2}$ and $\psi^\pm = (\psi^0 \pm \psi^i)/\sqrt{2}$. Eqs. (2.101) and (2.102) become

$$\partial X^\pm = e^{\pm 2\pi\epsilon_{1,2}} \bar{\partial} \bar{X}^\pm \Big|_{\sigma=0,\pi} , \quad (2.115)$$

$$\psi^\pm = \pm e^{\pm 2\pi\epsilon_{1,2}} \tilde{\psi}^\pm \Big|_{\sigma=0,\pi} . \quad (2.116)$$

Therefore, the only difference with respect to the magnetic field case is that the twist is imaginary rather than real, $\alpha_{1,2} = i\epsilon_{1,2}$.

The contributions to the partition function of the $\pm$ pair of twisted fields in the various sectors is essentially the same as in the case of rotations and boosts reported in Appendix B. The only difference is due to the additional bosonic zero modes $x^\pm$. Due to their canonical commutation relation, they behave as canonical variables conjugate to each other. They produce a phase space density of states equal to $\rho = \gamma/(2\pi^2)$, yielding an infinite degeneracy $\rho V_\pm$ [126]. The result for the interaction amplitudes between D-branes with electromagnetic fluxes is therefore similar to that obtained for moving D-branes, apart from a factor accounting for the afore mentioned additional degeneracy. For interesting discussions on open strings effective actions in electromagnetic fields, see also [112, 127, 128].

Chapter 3

Boundary state formalism

In this chapter, we introduce the boundary state encoding the couplings of a D-brane to closed strings. We show in particular that it allows to compute the asymptotic fields of D-branes and a direct computation of the interaction amplitude between two D-branes in the closed string channel.

3.1 World-sheet duality

As already mentioned, the cylinder amplitude giving the interaction between two D-branes can be interpreted either as a one-loop open string vacuum amplitude or as a closed string tree-level propagation [91, 92, 93]. In the open string channel, there is a direct prescription to compute the amplitude, which has been presented in the previous section. As shown, once the amplitude has been computed, it is possible to perform a modular transformation to understand the result in the closed string channel. We will present here a powerful method which allows to compute the amplitude directly in the closed string channel as the tree-level propagation between two closed string states representing the D-branes. More precisely, the circular world-sheet boundary to which a Dp-brane is attached can be interpreted as a closed string coherent state $|B_p\rangle$ implementing the b.c. defining the Dp-brane, called *boundary state* [94, 95, 96, 97]. This state encodes all the interactions between the Dp-brane and fundamental strings in the semiclassical approximation. In particular, the fundamental vertex involving an "in" Dp-brane emitting a closed string state and becoming an "out" Dp-brane, is encoded simply in the overlap $\langle B_p|\Psi\rangle$ between the boundary state $|B\rangle$ describing the Dp-brane semiclassical current in the eikonal approximation and the closed string state $|\Psi\rangle$. At least formally, the amplitude corresponding to an arbitrary world-sheet with n boundaries ending on D-branes can be computed by saturating the n-reggeon vertex with the boundary states describing the n D-branes [129]. Consider the cylinder in Fig. 3.1. with coordinates σ_1 and σ_2 grouped into the complex combinations $z_{open} = \sigma_1 + i\sigma_2$ and $z_{closed} = \sigma_2 + i\sigma_1$ in the open and closed string channels. To properly define the boundary state, it is convenient to start from the open string parameterization and turn to the closed string parameterization through a $\pi/2$ world-sheet Wick rotation ($z \to e^{i\pi/2}z = iz$) followed by a (convenient) parity transformation ($\sigma_2 \to -\sigma_2$). Indeed

$$z_{open} = \sigma_1 + i\sigma_2 \to -\sigma_2 + i\sigma_1 \to \sigma_2 + i\sigma_1 = z_{closed} \, . \tag{3.1}$$

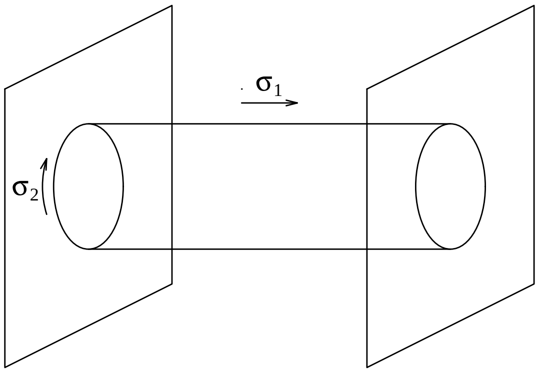


Figure 3.1: The cylinder amplitude.

Correspondingly, the world-sheet fields transform as two-dimensional scalars and spinors

$$\partial X^\mu \to e^{-i\frac{\pi}{2}} \partial X^\mu \ , \ \ \bar\partial \bar X^\mu \to e^{i\frac{\pi}{2}} \bar\partial \bar X^\mu \ , \ \ \psi \to e^{-i\frac{\pi}{4}} \psi \ , \ \ \tilde\psi \to e^{i\frac{\pi}{4}} \tilde\psi \ . \tag{3.2}$$

This allows a precise definition of the b.c. in the closed string channel.

Start from the open string parameterization, in which $\sigma_1 = \sigma \in [0, \pi]$ is the coordinate along the open string and $\sigma_2 = \tau \in [0, t]$ the periodic time of the loop. They are grouped into the complex world-sheet coordinate $z = \sigma + i\tau$. The b.c. at the two boundaries $\partial\Sigma_{1,2}$ at $\sigma = 0, \pi$ are the usual N,D and $\pm$ b.c. and generically

$$\left\{ \begin{array}{l} \partial X^\mu = (M_1)^\mu{}_\nu \bar\partial \bar X^\nu|_{\partial\Sigma_1} \\ \psi^\mu = \eta_1 (M_1)^\mu{}_\nu \tilde\psi^\nu\Big|_{\partial\Sigma_1} \end{array} \right. , \left\{ \begin{array}{l} \partial X^\mu = (M_2)^\mu{}_\nu \bar\partial \bar X^\nu|_{\partial\Sigma_2} \\ \psi^\mu = \eta_2 (M_2)^\mu{}_\nu \tilde\psi^\nu\Big|_{\partial\Sigma_2} \end{array} \right. . \tag{3.3}$$

Here $(M_{1,2})^\mu{}_\nu$ are diagonal matrices with ± 1 entries for the N or D b.c. and $\eta_{1,2} = \pm 1$ accounts for the two possible signs for the fermions. Only the relative sign $\eta_1\eta_2$ is insensitive to field redefinitions and corresponds to the two open string R ($\eta_1\eta_2 = 1$) and NS ($\eta_1\eta_2 = -1$) sectors. The bosonic and fermionic b.c. involve the same matrix $(M_{1,2})^\mu{}_\nu$, up to the signs $\eta_{1,2}$. This guarantees that at each boundary $\partial\Sigma_{1,2}$ ten-dimensional Lorentz invariance is broken into two factors corresponding to the $+$ and the $-$ entries in $(M_{1,2})^\mu{}_\nu$. For example, for the b.c. corresponding to a Dp-brane, the matrix $(M)^\mu{}_\nu$ has 1 in the first p+1 entries and -1 in the last 9$-$p, and SO(9,1) is broken to SO(p,1)$\times$SO(9$-$p). The bosonic fields are periodic in σ_2, whereas the fermion fields can pick up a sign η_3 and are antiperiodic ($\eta_3 = -1$) and periodic ($\eta_3 = 1$) in σ_2 in the $\pm$ spin structures corresponding to the 1/2 and $1/2(-1)^F$ parts of the open string GSO projection. Summarizing, around the cylinder

$$X^\mu \to X^\mu \ , \ \ \psi \to \eta_3 \psi \ , \ \ \text{around } \sigma_2 \ . \tag{3.4}$$

Now turn to the closed string parameterization, in which $\sigma_1 = \tau \in [0, l]$ is the propagation time and $\sigma_2 = \sigma \in [0, 2\pi]$ is the periodic coordinate along the closed string. Now the periodicity η_3 coming from the two parts of the open string GSO projection become simply the RR ($\eta_3 = 1$) and NSNS (η_3) sectors of the closed string in which the left and right fermions have the same periodicity η_3. The relative signs $\eta_1\eta_2 = \pm 1$ giving rise to the two open strings sectors now correspond to the two parts $1/4 + (-1)^F(-1)^{\tilde F}$ ($\eta_1\eta_2 = 1$) and

$1/2(-1)^F + 1/2(-1)^{\tilde{F}}$ ($\eta_1\eta_2 = -1$) of the closed string GSO projection. The b.c. defining the boundary state $|B\rangle$ are obtained from Eqs. (3.3) through the transformation (3.2). One obtains

$$\left\{\begin{array}{l} \partial X^\mu = -(M_1)^\mu{}_\nu \bar{\partial}\bar{X}^\nu|_{\partial\Sigma_1} \\ \psi^\mu = i\eta_1 (M_1)^\mu{}_\nu \tilde{\psi}^\nu\Big|_{\partial\Sigma_1} \end{array}\right. , \quad \left\{\begin{array}{l} \partial X^\mu = -(M_2)^\mu{}_\nu \bar{\partial}\bar{X}^\nu|_{\partial\Sigma_2} \\ \psi^\mu = i\eta_2 (M_2)^\mu{}_\nu \tilde{\psi}^\nu\Big|_{\partial\Sigma_2} \end{array}\right. . \tag{3.5}$$

To be complete and rigorous, one should also discuss the ghost and superghost part of the boundary state. This part can be determined requiring the boundary state to be BRST invariant invariant. As in the open string channel, the ghost and superghost contributions to amplitudes exactly cancel those of the unphysical pair of bosonic and fermionic fields.

3.2 Boundary states

It is at this point quite easy to construct the boundary state $|B_p\rangle$ representing a Dp-brane. We follow from now on the conventions of [95], setting the length of closed strings to 1 instead of 2π and the string tension equal to 1, that is $2\pi\alpha' = 1$. For later convenience, we shall work with the unusual complex coordinate $z = \sigma + i\tau$. This is just a convenient change of variable which restores the original sign in the bosonic b.c. but leaves the fermionic ones unchanged. With these conventions, the closed string mode expansions become

$$X^\mu(z) = \frac{x^\mu}{2} - \frac{z}{2}p^\mu + \frac{i}{\sqrt{4\pi}}\sum_{n>0}\frac{1}{\sqrt{n}}(a_n^\mu e^{2\pi n i z} - a_n^{\dagger\mu} e^{-2\pi n i z}) \,, \tag{3.6}$$

$$\bar{X}^\mu(\bar{z}) = \frac{x^\mu}{2} + \frac{\bar{z}}{2}p^\mu + \frac{i}{\sqrt{4\pi}}\sum_{n>0}\frac{1}{\sqrt{n}}(\tilde{a}_n^\mu e^{-2\pi n i \bar{z}} - \tilde{a}_n^{\dagger\mu} e^{2\pi n i \bar{z}}) \,, \tag{3.7}$$

$$\psi^\mu(z) = \sum_{n>0}(\psi_n^\mu e^{2\pi n i z} + \psi_n^{\dagger\mu} e^{-2\pi n i z}) \,, \tag{3.8}$$

$$\tilde{\psi}^\mu(\bar{z}) = \sum_{n>0}(\tilde{\psi}_n^\mu e^{-2\pi n i \bar{z}} + \tilde{\psi}_n^{\dagger\mu} e^{2\pi n i \bar{z}}) \,, \tag{3.9}$$

with the standard commutation relations $[a_m^\mu, a_n^{\dagger\nu}] = [\tilde{a}_m^\mu, \tilde{a}_n^{\dagger\nu}] = \eta^{\mu\nu}\delta_{mn}$ and $[x^\mu, p^\nu] = i\eta^{\mu\nu}$ for the bosons and anticommutation relations $\{\psi_m^\mu, \psi_n^{\dagger\nu}\} = \{\tilde{\psi}_m^\mu, \tilde{\psi}_n^{\dagger\nu}\} = \eta^{\mu\nu}\delta_{mn}$ for the fermions with appropriate moding in the RR and NSNS sectors. The Hamiltonian is

$$H = \frac{p^2}{2} + 2\pi\left\{\sum_{n=1}^{\infty} n\left(a_n^\dagger\cdot a_n + \tilde{a}_n^\dagger\cdot\tilde{a}_n\right) + \sum_{n>0}^{\infty} n\left(\psi_n^\dagger\cdot\psi_n + \tilde{\psi}_n^\dagger\cdot\tilde{\psi}_n\right) - b\right\} \,, \tag{3.10}$$

with integer or half-integer fermion moding and $b = 0, 1$ in the RR and NSNS sectors. The operators $(-1)^F$ and $(-1)^{\tilde{F}}$ appearing in the GSO projection are

$$(-1)^F = \left\{\begin{array}{ll} \eta_L \psi_0^{11}(-1)^{\sum_{n\geq 1}\psi_n^\dagger\cdot\psi_n} & , \ \text{RR} \\ -(-1)^{\sum_{n\geq 1/2}\psi_n^\dagger\cdot\psi_n} & , \ \text{NSNS} \end{array}\right. , \tag{3.11}$$

$$(-1)^{\tilde{F}} = \left\{\begin{array}{ll} \eta_R \tilde{\psi}_0^{11}(-1)^{\sum_{n\geq 1}\tilde{\psi}_n^\dagger\cdot\tilde{\psi}_n} & , \ \text{RR} \\ -(-1)^{\sum_{n\geq 1/2}\tilde{\psi}_n^\dagger\cdot\tilde{\psi}_n} & , \ \text{NSNS} \end{array}\right. , \tag{3.12}$$

where $\eta_L, \eta_R = \pm 1$ stand for the two possible chirality choices.

3.2.1 Static Dp-brane

The b.c. relate the left and right moving fields on the boundary corresponding to the Dp-brane, and translate into conditions relating the left and right modes. The boundary state is then defined as the eigenstate of these conditions and therefore reflects left and right movers into each others. Taking the Dp-brane at $\tau = 0$, one obtains

$$\left(p^\mu + (M_p)^\mu{}_\nu p^\nu\right)|B_p,\eta\rangle = 0 \,, \tag{3.13}$$

$$\left(a_n^\mu + (M_p)^\mu{}_\nu \tilde{a}_n^{\dagger\nu}\right)|B_p,\eta\rangle = \left(a_n^{\dagger\mu} + (M_p)^\mu{}_\nu \tilde{a}_n^\nu\right)|B_p,\eta\rangle = 0 \,, \tag{3.14}$$

$$\left(\psi_n^\mu + i\eta(M_p)^\mu{}_\nu \tilde{\psi}_n^{\dagger\nu}\right)|B_p,\eta\rangle = \left(\psi_n^{\dagger\mu} + i\eta(M_p)^\mu{}_\nu \tilde{\psi}_n^\nu\right)|B_p\rangle,\eta = 0 \,. \tag{3.15}$$

The solution for the boundary states can be factorized into a bosonic and a fermionic part

$$|B_p,\eta\rangle = |B_p\rangle_B \otimes |B_p,\eta\rangle_F \,. \tag{3.16}$$

Consider first the z.m.. For the bosons, Eq. (3.13) simply states that the boundary state carry no momentum k^α along the N directions, since the matrix $1/2(\mathbb{1} + M)^\mu_\nu$ projects onto the N directions. The z.m. part of the bosonic boundary state, $|\Omega_p\rangle_B$, is therefore a superposition of D momentum states $|k^i\rangle$. The precise wave function is determined by a stronger version of Eq. (3.13) requiring that the Dp-brane be localized at D position Y_i

$$\left(x^i - Y^i\right)|\Omega_p\rangle_B = 0 \,. \tag{3.17}$$

The solution is easily obtained from the Fock space vacuum $|0\rangle$ as

$$|\Omega_p\rangle_B = \delta^{(9-p)}\left(x^i - Y^i\right)|0\rangle = \int \frac{d^{9-p}k}{(2\pi)^{9-p}} e^{ik\cdot Y}|k^i\rangle \,. \tag{3.18}$$

For the fermions, there are z.m. only in the RR sector. In that case, the z.m. part of the boundary state satisfies the n=0 part of Eq. (3.15) which can be written

$$\left(\psi_0^\alpha + i\eta\tilde{\psi}_0^\alpha\right)|\Omega_p,\eta\rangle_{RR} = 0 \,, \quad \left(\psi_0^i - i\eta\tilde{\psi}_0^i\right)|\Omega_p,\eta\rangle_{RR} = 0 \,. \tag{3.19}$$

It follows that

$$\psi_0^{11}|\Omega_p,\eta\rangle_{RR} = (-1)^{p+1}\tilde{\psi}_0^{11}|\Omega_p,\eta\rangle_{RR} = |\Omega_p,-\eta\rangle_{RR} \,. \tag{3.20}$$

The state $|\Omega_p,\eta\rangle_{RR}$ can be explicitly constructed in various ways in the RR bi-spinor space. We shall present first the standard construction following [130, 131, 132], and then rely in Appendix D on an alternative construction better suited for fore-coming computations. We shall indicate with $|\alpha\rangle$ and $|\tilde{\alpha}\rangle$ the spinor states created out of the Fock vacuum from the spin fields S^α and $\tilde{S}^\alpha$. Recall that Type IIA and Type IIB theories differ by the relative chirality of these left and right spinors. The fermionic zero modes act as Γ-matrices, $\psi_0^\mu = \Gamma^\mu/\sqrt{2}$ and $\tilde{\psi}_0^\mu = \tilde{\Gamma}^\mu/\sqrt{2}$. Generically, the z.m. part of the RR boundary state will be of the form

$$|\Omega_p,\eta\rangle_{RR} = \mathcal{M}_{\alpha\beta}|\alpha\rangle|\tilde{\beta}\rangle \,. \tag{3.21}$$

Imposing the b.c. Eqs. (3.19) determines the matrix $\mathcal{M}$ to be

$$\mathcal{M} = C\Gamma^0 ....\Gamma^p \frac{1 - i\eta\Gamma^{11}}{1 - i\eta} \,. \tag{3.22}$$

In the NSNS sector there are no z.m. and one has simply an oscillator vacuum

$$|\Omega_p, \eta\rangle_{NSNS} = |0\rangle \ . \tag{3.23}$$

It is now easy to constructed the complete bosonic and fermionic boundary states from the vacua $|\Omega_p\rangle_B$ and $|\Omega_p, \eta\rangle_F$ through a Bogoliubov transformation implementing the b.c. (3.14) and (3.15) for the oscillator modes. One finds

$$|B_p\rangle_B = \exp\left\{\sum_{n=1}^{\infty}\left(M_{\mu\nu} a_n^{\dagger\mu} \tilde{a}_n^{\dagger\nu}\right)\right\}|\Omega_p\rangle_B \ , \tag{3.24}$$

$$|B_p, \eta\rangle_F = \exp\left\{-i\eta\sum_{n>0}^{\infty}\left(M_{\mu\nu} \psi_n^{\dagger\mu} \tilde{\psi}_n^{\dagger\nu}\right)\right\}|\Omega_p, \eta\rangle_F \ , \tag{3.25}$$

with appropriate moding and z.m. boundary state $|\Omega_p, \eta\rangle_F$ in the RR and NSNS sectors. The overall normalization of the boundary state is the only unknown quantity which has to be fixed by comparison with open string channel amplitude. We leave it unfixed for the moment.

The $(-1)^F$ and $(-1)^{\tilde{F}}$ operators entering the GSO projection act as follows

$$(-1)^F|B_p, \eta\rangle_{RR} = \eta_L|B_p, -\eta\rangle_{RR} \ , \tag{3.26}$$

$$(-1)^{\tilde{F}}|B_p, \eta\rangle_{RR} = \eta_R(-1)^{p+1}|B_p, -\eta\rangle_{RR} \ , \tag{3.27}$$

$$(-1)^F|B_p, \eta\rangle_{NSNS} = -|B_p, -\eta\rangle_{NSNS} \ , \tag{3.28}$$

$$(-1)^{\tilde{F}}|B_p, \eta\rangle_{NSNS} = -|B_p, -\eta\rangle_{NSNS} \ . \tag{3.29}$$

Consequently, the GSO-projected boundary state is

$$P\tilde{P}|B_p, \eta\rangle_{RR} = \frac{1 + \eta_L\eta_R(-1)^{p+1}}{4}|B_p, \eta\rangle_{RR} + \frac{\eta_L + \eta_R(-1)^{p+1}}{4}|B_p, -\eta\rangle_{RR} \ , \tag{3.30}$$

$$P\tilde{P}|B_p, \eta\rangle_{NSNS} = \frac{1}{2}|B_p, \eta\rangle_{NSNS} - \frac{1}{2}|B_p, -\eta\rangle_{NSNS} \ . \tag{3.31}$$

To obtain a non-vanishing result in the RR sector, one needs $\eta_L\eta_R = (-1)^{p+1}$, in agreement with the fact that even and odd p-branes can exist only in the Type IIA ($\eta_L\eta_R = -1$) and Type IIB ($\eta_L\eta_R = 1$) theories. The previous equations then reduce to

$$|\hat{B}_p, \eta\rangle = P\tilde{P}|B_p, \eta\rangle = \begin{cases} \frac{1}{2}\left(|B_p, \eta\rangle \pm |B_p, -\eta\rangle\right) & , \ \text{RR} \\ \frac{1}{2}\left(|B_p, \eta\rangle - |B_p, -\eta\rangle\right) & , \ \text{NSNS} \end{cases} \ . \tag{3.32}$$

Notice that the RR boundary state has a piece which depends on the overall chirality $\pm$ and encodes parity-violating couplings. The other piece in the RR sector, as well as the NSNS sector, are instead independent of η_L and encode parity-conserving couplings.

The boundary state encodes the couplings of the Dp-brane to all the tower of closed string states. Consider for instance the massless bosonic states $|\Psi\rangle$. In the RR sector, the generic polarization is a bi-spinor $\mathcal{H}_{\alpha\beta}$ which can be decomposed into antisymmetric tensors $H_{(n)}$ as

$$\mathcal{H}_{\alpha\beta} = \sum_{n=0}^{10}\frac{1}{n!}(CH_{(n)})_{\mu_1\ldots\mu_n}\Gamma^{\mu_1\ldots\mu_n}_{\alpha\beta} \ . \tag{3.33}$$

Due to the GSO projection, only forms with even or odd rank are present, depending on the chiralities of the two left and right R sectors. In the usual $(-1/2,-1/2)$ picture which can always be used when computing correlations on world-sheets without boundaries, $H_{(n)}$ are field-strengths $F_{(n)}$ and the GSO projection relates them by Hodge duality ${}^*F_{(n)} = F_{(10-n)}$. The appearance of the field strength rather than the potential reflects the fact that fundamental strings couple only non-minimally to the RR forms, and do not carry a true RR charge. Differently, in the $(-1/2,-3/2)$ or $(-3/2,-1/2)$ picture appropriate to soak the superghost zero mode anomaly of a disk corresponding to a world-sheet boundary ending on a D-brane, $H_{(n)}$ are potentials $C_{(p)}$, reflecting the fact that D-branes carry a non-zero RR charge. In the NSNS sector, the generic polarization is a two index tensor $\xi_{\mu\nu}$. It can be decomposed into trace, symmetric and antisymmetric parts

$$\xi^{(\phi)}_{\mu\nu} = \frac{1}{4}\left(\eta_{\mu\nu} - k_\mu l_\nu - k_\nu l_\mu\right) \ , \ \ \xi^{(h)}_{\mu\nu} = h_{\mu\nu} \ , \ \ \xi^{(b)}_{\mu\nu} = b_{\mu\nu} \ , \tag{3.34}$$

corresponding to the dilaton ϕ, the graviton $h_{\mu\nu}$ and antisymmetric Kalb-Ramond tensor $b_{\mu\nu}$. k^μ is the momentum of the state in the transverse D directions ($M^\mu{}_\nu k^\nu = 0$) and l^μ is an auxiliary vector satisfying $k \cdot l = 1$ and $l^2 = 0$. The corresponding RR and NSNS states are

$$|C\rangle_{RR} = \mathcal{H}_{\alpha\beta}\,|\alpha\rangle|\tilde{\beta}\rangle|k\rangle \ , \tag{3.35}$$

$$|\xi\rangle_{NSNS} = \xi_{\mu\nu}\,\psi^{\mu\dagger}_{1/2}\tilde{\psi}^{\dagger\nu}_{1/2}|k\rangle \ . \tag{3.36}$$

The conveniently normalized overlap $\langle\Psi\rangle_p = T_p\langle\hat{B}_p|\Psi\rangle$ then yields

$$\langle\xi\rangle_p = V_{p+1}\hat{T}_p\xi_{\mu\nu}M^{\mu\nu} \ , \tag{3.37}$$

$$\langle C\rangle_p = V_{p+1}\hat{T}_p\mathrm{Tr}[\Gamma^0...\Gamma^p C^{-1}\mathcal{H}] \ , \tag{3.38}$$

which, using Eqs. (3.33) with $H_{(n)} = C_{(p)}$ and (3.34), reproduce the correct Dp-brane couplings $\hat{T}_p$, $\hat{\mu}_p$ and $\hat{a}_p$ to massless RR and NSNS fields, given by Eqs. (1.76). One can also compute directly the asymptotic fields by inserting a closed string propagator Δ in the overlap and Fourier transforming. For details, see [130].

3.2.2 Rotated and boosted Dp-brane

The boundary state corresponding to a rotated or boosted Dp-brane can be constructed exactly in the same way as the static one, but start from rotated or boosted b.c.. Equivalently, the rotated or boosted boundary state can be obtained simply by applying to the static one a Lorentz transformation with negative angle or rapidity [132]. The closed string Lorentz generators are $J^{\mu\nu} = J^{\mu\nu}_B + J^{\mu\nu}_F$ with

$$J^{\mu\nu}_B = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty}\left(a^{\dagger\mu}_n a^\nu_n - a^{\dagger\nu}_n a^\mu_n + \tilde{a}^{\dagger\mu}_n\tilde{a}^\nu_n - \tilde{a}^{\dagger\nu}_n\tilde{a}^\mu_n\right) \ , \tag{3.39}$$

$$J^{\mu\nu}_F = \begin{cases} -\frac{i}{2}[\psi^\mu_0,\psi^\nu_0] - \frac{i}{2}[\tilde{\psi}^\mu_0,\tilde{\psi}^\nu_0] - i\sum_{n=1}^{\infty}\left(\psi^{\dagger\mu}_n\psi^\nu_n - \psi^{\dagger\nu}_n\psi^\mu_n + \tilde{\psi}^{\dagger\mu}_n\tilde{\psi}^\nu_n - \tilde{\psi}^{\dagger\nu}_n\tilde{\psi}^\mu_n\right) \ , & \text{RR} \\ -i\sum_{n=1/2}^{\infty}\left(\psi^{\dagger\mu}_n\psi^\nu_n - \psi^{\dagger\nu}_n\psi^\mu_n + \tilde{\psi}^{\dagger\mu}_n\tilde{\psi}^\nu_n - \tilde{\psi}^{\dagger\nu}_n\tilde{\psi}^\mu_n\right) \ , & \text{NSNS} \end{cases} \tag{3.40}$$

Consider for instance a rotation of angle $\pi\alpha$ in the plane of two N and D directions x^p and x^{p+1}. The boundary state $|B_p, \alpha\rangle$ for the rotated D-brane is obtained by applying the rotation $\exp\{-\pi\alpha J^{pp+1}\}$ to the boundary state $|B_p,\rangle$ of a static Dp-brane. Consider first the effect on the z.m. parts of the boundary state. The z.m. part of J_B rotates Eq. (3.17) and Eq. (3.18) becomes

$$|\Omega_p, \alpha\rangle_B = \delta(\cos\pi\alpha\, x^{p+1} - \sin\pi\alpha\, x^p)\delta^{(8-p)}\left(\vec{x} - \vec{Y}\right) = \int \frac{d^{9-p}k}{(2\pi)^{9-p}} e^{ik\cdot Y} |k(\alpha)\rangle \ , \tag{3.41}$$

where $k^\mu(\alpha) = (0, ..., 0, -\sin\pi\alpha\, k^{p+1}, \cos\pi\alpha\, k^{p+1}, k^{p+2}, ..., k^{9-p})$ is transverse to the rotated D-brane world-volume. The z.m. part of J_F affects Eq. (3.21) in the RR sector. In particular, the matrix $\mathcal{M}$ transforms to $\mathcal{M}(\alpha) = \Sigma_S(\alpha)\mathcal{M}\Sigma_S^{-1}(\alpha)$, where

$$\Sigma_S(\alpha) = \cos\frac{\pi\alpha}{2}\, \mathbb{1} - \sin\frac{\pi\alpha}{2}\, \Gamma^p\Gamma^{p+1} \ , \tag{3.42}$$

is the spinor representation of the rotation. One finds

$$\mathcal{M}(\alpha) = C\Gamma^0...\Gamma^{p-1}\left(\cos\pi\alpha\, \Gamma^p + \sin\pi\alpha\, \Gamma^{p+1}\right)\frac{1 - i\eta\Gamma^{11}}{1 - i\eta} \ . \tag{3.43}$$

Finally, the effect of J on the oscillator part of the boundary state amounts to transform the matrix $M^\mu{}_\nu$ to $M^\mu{}_\nu(\alpha) = (\Sigma_V(\alpha)M\Sigma_V^{-1}(\alpha))^\mu{}_\nu$, where

$$(\Sigma_V)^\mu{}_\nu(\alpha) = \begin{pmatrix} \cos\pi\alpha & \sin\pi\alpha \\ -\sin\pi\alpha & \cos\pi\alpha \end{pmatrix} \ . \tag{3.44}$$

One finds

$$M^\mu{}_\nu(\alpha) = \begin{pmatrix} \mathbb{1}_p & 0 & 0 & 0 \\ 0 & \cos 2\pi\alpha & -\sin 2\pi\alpha & 0 \\ 0 & -\sin 2\pi\alpha & -\cos 2\pi\alpha & 0 \\ 0 & 0 & 0 & -\mathbb{1}_{8-p} \end{pmatrix} \ . \tag{3.45}$$

A boost can be analyzed exactly in the same way.

Proceeding as in the static case, one can compute the couplings and the asymptotic fields for a rotated or boosted Dp-brane, finding the expected results dictated by Lorentz covariance.

3.2.3 Dp-brane with fluxes

The boundary state corresponding to a Dp-brane with constant electromagnetic fluxes can be constructed in a similar way. For instance, it can be obtained from that relative to a rotated or boosted Dp-brane by T-duality. As shown is Chapter 2, a magnetic flux in some N plane, say $F_{p-1p} = B$, amounts essentially to a rotation of opposite angle of left and right movers. The angle is given by the relation $2\pi\alpha' B = \tan\pi\alpha$. The boundary state Eq. (3.18) change only by an overall factor $\cos\pi\alpha$, whereas the matrices $\mathcal{M}$ entering Eq. (3.21) and the matrix M appearing in Eqs. (3.24) and (3.25) transform through the spinor and vector representations of the left and right rotations, $\mathcal{M}(\alpha) = \Sigma_S(\alpha)\mathcal{M}\Sigma_S^{-1}(-\alpha)$ and $M^\mu{}_\nu(\alpha) = (\Sigma_V(\alpha)M\Sigma_V^{-1}(-\alpha))^\mu{}_\nu$. One obtains

$$\mathcal{M}(\alpha) = C\Gamma^0...\Gamma^{p-2}\left(\cos\pi\alpha\,\Gamma^{p-1}\Gamma^p + \sin\pi\alpha\,\mathbb{1}\right)\frac{1-i\eta\Gamma^{11}}{1-i\eta}\ , \tag{3.46}$$

and

$$M^\mu{}_\nu(\alpha) = \begin{pmatrix} \mathbb{1}_{p-1} & 0 & 0 & 0 \\ 0 & \cos 2\pi\alpha & \sin 2\pi\alpha & 0 \\ 0 & -\sin 2\pi\alpha & \cos 2\pi\alpha & 0 \\ 0 & 0 & 0 & -\mathbb{1}_{9-p} \end{pmatrix} . \tag{3.47}$$

The case of an electric field is similar.

An important consequence of turning on a magnetic field is that the Dp-brane can then couple also to the RR (p−1)-form, beside the (p+1)-form. Turning on fluxes in n-planes, one finds couplings to the RR (p+1−2m)-forms with m≤n [122] (see also [133]). These couplings can be checked explicitly by computing overlaps of a generic RR state with the boundary state. They correspond to the Wess-Zumino terms in the Dp-brane effective action Eq. (1.72). A Dp-brane with electromagnetic fluxes in n planes can therefore be interpreted as a bound state n Dq-branes with q = p, p−2, ..., p−2n.

3.3 Interactions

The boundary state formalism allows to compute the cylinder amplitude directly in the closed string channel. Apart from an unknown normalization factor $\mathcal{N}$, the amplitude is obtained as the tree level propagation amplitude between the two GSO-projected boundary states $|\hat{B}_1, \eta_1\rangle$ and $|\hat{B}_2, \eta_2\rangle$ describing the D-branes on which the cylinder ends. The closed string propagator is conveniently written as

$$\Delta = \frac{1}{H} = \int_0^\infty dl e^{-lH}\ , \tag{3.48}$$

so that the amplitude reads

$$\mathcal{A} = \mathcal{N}\int_0^\infty dl\left\{\langle\hat{B}_1,\eta_1|e^{-lH}|\hat{B}_2,\eta_2\rangle_{NSNS} + \langle\hat{B}_1,\eta_1|e^{-lH}|\hat{B}_2,\eta_2\rangle_{RR}\right\}\ . \tag{3.49}$$

The GSO projection applied to a boundary state with parameter η produces a combination of two boundary states with parameters $\pm\eta$ according to Eqs. (3.32). As it must be, $\langle B_1,\eta|e^{-lH}|B_2,\eta'\rangle$ depends only on $\eta\eta' = \pm 1$, so that the amplitude receives four independent contributions coming from the two possible relative signs in each sector. These four contributions correspond to the four spin-structures that we shall call R+, R−, NS+ and NS−. In each of the two sectors SS, where S=R,NS, the contribution of the spin structure $S\pm$ is indicated as

$$\langle B_1|e^{-lH}|B_2\rangle_{S\eta\eta'} = \langle B_1,\eta|e^{-lH}|B_2,\eta'\rangle_{SS}\ , \tag{3.50}$$

and the amplitude can be written as

$$\mathcal{A} = \mathcal{N}\int_0^\infty dl\frac{1}{2}\left\{\langle B_1|e^{-lH}|B_2\rangle_{NS+} - \langle B_1|e^{-lH}|B_2\rangle_{NS-} + \langle B_1|e^{-lH}|B_2\rangle_{R+} \pm\langle B_1|e^{-lH}|B_2\rangle_{R-}\right\}\ . \tag{3.51}$$

A suggestive and concise way of writing this is the following

$$\mathcal{A} = \mathcal{N}\int_0^\infty dl \frac{1}{2}\sum_s (\pm) Z_s(l) \ , \tag{3.52}$$

where we have defined the "partition function" in the spin structure s as

$$Z_s(l) = \langle B_1|e^{-lH}|B_2\rangle_s \ . \tag{3.53}$$

The name partition function is in this case an abuse of language, finding its significance in the fact that the above expression encodes all the results of Polyakov's path-integral on the cylinder. As its open string analog, the partition function splits into the product of a bosonic and a fermionic parts $Z_B(l)$ and $Z_F(l)$ corresponding to the bosonic and fermionic components of the boundary state. Each of these can further be decomposed into z.m. and oscillator contributions $Z_0(l)$ and $Z_{osc}(l)$.

As an example of the power of the boundary state formalism, we shall briefly summarize the computation of these partition functions in the case of two parallel D-branes. Consider first the bosonic z.m. contribution. For the case of two static and parallel Dp and D(p+2n)-branes, one finds

$$Z_0^B(l) = V_{p+1}\int \frac{d^{9-2n-p}\vec{k}}{(2\pi)^{9-n-p}} e^{i\vec{k}\cdot\vec{r}} e^{-\frac{k^2}{2}} = V_{p+1}(2\pi l)^{-\frac{9-2n-p}{2}} e^{-\frac{r^2}{2l}} \ , \tag{3.54}$$

where $\vec{r} = \vec{Y}_1 - \vec{Y}_2$ is the distance separating the Dp and the D(p+2n)-branes in the (9−2n−p)-dimensional space. For constant velocities $v_{1,2} = \tanh \pi\epsilon_{1,2}$ in some D direction, say x^9, this becomes

$$Z_0^B(l,\epsilon) = \frac{V_p}{\sinh \pi\epsilon}\int \frac{d^{8-n-p}\vec{k}}{(2\pi)^{8-2n-p}} e^{i\vec{k}\cdot\vec{b}} e^{-\frac{k^2}{2}} = \frac{V_p}{\sinh \pi\epsilon}(2\pi l)^{-\frac{8-2n-p}{2}} e^{-\frac{b^2}{2l}} \ , \tag{3.55}$$

where $\vec{b} = \vec{Y}_1 - \vec{Y}_2$ is know the impact parameter and $\epsilon = \epsilon_1 - \epsilon_2$ the relative rapidity. The effect of rotations is similar.

Consider now the fermionic z.m. contribution. Using

$$\langle \Omega_p, \eta|\Omega_{p+n}, \eta'\rangle_{RR} = 2^5 \delta_{\eta\eta'}\delta_{n,0} \ , \tag{3.56}$$

$$\langle \Omega_p, \eta|\Omega_{p+n}, \eta'\rangle_{NSNS} = 1 \ , \tag{3.57}$$

one finds

$$Z_0^{NS\pm} = 1 \ , \ \ Z_0^{R+} = 2^5\delta_{n,0} \ , \ \ Z_0^{R-} = 0 \ . \tag{3.58}$$

Finally, consider the oscillator contribution. This can be computed in the general case in terms of the matrix $(M_{1,2})^\mu{}_\nu$ characterizing each of the two D-branes. For the ten bosons, one finds

$$Z_{osc}^B(l) = \prod_{n=1}^\infty \det{}^{-1}\left(1\!\!1 + q^{2n} M_1^T M_2\right) \ , \tag{3.59}$$

where $q = e^{-2\pi l}$. Similarly, the ten fermions give the following contributions in the four spin-structures

$$Z_{osc}^{R\pm}(l) = \prod_{n=1}^\infty \det\left(1\!\!1 \pm q^{2n} M_1^T M_2\right) \ , \tag{3.60}$$

$$Z_{osc}^{NS\pm}(l) = \prod_{n=1}^\infty \det\left(1\!\!1 \pm q^{2n-1} M_1^T M_2\right) \ . \tag{3.61}$$

The ghost contributions can be summarized as follows. As for their oscillator part, the ghosts and superghosts give a contribution which is the inverse of that of a pair of "normal" bosons and fermions, corresponding to a 2×2 block M_2 in M such that $M_2^T M_2 = \mathbb{1}_2$. This amounts to use an 8×8 light-cone matrix M in Eqs. (3.59), (3.60) and (3.61). As for the z.m. contributions, the bosonic ghosts play no role. More precisely, they are "inserted" and have be traded for an explicit integration over the world-sheet modulus l. The superghosts are more subtle to treat. In the NSNS sector, nothing happens since the superghost are antiperiodic and have no zero modes. In the RR sector instead, the superghosts are periodic and have z.m.. As for the fermions, these z.m. are fake in the RR+ spin structure, in the sense that they are z.m. only with respect to the string Fourier decomposition but not on the covering torus. Their effect is simply to lower the factor 2^5 in Eq. (3.58) by a factor 2, leaving 2^4. In the RR$-$ spin-structure, the superghost z.m. are true z.m. also on the covering torus. Naively, their determinant would give an infinite factor $1/0 = \infty$ coming from these z.m.. However, from an odd spin-structure path-integral point of view, the superghost determinant was born as a "primed" determinant with the z.m. excluded, since it corresponds to the jacobian of the super-diffeomorphism gauge fixing necessary to gauge away the non-harmonic part of the world-sheet gravitino. However, in this approach it remains an integration over the harmonic zero modes part of the world-sheet gravitino, which are nothing but the supermoduli. Since the gravitino couples to the supercurrent, this leads to the well-known super-Teichmüller insertions of the world-sheet supercurrent.

A simple way out of the subtlety associated to the odd spin-structure z.m. is proposed and discussed in great detail in [131], and consists in giving a regularization prescription for canceling the z.m. contributions of superghosts and longitudinal unphysical fermions. Heuristically, in the RR+ spin-structure, each pair of fermionic z.m. gives a factor 2 for a NN,NN or DD,DD plane, and 0 for a ND,ND or DN,DN plane. Conversely, in the RR$-$ spin-structure, each pair of fermionic z.m. gives a factor 0 for a NN,NN or DD,DD plane and a factor 2 for a ND,ND or DN,DN plane. The superghosts z.m. contribute instead 1/2 for the RR+ spin-structure and 1/0 in the RR$-$ spin-structure. Therefore, canceling the 0 of the superghosts with the 0 of one pair of fermions, one finds for the Dp-D(p+2n) system a total of $2^{4-n}0^n$ in the RR+ and $2^n 0^{4-n}$ in the RR$-$ spin-structure. This construction seems to differ from the usual path-integral approach but allows to describe correctly the D0-D8 system, which involves a peculiar odd spin-structure interaction [131]. An analogous subtlety arises also for the Dp-D(6$-$p) system in relative motion, in which the odd spin-structure encodes the magnetic interaction [101]. We shall see that in this case the path-integral approach with supercurrents insertion suggests a simple prescription to obtain directly the peculiar magnetic phase-shift [101], whereas the regularization procedure described above produce a result which is difficult to interpret as a phase-shift [134].

To complete the discussion, consider also the case in which additional vertex operators are inserted on the cylinder world-sheet, focusing on the case in which only one of them is present. The total superghost charge is 0, and in the even spin-structures the vertex operator has to be taken in the 0 picture. In the odd spin-structre instead, the supercurrent times $\delta(\beta)$ acts as a picture-changing operator and the vertex operator has to be taken in the -1 picture. Notice that also in the even spin-structures one could formally take the vertex operator in the -1 picture and insert a picture-changing operator, giving altogether the 0 picture vertex operator. We have not been able to found a clear discussion of the corresponding construction for the odd spin-structure, and we will therefore assume that also in this case the combination of the -1 picture vertex operator and the picture-changing operator will give the 0 picture operator, while the superghost determinant is also in this case the "primed" one, as for the ghosts. For a related discussion see [95].

Using the above results, it is straightforward to reproduce the amplitudes computed in Chapter 3 in their closed string channel version. In order to obtain the correct normalization, one has to take

$$\mathcal{N} = \frac{\hat{T}_p^2}{2^4} \, . \tag{3.62}$$

Notice that the partition function defined here within the boundary state formalism differs by a factor $2^{\#}$ from that used in Chapter 2 in the closed channel factorization, $\#$ being the number of direction with twisted b.c.. One could redefine the partition function to match it with the definition obtained by performing a modular transformation form the open string channel, but then one would get an overall normalization of the boundary state amplitude which would depend on the number of twisted direction. We therefore prefer use the above convention.

Chapter 4

Compactification and point-like D-branes

In this chapter we discuss D-branes and their dynamics in the framework of some simple compactification schemes. We study several examples of D-branes wrapped on the compact part of spacetime, yielding point-like objects in the lower-dimensional non-compact part of spacetime. We compute the phase-shift for two of these point-like objects and perform a detailed analysis of the amplitude for emitting a massless closed string state during the interaction. Comparison with field theory computations allows the precise determination of the couplings of the various point-like configurations to four-dimensional massless fields. We follow mainly [99, 100]. See also [135, 136, 137]

4.1 Toroidal and orbifold compactifications

One of the most important problems in string theory is the fact that it is a consistent theory only in ten dimensions. A way to reach a phenomenologically more realistic theory is to imagine ten-dimensional spacetime $\mathcal{M}_{10}$ to be the product $\mathcal{M}_{10} = \mathbb{R}^{3,1} \times \mathcal{M}_6$ of ordinary four-dimensional flat Minkowski space $\mathbb{R}^{3,1}$ and some compact manifold $\mathcal{M}_6$, whose typical size L is much shorter than the length scale probed by present days experiments. Requiring $\mathcal{M}_6$ to be Ricci-flat ensures that this ansatz is compatible with conformal invariance and is therefore an acceptable solution of string theory. The ten-dimensional fields of string theory decompose into their four-dimensional content by harmonic analysis on $\mathcal{M}_6$. Massless fields in four dimensions are in one-to-one correspondence with zero modes of the wave operators on $\mathcal{M}_6$, whereas higher harmonics give rise to massive Kaluza-Klein modes. In addition to these, there are also winding modes coming from strings wrapped on one-cycles of $\mathcal{M}_6$, whose mass scales with the inverse of the typical size of $\mathcal{M}_6$, $m \sim \alpha'/\mathrm{L}$. Consequently, the size L cannot be too small, since the proliferation of light winding modes would in some sense redecompactify the theory in a T-dual version. The fraction of the original ten-dimensional supersymmetry inherited by the four-dimensional low-energy effective theory depends instead on the number of covariantly constant spinors on $\mathcal{M}_6$.

Consider in particular the two Type II theories compactified on a six-torus $\mathcal{M}_6 = T^6$. Call Γ_6 the lattice defining T^6 as the quotient $T^6 = \mathbb{R}^6/\Gamma_6$. The fields of the corresponding ten-dimensional LEEA decompose by simple dimensional reduction, and the original N=2 D=10 SUGRA reduces to N=8 D=4 SUGRA. As for the fields, the N=2 D=10 gravitational multiplet becomes simply the N=8 D=4 gravitational multiplet. D=4 theories with less supersymmetry can be obtained through the so-called *orbifold* construction, obtained by gauging some discrete symmetry of Γ_6. In particular, we will be interested in the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifold compactifications, $\mathcal{M}_6 = T^2 \times T^4/\mathbb{Z}_2$ and $\mathcal{M}_6 = T^6/\mathbb{Z}_3$, with N=4 and N=2 supersymmetry.

Consider first the construction of the $T^4/\mathbb{Z}_2$ orbifold. One starts with a four-torus T^4 which is the product $T^4 = T^2_1 \times T^2_2$ of two identical two-tori T^2_i with modulus $\tau = i$. Each $T^2_i = \mathbb{R}/\Gamma_2$, defined by the equivalence $z_i = z_i + m + n\tau$, is symmetric with respect to $\mathbb{Z}_2$ reflections $g : z_i \to -z_i$. The Hamiltonian is invariant as well, so that one can gauge this $\mathbb{Z}_2$ symmetry by projecting the Hilbert space of the theory onto $\mathbb{Z}_2$-invariant states. This is done using the projector $P = 1/2(1+g)$. In particular, only 1/2 of the 32 supercharges survives this projection, so that one has a N=4 residual supersymmetry in D=4. Modular invariance at the one-loop level requires the inclusion of twisted sectors in the Hilbert space, in which strings are closed only up to a $\mathbb{Z}_2$ gauge transformation. The $\mathbb{Z}_2$ actions is not free, but has 2^2 fixed points in $\hat{z}_i = k/2e^{i\pi/4}$ with k=0, 1, where the space $T^4/\mathbb{Z}_2$ is singular and no longer a manifold.

The construction of the $T^6/\mathbb{Z}_3$ orbifold is similar. One starts this time with a six-torus T^6 which is the product $T^6 = T^2_1 \times T^2_2 \times T^2_3$ of three identical two-tori T^2_i with modulus $\tau = e^{2\pi i/3}$. Each $T^2_i = \mathbb{R}/\Gamma_2$, defined by the equivalence $z_i = z_i + m + n\tau$, is now symmetric with respect to $\mathbb{Z}_3$ rotations $g : z_i \to e^{2\pi i/3} z_i$. As before, the Hamiltonian is invariant as well, so that one can gauge this $\mathbb{Z}_3$ symmetry by projecting the Hilbert space of the theory onto $\mathbb{Z}_3$-invariant states with $P = 1/3(1+g+g^2)$. In particular, only 1/8 of the 32 supercharges survives this projection, so that one has a N=2 residual supersymmetry in D=4. Modular invariance at the one-loop level again requires the inclusion of twisted sectors in the Hilbert space, in which strings are closed only up to a $\mathbb{Z}_3$ gauge transformation. As before, the $\mathbb{Z}_3$ actions is not free, but has 3^3 fixed points in $\hat{z}_i = k/3e^{i\pi/6}$ with k=0,1,2, where the space $T^6/\mathbb{Z}_3$ is no longer a manifold.

The need of including twisted sectors can be understood quite in general. Consider in fact a generic $\mathbb{Z}_n$ projection $P = 1/N(1 + g + ... + g^{N-1})$. The one-loop partition function of the projected theory is

$$Z(\beta) = \mathrm{Tr}[P e^{-\beta H}] = \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{Tr}[g^n e^{-\beta H}] \; . \tag{4.1}$$

In the notation of Appendix B for partition functions on the torus, now for a complex pair of fields, this corresponds to summing the following b.c.

$$Z(\beta) = \frac{1}{N} \sum_{n=0}^{N-1} {}^{g^n}\underset{1}{\Box} \quad . \tag{4.2}$$

It is clear that this cannot be modular invariant, unless one adds all the possible twists along the other cycle of the torus,

$$Z(\beta) = \frac{1}{N} \sum_{m,n=0}^{N-1} {}^{g^n}\underset{h^m}{\Box} \quad , \tag{4.3}$$

corresponding exactly to adding the contribution of all the twisted sectors $\mathcal{H}_m$ corresponding to a twist h^n

$$Z(\beta) = \sum_{m=0}^{N-1} \mathrm{Tr}_{\mathcal{H}_m}[Pe^{-\beta H}] = \frac{1}{N} \sum_{m,n=0}^{N-1} \mathrm{Tr}[g^n h^m e^{-\beta H}] \ . \tag{4.4}$$

Consider now a generic $\mathbb{Z}_N$ orbifold twisted sector in which strings close only up to $g^m \in \mathbb{Z}_N$. The complex coordinate $Z^i, Z^{i*} = (X^i \pm iX^{i+1})/\sqrt{2}$, has twisted periodicity conditions

$$Z^i(\sigma + 1) = e^{2\pi i\alpha} Z^i(\sigma) \ , \ \ Z^{i*}(\sigma + 1) = e^{-2\pi i\alpha} Z^{i*}(\sigma) \ , \tag{4.5}$$

with $\alpha = m/N$. The mode expansion is

$$Z^i(z) = \frac{z^i}{2} + \frac{i}{\sqrt{4\pi}} \left\{ \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+\alpha}} c_n^i e^{2\pi(n+\alpha)iz} - \sum_{n=1}^{\infty} \frac{1}{\sqrt{n-\alpha}} c_n^{\dagger i} e^{-2\pi(n-\alpha)iz} \right\} \ , \tag{4.6}$$

$$\tilde{Z}^i(\bar{z}) = \frac{z^i}{2} + \frac{i}{\sqrt{4\pi}} \left\{ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n-\alpha}} \tilde{c}_n^i e^{-2\pi(n-\alpha)i\bar{z}} - \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+\alpha}} \tilde{c}_n^{\dagger i} e^{2\pi(n+\alpha)i\bar{z}} \right\} \ , \tag{4.7}$$

$$Z^{i*}(z) = \frac{z^{i*}}{2} + \frac{i}{\sqrt{4\pi}} \left\{ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n-\alpha}} c_n^{i*} e^{2\pi(n-\alpha)iz} - \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+\alpha}} c_n^{\dagger i*} e^{-2\pi(n+\alpha)iz} \right\} \ , \tag{4.8}$$

$$\tilde{Z}^{i*}(\bar{z}) = \frac{z^{i*}}{2} + \frac{i}{\sqrt{4\pi}} \left\{ \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+\alpha}} \tilde{c}_n^{i*} e^{-2\pi(n+\alpha)i\bar{z}} - \sum_{n=1}^{\infty} \frac{1}{\sqrt{n-\alpha}} \tilde{c}_n^{\dagger i*} e^{2\pi(n-\alpha)i\bar{z}} \right\} \ , \tag{4.9}$$

with the commutation relations $[c_m^i, c_n^{\dagger i*}] = [\tilde{c}_m^i, \tilde{c}_n^{\dagger i*}] = \delta_{m,n}$. In this notation, the $\dagger$ operator is meant to indicate negative frequency mode creation operators, and acts independently from the $*$ operation related to the complexification of the fields. The zero modes z^i exist only at fixed points of the orbifold, for which $z^i = gz^i$, and no momentum nor winding is possible. Similarly, the complex combinations $\chi^i, \chi^{i*} = (\psi^i \pm i\psi^{i+1})/\sqrt{2}$ of fermion fields have the following periodicities in the RR and NSNS sectors

$$\chi^i(\sigma + 1) = \pm e^{2\pi i\alpha} \chi^i(\sigma) \ , \ \ \chi^{i*}(\sigma + 1) = \pm e^{-2\pi i\alpha} \chi^{i*}(\sigma) \ , \tag{4.10}$$

with $\alpha = m/N$. The corresponding mode expansions are

$$\chi^i(z) = \sum_{n\geq 0}^{\infty} \chi_n^i e^{2\pi(n+\alpha)iz} + \sum_{n>0}^{\infty} \chi_n^{\dagger i} e^{-2\pi(n-\alpha)iz} \ , \tag{4.11}$$

$$\tilde{\chi}^i(\bar{z}) = \sum_{n>0}^{\infty} \tilde{\chi}_n^i e^{-2\pi(n-\alpha)i\bar{z}} + \sum_{n\geq 0}^{\infty} \tilde{\chi}_n^{\dagger i} e^{2\pi(n+\alpha)i\bar{z}} \ , \tag{4.12}$$

$$\chi^{i*}(z) = \sum_{n>0}^{\infty} \chi_n^{i*} e^{2\pi(n-\alpha)iz} + \sum_{n\geq 0}^{\infty} \chi_n^{\dagger i*} e^{-2\pi(n+\alpha)iz} \ , \tag{4.13}$$

$$\tilde{\chi}^{i*}(\bar{z}) = \sum_{n\geq 0}^{\infty} \tilde{\chi}_n^{i*} e^{-2\pi(n+\alpha)i\bar{z}} + \sum_{n>0}^{\infty} \tilde{\chi}_n^{\dagger i*} e^{2\pi(n-\alpha)i\bar{z}} \ , \tag{4.14}$$

with the anticommutation relations $\{\chi_m^i, \chi_n^{\dagger i*}\} = \{\tilde{\chi}_m^i, \tilde{\chi}_n^{\dagger i*}\} = \delta_{m,n}$, and n integer or half-integer in the RR and NSNS sectors. The total contribution to the Hamiltonian from a twisted pair of bosons and fermions is

$$H = \frac{p^2}{2} + 2\pi \left\{ \sum_{n=0}^{\infty} (n+\alpha) \left(c_n^{\dagger i *} c_n^i + \tilde{c}_n^{\dagger i *} \tilde{c}_n^i \right) + \sum_{n=1}^{\infty} (n-\alpha) \left(c_n^{\dagger i} c_n^{i*} + \tilde{c}_n^{\dagger i} \tilde{c}_n^{i*} \right) \right. \tag{4.15}$$

$$\left. + \sum_{n=0}^{\infty} (n+\alpha) \left(\chi_n^{\dagger i *} \chi_n^{i*} + \tilde{\chi}_n^{\dagger i} \tilde{\chi}_n^{i} \right) + \sum_{n=1}^{\infty} (n-\alpha) \left(\chi_n^{\dagger i} \chi_n^{i} + \tilde{\chi}_n^{\dagger i *} \tilde{\chi}_n^{i*} \right) - b(\alpha) \right\} ,$$

with n integer or half-integer for the fermions in the RR and NSNS sectors. The total normal constant $b(\alpha)$ ordering can be computed using the ζ-function regularization Eq. (2.62). One finds, for each twisted pair of fields which were initially periodic (P) or antiperiodic (A), the following result

$$b(\alpha) = \mp \begin{cases} \frac{1}{2} \left[-\frac{1}{6} + \alpha(1-\alpha) \right] & , \ \mathrm{P} \\ \frac{1}{2} \left(\frac{1}{12} - \alpha^2 \right) & , \ \mathrm{A} \end{cases} , \tag{4.16}$$

where the two overall signs refer to bosons and fermions respectively.

4.2 Interaction of point-like D-branes

We are interested in D-brane configurations which are point-like objects from the four-dimensional point of view. In particular, the study of their dynamics will allow a first important classification of their properties. Relying on simple extensions of the results of Appendix C, it is possible to recognize the contributions of the exchange of the various massless fields of the low-energy effective SUGRA from their dependence on the rapidity. Due to the simple block-diagonal ansatz for the metric of the target spacetime, the boundary state, as well as the partition function, will split into a contribution related to the universal four-dimensional Minkowski part $\mathbb{R}^{3,1}$ of spacetime, and an internal contribution related to the six-dimensional compact part $\mathcal{M}_6$ of spacetime which will depend on the compactification scheme and on which ten-dimensional D-brane has been wrapped on $\mathcal{M}_6$ and in which way. The ghost and superghost contributions are also universal, and will therefore be considered together with the universal Minkowski contribution.

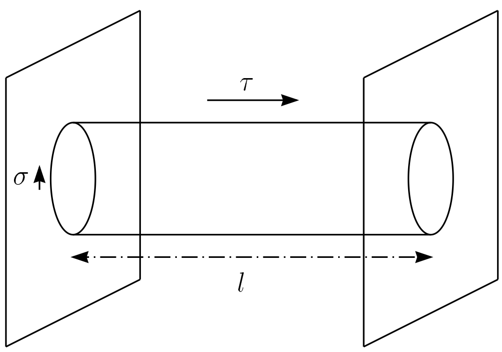


Figure 4.1: The cylinder amplitude.

As shown in Chapter 3, the interaction amplitude between two identical Dp-branes in relative motion is given by a cylindrical world-sheet as in Fig. 4.1 and reads

$$\mathcal{A} = \frac{\hat{T}_p^2}{2^4} \int_0^\infty dl \frac{1}{2} \sum_s (\pm) Z_s(l,\epsilon) \ , \tag{4.17}$$

in terms of the partition functions in the various spin-structures

$$Z_s(l,\epsilon) = \langle B_1, \epsilon_1, \vec{Y}_1 | e^{-lH} | B_2, \epsilon_1, \vec{Y}_2 \rangle_s \ . \tag{4.18}$$

Here $\vec{Y}_{1,2}$ are the positions in the (x^2, x^3) transverse plane and $v_{1,2} = \tanh \pi\epsilon_{1,2}$ the constant velocities in the x^1 direction. As discussed above, $Z_s(l,\epsilon)$ will split into a universal non-compact part $Z_s^{(nc)}(l,\epsilon)$, containing also the ghost and superghost contributions, and a compact part $Z_s^{(c)}(l)$. The details of the construction of the boundary state in each of the cases that we will consider, as well as the explicit computation of the partition functions entering the interaction amplitudes, are reported in Appendix D.

Consider first the universal Minkowski part. Since we are looking for point-like configurations in four dimensions, we impose N b.c. in the time direction and D b.c. in all the three space directions. As shown in Appendix D, the total bosonic and fermionic contributions to the partition function are

$$Z_B^{(nc)}(l,\epsilon) = 2 \frac{e^{-\frac{b^2}{2l}}}{(2\pi l)} \frac{\eta(2il)}{\vartheta_1(i\epsilon|2il)} \ , \tag{4.19}$$

$$Z_{Fs}^{(nc)}(l,\epsilon) = \frac{\vartheta_\alpha(i\epsilon|2il)}{\eta(2il)} \ , \tag{4.20}$$

with $\alpha = 2$ for s=R+ and $\alpha = 3,4$ for s=NS±. For s=R− the result vanishes.

Consider now the internal part. Due to the periodicity of the compact coordinates, the bosonic zero modes along these directions get drastically modified. Consider first the position zero mode x^i. For a D direction, the b.c. is a periodic δ-function, instead of a usual one. Decomposing in Fourier modes, this translate into the fact that the boundary state is now a discrete superposition of closed string states with zero winding but arbitrary momentum belonging to the momentum lattice Γ_6^*. Similarly, for a N direction, the b.c. allow now for a non-zero Wilson line, which is noting but the T-dual of the position. Correspondingly, the boundary state has to be a discrete superposition of closed string states with zero momentum but arbitrary winding belonging to the winding lattice Γ_6. For simplicity, we will neglect the role of Kaluza-Klein and winding modes, since their contribution will not be relevant for the following discussions. In interactions, this contribution is just an overall multiplicative correction to the partition function, which can be usually resummed to give a modular function, that we shall simply omit to write in order to avoid heavy expressions. In any case, neglecting these contributions is justified at energy corresponding to a length scale much above the typical size L of the compact part of spacetime, as well as its T-dual $\alpha'/$L. By doing so, the contribution of the bosonic z.m. in the compact directions amounts to the normalization $V_p^2/V_{\mathcal{M}_6}$. This factor is just what is needed to change the mass density T_p of the Dp-brane into the mass $M = T_p V_p$ of the corresponding four-dimensional point-like object, and the ten dimensional coupling $\kappa_{(10)}^2$ into the four-dimensional one, obtained by dividing by $V_{\mathcal{M}_6}$, $\kappa_{(4)}^2 = \kappa_{(10)}^2 / V_{\mathcal{M}_6}$. Summarizing, the ten-dimensional factor $V_p \hat{T}_p^2$ becomes in four dimensions $\hat{T}_p^2 V_p^2/V_{\mathcal{M}_6} = \hat{M}^2$. The amplitude stays dimensionless, since the propagator changes from $\Delta_{(9-p)}$ to $\Delta_{(2)}$. For orbifold twisted sectors, there is no momentum nor winding, and one finds the same factor as above. Notice that since $V_{T^6/\mathbb{Z}_N} = V_{T^6}/N$, $\hat{M}$ and the four-dimensional coupling $\kappa_{(4)}^2$ depend on the compactification scheme. For $\mathbb{Z}_N$ compactification, they are N times bigger than for toroidal compactification.

For orbifold compactifications, one has to project the boundary state onto its invariant part through the orbifold projector P. In the $\mathbb{Z}_N$ case, $P = 1/N(1 + g + ... + g^{N-1})$, and therefore one can first compute the partition function for a generic relative orbifold twist g^n and then average over all the elements of $\mathbb{Z}_N$. Twisted sectors have to be considered only at orbifold fixed-points. More precisely, twisting is consitent only with N,N or D,D b.c. in the corresponding plane, and not with N,D or D,N b.c. breaking rotation invariance. In the D,D case, the position Y of the D-brane must coincide with a fixed-point, whereas in the N,N case one should sum over all the fixed points in the world-volume of the D-brane (we will not consider this case). In each of the twisted sectors, if any, one has to project onto the $\mathbb{Z}_N$-invariant part. As in the untwisted sector, this can be done by first considering an arbitrary orbifold twist g^n and then averaging over all the elements of $\mathbb{Z}_N$.

We will consider in parallel compactifications with $\mathcal{M}_6 = T^6$, $T^2 \times T^4/\mathbb{Z}_2$ and $T^6/\mathbb{Z}_3$, by adopting the following strategy. We will first compute the contribution to the partition function of the fields along the compact directions, for arbitrary relative twists $e^{2\pi i w_a}$, $a = 4, 6, 8$ in all of the three pairs of compact coordinates. By appropriate choices, it will then be easy to specialize to the various cases of interest. For instance, the toroidal case is obtained simply by setting all the twists to zero, $w_4 = w_6 = w_8 = 0$. The $\mathbb{Z}_2$ case is obtained by setting the first twist to zero, $w_4 = 0$, and averaging over $w_6 = w_8 = 0, 1/2$. Finally, the $\mathbb{Z}_3$ orbifold case is obtained by averaging over $w_4 = w_6 = w_8 = 0, 1/3, 2/3$. Similarly, for the analysis of orbifold twisted sectors, on can consider a general situation with three twists α_a, $a = 4, 6, 8$ in all the three pairs of compact coordinates. For the $\mathbb{Z}_2$ orbifold, there is a single twisted sector with $\alpha_4 = 0$ and $\alpha_6 = \alpha_8 = 1/2$, whereas for the $\mathbb{Z}_3$ orbifold, there are two identical twisted sectors with $\alpha_4 = \alpha_6 = \alpha_8 = 1/3$ and $\alpha_4 = \alpha_6 = \alpha_8 = 2/3$.

Again, some details about the construction of the boundary state and the computation of the corresponding partition functions are reported in Appendix D.

4.2.1 Dimensionally reduced D0-branes

The first and most obvious way to obtain a point-like object in D=4 is to start with a point-like object in D=10, that is a D0-brane. We take therefore D b.c. in all the six compact directions. As explained above, if the D0-brane sits at an orbifold fixed-point, one has to consider also twisted sectors.

Untwisted sector

The total bosonic and fermionic contributions in the untwisted sector are

$$Z_B^{(c)}(l) = \frac{1}{V_{\mathcal{M}_6}} \frac{1}{\eta^6(2il)} \,, \tag{4.21}$$

$$Z_{sF}^{(c)}(l) = \frac{\vartheta_\alpha^3(0|2il)}{\eta^3(2il)} \,, \tag{4.22}$$

with $\alpha = 2$ for s=R+ and $\alpha = 3, 4$ for s=NS±.

Collecting Eqs. (4.19), (4.20), (4.21) and (4.22), the untwisted sector contribution to the interaction amplitude between two D0-branes is found to be

$$\mathcal{A} = \frac{\hat{M}^2}{2^4} \int_0^\infty \frac{dl}{2\pi l} e^{-\frac{b^2}{2l}} Z(l,\epsilon) \;, \tag{4.23}$$

with

$$Z(l,\epsilon) = \frac{\vartheta_3(i\epsilon|2il)\vartheta_3^3(0|2il) - \vartheta_2(i\epsilon|2il)\vartheta_2^3(0|2il) - \vartheta_4(i\epsilon|2il)\vartheta_4^3(0|2il)}{\vartheta_1(i\epsilon|2il)\eta^9(2il)} \;. \tag{4.24}$$

Notice that this is independent of the compactification scheme. Using the Riemann identity Eq. (A.21), it is easy to show that $\mathcal{A} \sim v^3$ for $\pi\epsilon \simeq v \ll 1$, as a consequence of supersymmetry. The result for the $\mathbb{Z}_2$ orbifold compactification is in agreement, in the orbifold limit, with the phase-shift for D0-branes on $\mathcal{K}_3$ computed in [138] in terms of characters of the internal N=4 superconformal field theory. In the large distance limit $b \gg l_s$, only world-sheets with $l \to \infty$ contribute to the amplitude. Since

$$Z(l,\epsilon) \underset{l\to\infty}{\longrightarrow} \frac{6 + 2\cosh 2\pi\epsilon - 8\cosh \pi\epsilon}{\sinh \pi\epsilon} \;, \tag{4.25}$$

one finds

$$\mathcal{A} \underset{b\gg l_s}{\longrightarrow} \hat{M}^2 \left(\frac{3}{4} + \frac{1}{4}\cosh 2\pi\epsilon - \cosh \pi\epsilon\right) \int_{-\infty}^{\infty} d\tau \Delta_{(3)}(r(\tau)) \;, \tag{4.26}$$

where $r(\tau) = \sqrt{b^2 + \sinh^2 \pi\epsilon\, \tau^2}$ is the true four-dimensional distance.

Twisted sectors

The total bosonic and fermionic contributions in a generic twisted sector are

$$Z_B^{(c)}(l,\alpha_a) = \frac{1}{V_{\mathcal{M}_6}} \prod_a \frac{\eta(2il)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}-\alpha_a \\ \frac{1}{2}\end{smallmatrix}\right](0|2il)} \;, \tag{4.27}$$

$$Z_{sF}^{(c)}(l) = \prod_a \frac{\vartheta\left[\begin{smallmatrix}a-\alpha_a \\ b\end{smallmatrix}\right](0|2il)}{\eta(2il)} \;, \tag{4.28}$$

with $a = 1/2$, $b = 0, 1/2$ for s=R±, and $a = 0$, $b = 0, 1/2$ for s=NS±.

Collecting Eqs. (4.19), (4.20), (4.27) and (4.28), the twisted sector contribution to the interaction amplitude between two D0-branes is found to be

$$\mathcal{A} = \frac{\hat{M}^2}{2^4} \int_0^\infty \frac{dl}{2\pi l} e^{-\frac{b^2}{2l}} Z(l,\epsilon) \;, \tag{4.29}$$

with

$$Z(l,\epsilon) = \begin{cases} \displaystyle\sum_{a,b=0}^{\frac{1}{2}} (-1)^{2(a+b)} \frac{\vartheta\left[\begin{smallmatrix}a \\ b\end{smallmatrix}\right](i\epsilon|2il)\vartheta\left[\begin{smallmatrix}a \\ b\end{smallmatrix}\right](0|2il)\vartheta^2\left[\begin{smallmatrix}a-\frac{1}{2} \\ b\end{smallmatrix}\right](0|2il)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2} \\ \frac{1}{2}\end{smallmatrix}\right](i\epsilon|2il)\vartheta^2\left[\begin{smallmatrix}0 \\ \frac{1}{2}\end{smallmatrix}\right](0|2il)\eta^3(2il)} & , \; T^2 \times T^4/\mathbb{Z}_2 \\ \displaystyle\sum_{a,b=0}^{\frac{1}{2}} (-1)^{2(a+b)} \frac{\vartheta\left[\begin{smallmatrix}a \\ b\end{smallmatrix}\right](i\epsilon|2il)\vartheta^3\left[\begin{smallmatrix}a-\frac{1}{3} \\ b\end{smallmatrix}\right](0|2il)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2} \\ \frac{1}{2}\end{smallmatrix}\right](i\epsilon|2il)\vartheta^3\left[\begin{smallmatrix}\frac{1}{6} \\ \frac{1}{2}\end{smallmatrix}\right](0|2il)\eta^3(2il)} & , \; T^6/\mathbb{Z}_3 \end{cases} \;. \tag{4.30}$$

The odd spin-structure a=b=1 never contributes, because of the fermionic zero modes in the transverse plane. Using the Riemann identity Eq. (A.7), it is easy to show that $\mathcal{A} \sim v$ for $\pi\epsilon \simeq v \ll 1$, as a consequence of supersymmetry. The fact that one has $\mathcal{A} \sim v$ rather than $\mathcal{A} \sim v^3$ exhibits the fact that orbifold compactifications lead to low-energy effective theories with less supersymmetry with respect to toroidal compactification. In the large distance limit $b \gg l_s$, only world-sheets with $l \to \infty$ contribute to the amplitude. One finds

$$Z(l,\epsilon) \underset{l\to\infty}{\longrightarrow} \begin{cases} \dfrac{4 - 4\cosh \pi\epsilon}{\sinh \pi\epsilon} & , \; T^2 \times T^4/\mathbb{Z}_2 \\ \dfrac{2 - 2\cosh \pi\epsilon}{\sinh \pi\epsilon} & , \; T^6/\mathbb{Z}_3 \end{cases} \, , \tag{4.31}$$

and since there is only one twisted sector in the $\mathbb{Z}_2$ case and two identical ones in the $\mathbb{Z}_3$ case, one has in total

$$\mathcal{A} \underset{b\gg l_s}{\longrightarrow} \frac{\hat{M}^2}{4} (1 - \cosh \pi\epsilon) \int_{-\infty}^{\infty} d\tau \Delta_{(3)}(r(\tau)) \, , \tag{4.32}$$

both in the $\mathbb{Z}_2$ and the $\mathbb{Z}_3$ cases.

4.2.2 Wrapped D3-branes

Another way to obtain a point-like object in D=4 is to start from a p-extended object in D=10, a Dp-brane, and wrap it on some p-cycle of $\mathcal{M}_6$. We shall consider the particular case of a D3-brane. This is achieved by taking N b.c. in three of the six compact directions, say x^a, $a = 4,6,8$ and D in the other three, x^{a+1}, $a = 4,6,8$. Each of the three complex combinations Z^a of compact coordinates has therefore mixed boundary conditions, in the sense that they correspond to a NN,DD plane, and will be sensitive to orbifold rotations.

Untwisted sector

The total bosonic and fermionic contributions in the untwisted sector are

$$Z_B^{(c)}(l,w_a) = \frac{V_3^2}{V_{\mathcal{M}_6}} \eta^3(2il) \prod_a \frac{2 \sin 2\pi w_a}{\vartheta_1(2w_a|2il)} \, , \tag{4.33}$$

$$Z_{sF}^{(c)}(l,w_a) = \frac{\vartheta_\alpha^3(2w_a|2il)}{\eta^3(2il)} \, , \tag{4.34}$$

with $\alpha = 1,2$ for s=R$\pm$ and $\alpha = 3,4$ for s=NS$\pm$.

Collecting Eqs. (4.19), (4.20), (4.33) and (4.34), the untwisted sector contribution to the interaction amplitude between two wrapped D3-branes is found to be

$$\mathcal{A} = \frac{\hat{M}^2}{2^4} \int_0^\infty \frac{dl}{2\pi l} e^{-\frac{b^2}{2l}} \frac{1}{N} \sum_{\{w_a\}} Z(l,\epsilon,w_a) \, , \tag{4.35}$$

with

$$Z(l,\epsilon,w_a) = \sum_\alpha (-1)^{1+\alpha} \frac{\vartheta_\alpha(i\epsilon|2il) \prod_a \vartheta_\alpha(2w_a|2il)}{\vartheta_1(i\epsilon|2il) \prod_a \vartheta_1(2w_a|2il)} \prod_a (2 \sin 2\pi w_a) \, . \tag{4.36}$$

Using the Riemann identity Eq. (A.21), it is easy to show that each contribution to the amplitude at fixed relative twist w_a vanishes at least like v for $\pi\epsilon \simeq v \ll 1$. In order this to be true, it is crucial that $\pm w_4 \pm w_6 \pm w_8 = n$ for some combination of signs. This is precisely the condition that one has to impose on the orbifold action in order that at least a couple of supercharges survive the orbifold projection [15, 16]. Averaging over the allowed twists, one finds $\mathcal{A} \sim v^3$ for toroidal and $\mathbb{Z}_2$ orbifold compactification, but $\mathcal{A} \sim v$ for $\mathbb{Z}_3$ orbifold compactification, in the limit $\pi\epsilon \simeq v \ll 1$. In the large distance limit $b \gg l_s$, only world-sheets with $l \to \infty$ contribute to the amplitude. For a fixed orbifold relative twist, one finds

$$Z(l,\epsilon,w_a) \underset{l\to\infty}{\longrightarrow} \frac{2\sum_a \cos 4\pi w_a + 2\cosh 2\pi\epsilon - 8\prod_a \cos 2\pi w_a \cosh \pi\epsilon}{\sinh \pi\epsilon} \; . \tag{4.37}$$

Averaging over $w_4 = 0$, $w_6 = w_8 = 0, 1/2$ in the $\mathbb{Z}_2$ case, and over $w_4 = w_6 = w_8 = 0, 1/3, 2/3$ in the $\mathbb{Z}_3$ case, one finds finally

$$\mathcal{A} \underset{b\gg l_s}{\longrightarrow} \begin{cases} \hat{M}^2 \left(\frac{3}{4} + \frac{1}{4}\cosh 2\pi\epsilon - \cosh \pi\epsilon\right) \int_{-\infty}^{\infty} d\tau \Delta_{(3)}(r(\tau)) & , \; T^6 \; , \; T^2 \times T^4/\mathbb{Z}_2 \\ \frac{\hat{M}^2}{4} \left(\cosh 2\pi\epsilon - \cosh \pi\epsilon\right) \int_{-\infty}^{\infty} d\tau \Delta_{(3)}(r(\tau)) & , \; T^6/\mathbb{Z}_3 \end{cases} . \tag{4.38}$$

Twisted sectors

As discussed above, there is no contribution from orbifold twisted sectors. This means that there is no coupling to twisted closed string states.

4.2.3 Non-relativistic behavior versus supersymmetry

We have seen in Chapter 2 that the non-relativistic behavior of the phase-shift for two D-branes is intimately related to the supersymmetry preserved by the composite configuration. In the following, we will discuss the potential $V(r)$, rather than the phase-shift $\mathcal{A}$ which is its integral over the trajectory,

$$\mathcal{A} = \int_{-\infty}^{\infty} d\tau V(r(\tau)) \; . \tag{4.39}$$

Recalling the results of Chapter 2, we see that $V \sim v^4$ for the Dp-Dp system preserving 16 supersymmetries and $V \sim v^2$ for the Dp-D(p+4) system, preserving 8 supersymmetries. Be a reasoning analogous to that discussed in detail in Chapter 6, one can show that the potential for a system preserving 2n supersymmetries, corresponding to n Green-Schwarz fermionic z.m. on the cylinder, vanishes at least like $V \sim v^{n/2}$. Indeed, in the open string channel, the vanishing of the static potential is due the path-integral over these z.m., representing the fermionic degeneracy associated to the residual supersymmetry. In order to get a non-vanishing result, one has to perturb with some background breaking supersymmetry, which in the string conformal field theory will correspond to an interaction term involving fermionic z.m.. The leading behavior in the perturbation parameter is obtained by bringing down from the exponential of the interaction in the path-integral the appropriate power of the interaction needed to provide the n fermionic z.m. to be soaked. A constant velocity corresponds to an electromagnetic-like interaction involving at most two fermionic z.m., and therefore the path-integral will be non-vanishing at order $v^{n/2}$ as stated. The results for the Dp-Dp and Dp-D(p+4) systems are recovered with n equal to 8 and 4. This general result corresponds to the minimal cancellation imposed by supersymmetry, but one can have additional accidental cancellations occurring in some special cases. For example, the potential is even in the velocity by parity symmetry, and therefore when the exponent $n/2$ is odd, it jumps automatically to $n/2+1$ which is then even.

For the point-like configurations analyzed in this section, the situation is the following. Since we always consider the interaction between identical D-branes, the corresponding composite system preserve 1/2 of the relevant four-dimensional theory, which has a number of supersymmetries which depends on the compactifications scheme. For compactification on T^6, $T^2 \times T^4/\mathbb{Z}_2$ and $T^6/\mathbb{Z}_3$, one has 32, 16 and 8 supersymmetries respectively. According to the discussion above, n is 8, 4 and 2 in the three cases and the potential should vanish at least as v^4, v^2 and v^2 respectively. Indeed, one finds the following behaviors.

Dimensionally reduced D0-branes

$$V^{(untw.)} \sim v^4 \ , \quad V^{(tw.)} \sim v^2 \ . \tag{4.40}$$

Wrapped D3-branes

$$V^{(untw.)} \sim \left\{ \begin{array}{l} v^4 \ , \quad T^6 \ , \quad T^2 \times T^4/\mathbb{Z}_2 \\ v^2 \ , \quad T^6/\mathbb{Z}_3 \end{array} \right. \quad , \quad V^{(tw.)} = 0 \ . \tag{4.41}$$

4.2.4 Field theory interpretation

In order to give a field theory interpretation of the interaction amplitudes that we have obtained, we will need a generalization to four dimensions of the results of Appendix C. For later convenience, we work in momentum space. The kinematics for Feynman diagrams is the following. In the eikonal approximation, the momenta of the two point-like D-branes are (setting their mass to 1)

$$B^\mu_{1,2} = \Big(\cosh \pi\epsilon_{1,2}, \sinh \pi\epsilon_{1,2}, \vec{0}\Big) \ . \tag{4.42}$$

In the eikonal approximation, these D-branes can emit the momenta

$$k^\mu(\epsilon_1) = \Big(\sinh \pi\epsilon_1\, k^1, \cosh \pi\epsilon_1\, k^1, \vec{k}_T\Big) \ , \tag{4.43}$$

$$q^\mu(\epsilon_2) = \Big(\sinh \pi\epsilon_2\, q^1, \cosh \pi\epsilon_2\, q^1, \vec{q}_T\Big) \ , \tag{4.44}$$

transverse to themselves, $k \cdot B_1 = q \cdot B_2 = 0$. Momentum conservation requires $k^\mu = q^\mu$, implying $k^1 = q^1 = 0$ and $\vec{k}_T = \vec{q}_T$.

Since they are point-like, the two D-branes can exchange scalars, vectors and gravitons. The corresponding sources are, neglecting corrections due to the small momentum transfer,

$$S_{1,2} = \hat{a} \ , \quad J^\mu_{1,2} = \hat{e} B^\mu_{1,2} \ , \quad T^{\mu\nu}_{1,2} = \hat{m} B^\mu_{1,2} B^\nu_{1,2} \ . \tag{4.45}$$

They are conserved, thanks to the property $k \cdot B_1 = q \cdot B_2 = 0$. Using the propagators of Appendix C, the scalar, vector and gravitational fields generated by the D-brane are found to be

$$\begin{aligned} \phi_1 &= \frac{\hat{a}}{k^2} \; , \;\; A_1^\mu = \frac{\hat{e}}{k^2} J_1^\mu \; , \;\; h_1^{\mu\nu} = -\frac{\hat{m}}{k^2} \left(T_1^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} T_1 \right) \; , \\ \phi_2 &= \frac{\hat{a}}{q^2} \; , \;\; A_2^\mu = \frac{\hat{e}}{q^2} J_2^\mu \; , \;\; h_2^{\mu\nu} = -\frac{\hat{m}}{q^2} \left(T_2^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} T_2 \right) \; . \end{aligned} \tag{4.46}$$

The phase-shift is obtained by introducing the fields emitted by one of the D-brane, say the first, in the effective lagrangian describing the coupling of the other brane, say the second. The contribution from scalar exchange is encoded in

$$\mathcal{L}_{(\phi)} = \phi S \; , \tag{4.47}$$

and one finds

$$\mathcal{A}_{(\phi)} = \hat{a}^2 \int_{-\infty}^{\infty} d\tau \Delta_{(3)}(r(\tau)) \; . \tag{4.48}$$

Similarly, the contribution from vector exchange is encoded in

$$\mathcal{L}_{(\mathcal{A})} = -A_\mu J^\mu \; , \tag{4.49}$$

and one finds

$$\mathcal{A}_{(A)} = -\hat{e}^2 \cosh \pi\epsilon \int_{-\infty}^{\infty} d\tau \Delta_{(3)}(r(\tau)) \; . \tag{4.50}$$

Finally, the contribution from vector exchange is encoded in

$$\mathcal{L}_{(h)} = -\frac{1}{2} h_{\mu\nu} T^{\mu\nu} \; , \tag{4.51}$$

and one finds

$$\mathcal{A}_{(h)} = \frac{\hat{m}^2}{4} \cosh 2\pi\epsilon \int_{-\infty}^{\infty} d\tau \Delta_{(3)}(r(\tau)) \; . \tag{4.52}$$

Therefore, the phase-shift between two moving point-like objects coupling with charge $\hat{a}$ to a scalar, charge $\hat{e}$ to a vector and mass $\hat{m}$ to the graviton, is

$$\mathcal{A} = \left(\hat{a}^2 + \frac{\hat{m}^2}{4} \cosh 2\pi\epsilon - \hat{e}^2 \cosh \pi\epsilon \right) \int_{-\infty}^{\infty} d\tau \Delta_{(3)}(r(\tau)) \; . \tag{4.53}$$

Notice that the scalar, vector and graviton exchange give contributions proportional to 1, $\cosh \pi\epsilon$ and $\cosh 2\pi\epsilon$. This peculiar dependence on the rapidity allows to recognize unambiguously which kind of particles are exchanged between the two point-like objects in the various constructions of the previous section. Comparing with the large distance limit of the phase-shifts computed above, one can determine the coupling $\hat{a}$, $\hat{e}$ and $\hat{m}$ in each case. One finds the following results.

Dimensionally reduced D0-brane

In the untwisted sector, we recognize the exchange of scalars, vectors and gravitons, with couplings

$$\hat{a} = \frac{\sqrt{3}}{2} \hat{M} \; , \;\; \hat{e} = \hat{M} \; , \;\; \hat{m} = \hat{M} \; . \tag{4.54}$$

This is interpreted as the coupling to the gravitational multiplet and possibly vector multiplets, of the relevant Type IIA SUGRA compactification. In the twisted sector instead, only scalars and vectors are exchanged, with couplings

$$\hat{a} = \frac{1}{2}\hat{M} \ , \quad \hat{e} = \frac{1}{2}\hat{M} \ . \tag{4.55}$$

This corresponds to the coupling to the additional vector multiplets arising from twisted closed string states.

From the SUGRA point of view, this configuration should correspond to a vertical dimensional reduction [51] of the ten-dimensional 0-brane solution down to four dimensions, which is singular because of its coupling to the dilaton. The coupling to scalars is indicative for a singular solution of the corresponding N=2, 4 and 8 SUGRAs with no horizon and zero entropy.

Wrapped D3-brane

In the untwisted sector, we recognize the exchange of vectors and gravitons for all the compactification schemes. The scalar exchange seems instead to be absent for the $\mathbb{Z}_3$ compactification. This is interpreted as the coupling to the gravitational multiplet of the relevant Type IIB SUGRA only, which has scalars only for N=4 and N=8 supersymmetry corresponding to toroidal and $\mathbb{Z}_2$ compactification, but not in the N=2 case corresponding to the $\mathbb{Z}_3$ compactification. The couplings are

$$\begin{cases} \hat{a} = \frac{\sqrt{3}}{2}\hat{M} \ , \quad \hat{e} = \hat{M} \ , \quad \hat{m} = \hat{M} \ , \quad T^6 \ , \quad T^2 \times T^4/\mathbb{Z}_2 \\ \hat{a} = 0 \ , \quad \hat{e} = \frac{1}{2}\hat{M} \ , \quad \hat{m} = \hat{M} \qquad , \quad T^6/\mathbb{Z}_3 \end{cases} . \tag{4.56}$$

The only vector multiplets arising in these Type IIB compactifications come from the twisted sectors. Since in this sector the amplitude is zero, we conclude that our configuration does not couple to them.

From the SUGRA point of view, this configuration should correspond to a diagonal dimensional reduction [51] of the ten-dimensional 3-brane solution down to four dimensions, which is non-singular because of the absence of coupling to the dilaton. The absence of any coupling to scalars in the $\mathbb{Z}_3$ case is suggestive for a regular N=2 charged Reissner-Nordström (R-N) black hole solution with a horizon and a non-vanishing entropy, whereas the presence of couplings to scalars for toroidal and $\mathbb{Z}_2$ compactifications suggests singular solutions of the corresponding N=4 and 8 SUGRAs with no horizon and zero entropy.

The four dimensional SUGRA solutions corresponding to the dimensionally reduced D0-brane and the wrapped D3-brane can be obtained as particular cases of a more general one. Consider indeed a generic four-dimensional action of the following type

$$\mathcal{S} = \frac{1}{2\kappa_{(4)}^2}\int d^4x\sqrt{g}\left(R - \frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2\cdot 2!}e^{-b\phi}F_{(2)}^2\right) \ . \tag{4.57}$$

Taking $b \neq 0$ corresponds to a theory with a non-linearly coupled vector field, which should be relevant for the dimensionally reduced D0-brane, whereas taking $b = 0$ corresponds to a truncated theory with a decoupled vector field, which should be relevant for the wrapped D3-brane. The general electric extremal solution of this action is [49, 50, 51]

$$\phi = \alpha \ln H(r) \ , \quad ds^2 = -H(r)^{-\frac{\beta}{2}} dt^2 + H(r)^{\frac{\beta}{2}} d\vec{x} \cdot d\vec{x} \ , \quad A_0 = \gamma \left(H(r)^{-1} - 1 \right) \ . \tag{4.58}$$

where

$$\alpha = \frac{2b}{1+b^2} \ , \quad \beta = \frac{4}{1+b^2} \ , \quad \gamma = \frac{2}{\sqrt{1+b^2}} \ . \tag{4.59}$$

and satisfy $\alpha^2 + \frac{\beta^2}{4} - \gamma^2 = 0$ as a consequence of BPS saturation. $H(r)$ satisfies the three-dimensional Laplace equation, being therefore of the form $H(r) = 1 + 2\kappa_{(4)} N \Delta_3(r)$, with arbitrary N. The relevant asymptotic long range fields are

$$\begin{cases} \phi = 2\kappa_{(4)}^2 \, a \, \Delta_3(r) \\ h_{00} = \kappa_{(4)}^2 \, m \, \Delta_3(r) \ , \quad h_{ij} = \kappa_{(4)}^2 \, m \, \delta_{ij} \Delta_3(r) \\ A_0 = 2\kappa_{(4)}^2 \, e \, \Delta_3(r) \end{cases} \tag{4.60}$$

with

$$a = \alpha M \ , \quad m = \beta M \ , \quad e = \gamma M \ . \tag{4.61}$$

The corresponding hatted quantities are as usual defined by multiplying the non-hatted ones by $\sqrt{2}\kappa_{(4)}$. The case of the D0-brane is obtained by taking $b = \sqrt{3}$ and $N = M$, which leads to $\hat{a} = \sqrt{3}/2 \, \hat{M}$, $\hat{m} = \hat{M}$ and $\hat{e} = \hat{M}$. The case of the D3-brane is instead obtained by taking $b = 0$ and $N = M/4$, leading to $\hat{a} = 0$, $\hat{m} = \hat{M}$ and $\hat{e} = \hat{M}/2$. A crucial difference between the two, responsible for the absence of horizon and the vanishing entropy for the former, and the finite horizon and entropy for the latter, is the power of the harmonic function in the metric, $\pm 1/2$ and ± 2 respectively.

4.3 Closed string radiation

The amplitude for the emission of a closed string state from two interacting point-like D-brane configurations is described by a one-point function on the cylinder joining the two D-brane configurations, as depicted in Fig. 4.2. In the boundary state formalism, this is

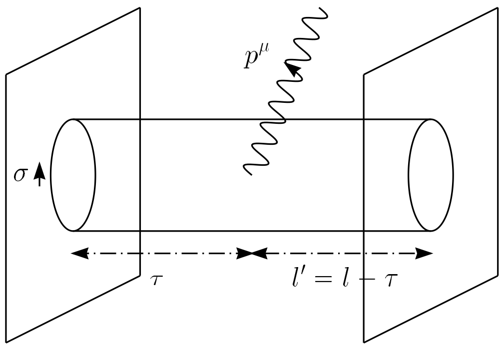


Figure 4.2: The cylinder emission amplitude.

given by the expectation value of the appropriate vertex operator between the two boundary states describing the D-branes. The amplitude is indeed

$$\mathcal{A} = \frac{\hat{T}_p^2}{2^4} \int_0^\infty dl \int dz \int d\bar{z} \frac{1}{2} \sum_s (\pm) \langle\langle V(z,\bar{z}) \rangle\rangle_s \ , \tag{4.62}$$

where the symbol $\langle\langle ... \rangle\rangle_s$ denotes the non-connected correlation function obtained as

$$\langle\langle \mathcal{O} \rangle\rangle_s = \langle B, \epsilon_1, Y_1 | e^{-lH} \mathcal{O} | B, \epsilon_2, Y_2 \rangle_s \ , \tag{4.63}$$

whereas the symbol $\sum_s(\pm)$ represents the sum over the spin-structures with the appropriate signs. The integral over the position of the vertex operator is $\int dz \int d\bar{z} = \int_0^1 d\sigma \int_0^l d\tau$. On symmetry grounds, the correlation does not depend on σ, and therefore on can omit the corresponding integration.

We shall consider the emission of massless NSNS states with momentum p^μ in four-dimensional part of spacetime. The polarization can be either along the four non-compact directions, $\xi_{\mu\nu}$, representing four-dimensional axions, dilatons and gravitons, or in the six compact directions, ξ_{ab}, representing four-dimensional scalars arising upon compactification. In the following, we shall concentrate on the former case, the latter being a straightforward generalization that we shall not discuss. The vertex operator for a massless NSNS state with four-dimensional momentum p^μ and polarization $\xi_{\mu\nu}$, with μ running from 0 to 3, can be taken to be

$$V = \xi_{ij} (\partial X^i - \frac{1}{2} p \cdot \psi \psi^i)(\bar{\partial} X^j + \frac{1}{2} p \cdot \bar{\psi} \bar{\psi}^j) e^{ip\cdot X} \ , \tag{4.64}$$

with $z = \sigma + i\tau$ and $\partial = \partial_z$. Exploiting gauge invariance, we have chosen purely space-like transverse polarizations ξ_{ij}, satisfying $p^i \xi_{ij} = 0$. The four-dimensional axion (a), dilaton (ϕ) and graviton (h) are described by the following polarizations

$$\begin{aligned} \xi_{ij}^{(a)} &= \frac{1}{2} \epsilon_{ijk} \frac{p^k}{|\vec{p}|} \ , \quad \xi_{ij}^{(\phi)} = \delta_{ij} - \frac{p^i p^j}{\vec{p}^2} \\ \xi_{ij}^{(h)} &= h_{ij} \ , \quad h_{ij} = h_{ji} \ , \quad h^i{}_i = 0 \ . \end{aligned} \tag{4.65}$$

The bosonic and fermionic contributions to the correlation always factorize, as they do for the boundary state. Consequently, the correlation that we have to evaluate reads

$$\begin{aligned} \langle\langle V \rangle\rangle_s = \xi_{ij} \Big\{ & \langle\langle \partial X^i \bar{\partial} X^j e^{ip\cdot X} \rangle\rangle + \frac{1}{2} \Big[\langle\langle \partial X^i e^{ip\cdot X} \rangle\rangle \langle\langle p \cdot \bar{\psi} \bar{\psi}^j \rangle\rangle_s - \langle\langle \bar{\partial} X^j e^{ip\cdot X} \rangle\rangle \langle\langle p \cdot \psi \psi^i \rangle\rangle_s \Big] \\ & - \frac{1}{4} \langle\langle p \cdot \psi \psi^i p \cdot \bar{\psi} \bar{\psi}^j \rangle\rangle_s \Big\} \ . \end{aligned} \tag{4.66}$$

In order to evaluate this matrix element, it will be convenient to treat separately the bosonic z.m. part, which will also fix the kinematics. Therefore, we split the bosonic fields as $X^\mu = X_0^\mu + X_{osc}^\mu$, and correspondingly factorize the bosonic part of the boundary state as $|B\rangle_B = |B_0\rangle_B \otimes |B_{osc}\rangle_B$. Recall that $X_0^\mu = x^\mu - i\tau p^\mu$, $\partial X_0^\mu = -p^\mu/2$ and $\bar{\partial} X_0^\mu = p^\mu/2$, in terms of the center of mass position and momentum operators x^μ and p^μ. The four-dimensional momentum content of the two boundary states is

$$\begin{aligned} k^\mu(\epsilon_1) &= \Big(\sinh \pi\epsilon_1 \, k^1, \cosh \pi\epsilon_1 \, k^1, \vec{k}_T\Big) \ , \\ q^\mu(\epsilon_2) &= \Big(\sinh \pi\epsilon_2 \, q^1, \cosh \pi\epsilon_2 \, q^1, \vec{q}_T\Big) \ . \end{aligned} \tag{4.67}$$

whereas the four-momentum p^μ of the emitted massless particle can be parameterized as

$$p^\mu = (p, \cos\theta\, p, \vec{p}_T = \vec{n}\sin\theta\, p) \ . \tag{4.68}$$

Acting on the second boundary state, the exponential wave function factor of the vertex operator shifts the momentum $q^\mu(\epsilon_2)$ by p^μ, due to the position operator. The momentum conservation δ-function is therefore

$$\begin{aligned}\langle k(\epsilon_1)^\mu | (p+q(\epsilon_2))^\mu\rangle &= (2\pi)^4\delta^{(4)}\left(p^\mu - k(\epsilon_2)^\mu + q(\epsilon_1)^\mu\right) \\ &= \frac{(2\pi)^4}{\sinh\pi\epsilon}\delta\left(k_1 - \frac{p^{(2)}}{\sinh\pi\epsilon}\right)\delta\left(q_1 - \frac{p^{(1)}}{\sinh\pi\epsilon}\right)\delta^{(2)}\left(\vec{p}_T - \vec{k}_T + \vec{q}_T\right) \ . \end{aligned} \tag{4.69}$$

The quantities $p^{(1,2)}$ are the energies of the outgoing particle in the rest frame of the first and the second D-brane respectively

$$p^{(1,2)} = (\cosh\pi\epsilon_{1,2} - \sin\pi\epsilon_{1,2}\cos\theta)\, p \ . \tag{4.70}$$

Eq. (4.69) implies a very particular and restricted kinematics. Indeed, the energies and longitudinal momenta carried by the two boundary states is completely fixed by the momentum of the emitted particle. With reference to Eqs. (4.67)

$$\begin{aligned} k^0 &= \frac{v_1}{v_1 - v_2}(1 - v_2\cos\theta)\, p \ , \quad q^0 = \frac{v_2}{v_1 - v_2}(1 - v_1\cos\theta)\, p \ , \\ k^1 &= \frac{1}{v_1 - v_2}(1 - v_2\cos\theta)\, p \ , \quad q^1 = \frac{1}{v_1 - v_2}(1 - v_1\cos\theta)\, p \ , \end{aligned} \tag{4.71}$$

whereas the transverse momenta are subject to the usual momentum conservation

$$\vec{k}_T - \vec{q}_T = \vec{p}_T \ . \tag{4.72}$$

The z.m. part of the exponential of the Hamiltonian acting on the first boundary state gives an exponential momentum factor, $\langle k^\mu(\epsilon_1)| e^{-lH_0} = \exp\{-lk^2(\epsilon_1)/2\}\langle k^\mu(\epsilon_1)|$. Similarly, acting on the second boundary state, the z.m. part of the exponential wave function factor in the vertex operator gives an additional exponential factor due to the momentum operator, $\exp\{ip\cdot X_0\}|q^\mu(\epsilon_2)\rangle = \exp\{\tau p\cdot q(\epsilon_2)\}|q^\mu(\epsilon_2) + p^\mu\rangle$. Finally, ∂X_0^μ or/and $\bar\partial X_0^\mu$ insertions give, acting for example on the first boundary state, simply additional momentum factors, $\langle k^\mu(\epsilon_1)|\partial X_0^\mu = -k^\mu(\epsilon_1)/2\langle k^\mu(\epsilon_1)|$, $\langle k^\mu(\epsilon_1)|\bar\partial X_0^\mu = k^\mu(\epsilon_1)/2\langle k^\mu(\epsilon_1)|$. It is very convenient to transform the modular integral over l and the vertex position integral over τ into integrals over the proper times τ and $l' = l - \tau$ of the closed strings emitted by the two D-branes. Doing so, one has

$$\int_0^\infty dl \int_0^l d\tau = \int_0^\infty d\tau \int_0^\infty dl' \ . \tag{4.73}$$

$l' = 0$, $\tau = l$ corresponds to the boundary attached to the first brane, whereas $\tau = 0$, $l' = l$ corresponds to the boundary attached to the second brane. In the new variables, the exponential factor coming from the bosonic z.m. becomes $\exp\{-\tau q^2(\epsilon_2)/2 - l'k^2(\epsilon_1)/2\}$, where

$$k^2(\epsilon_1) = \vec{k}_T^2 + \frac{p^{(2)2}}{\sinh^2\pi\epsilon} \ , \quad q^2(\epsilon_2) = (\vec{k}_T - \vec{p}_T)^2 + \frac{p^{(1)2}}{\sinh^2\pi\epsilon} \ . \tag{4.74}$$

From now on, we will abbreviate $k^\mu(\epsilon_1) = k^\mu$ and $q^\mu(\epsilon_2) = q^\mu$. Using Eq. (4.69) and carrying out the momentum integrations, the z.m. contributions of the various bosonic correlations entering the amplitude are found to be, in terms of the proper times τ and l',

$$\langle\langle(\partial X^\mu)^m(\bar\partial X^\nu)^n e^{ip\cdot X}\rangle\rangle_0 = \frac{1}{\sinh\pi\epsilon}\int\frac{d^2\vec k_T}{(2\pi)^2}e^{i\vec k_T\cdot\vec b}e^{-\frac{q^2}{2}\tau}e^{-\frac{k^2}{2}l'}\left(-\frac{k^\mu}{2}\right)^m\left(\frac{k^\nu}{2}\right)^n \;, \quad (4.75)$$

with $m,n = 0,1$. Neglecting Kaluza-Klein and winding modes, as in previous section, the compact bosonic z.m. contribute only a factor $V_p^2/V_{\mathcal{M}_6}$, turning $\hat T_p^2$ into $\hat M^2$.

The other bosonic correlations, involving the remaining oscillator part of the fields, as well as the fermionic correlation functions, have to be computed explicitly. Actually, it is convenient to work with connected Green functions, indicated as $\langle...\rangle$, rather than the non-connected correlations $\langle\langle...\rangle\rangle$. These are defined by factorizing the partition functions

$$Z_s(l) = \langle B,\epsilon_1|e^{-lH}|B,\epsilon_2\rangle_s \;, \quad (4.76)$$

as

$$\langle\mathcal{O}\rangle = \frac{\langle B,\epsilon_1|e^{-lH}\mathcal{O}|B,\epsilon_2\rangle_s}{\langle B,\epsilon_1|e^{-lH}|B,\epsilon_2\rangle_s} \;. \quad (4.77)$$

With these definitions, one has simply $\langle\langle\mathcal{O}\rangle\rangle = \langle\mathcal{O}\rangle Z_s$. Actually, there is a subtlety in the odd spin-structure, where the partition function can vanish because of the fermionic zero modes. To cope with this difficulty, it will be enough to first select operators with enough fermion fields to give a non-vanishing non-connected correlation $\langle\langle...\rangle\rangle$, and define the corresponding connected correlation $\langle...\rangle$ by factorizing the partition function with all the dangerous fermionic z.m. inserted, in order to have a non-vanishing result. Since we have already taken into account the bosonic zero modes, with partition function we will now mean the contributions of the bosonic oscillators and the fermions

$$Z^B_{osc} = \langle B,\epsilon_1|e^{-lH}|B,\epsilon_2\rangle^B_{osc} \;, \quad (4.78)$$

$$Z^F_s = \langle B,\epsilon_1|e^{-lH}|B,\epsilon_2\rangle^F_s \;. \quad (4.79)$$

Furthermore, one can use Wick's theorem to reduce all the correlations to the following connected two-point functions

$$\langle X^\mu X^\nu\rangle_{osc} = \frac{\langle B,\epsilon_1|e^{-lH}X^\mu X^\nu|B,\epsilon_2\rangle^B_{osc}}{\langle B,\epsilon_1|e^{-lH}|B,\epsilon_2\rangle^B_{osc}} \;, \quad (4.80)$$

$$\langle\psi^\mu\psi^\nu\rangle_s = \frac{\langle B,\epsilon_1|e^{-lH}\psi^\mu\psi^\nu|B,\epsilon_2\rangle^F_s}{\langle B,\epsilon_1|e^{-lH}|B,\epsilon_2\rangle^F_s} \;. \quad (4.81)$$

In the odd spin-structure, the insertion of the fermionic z.m. is understood both in the numerator and in the denominator, as well as in the partition functions.

Finally, the general amplitude is written as

$$\mathcal{A} = \frac{\hat M^2}{4\sinh\pi\epsilon}\int_0^\infty d\tau\int_0^\infty dl'\int\frac{d^2\vec k_T}{(2\pi)^2}e^{i\vec k_T\cdot\vec b}e^{-\frac{q^2}{2}\tau}e^{-\frac{k^2}{2}l'}\langle e^{ip\cdot X}\rangle_{osc}\mathcal{N} \quad (4.82)$$

where

$$\mathcal{N} = \frac{1}{4}Z^B_{osc}\sum_s Z^F_s\mathcal{M}_s \;, \quad (4.83)$$

and

$$\mathcal{M}_s = \xi_{ij}\left\{\langle\partial X^i\bar\partial X^j\rangle_{osc} - \langle\partial X^i p\cdot X\rangle_{osc}\langle\bar\partial X^j p\cdot X\rangle_{osc}\right.$$
$$+\frac{1}{4}\left(\langle p\cdot\psi p\cdot\bar\psi\rangle_s\langle\psi^i\bar\psi^j\rangle_s - \langle p\cdot\psi\psi^i\rangle_s\langle p\cdot\bar\psi\bar\psi^j\rangle_s + \langle p\cdot\bar\psi\psi^i\rangle_s\langle p\cdot\psi\bar\psi^j\rangle_s\right)$$
$$+\frac{i}{2}\left(\langle\partial X^i p\cdot X\rangle_{osc}\langle p\cdot\bar\psi\bar\psi^j\rangle_s - \langle\bar\partial X^j p\cdot X\rangle_{osc}\langle p\cdot\psi\psi^i\rangle_s\right)$$
$$-\frac{1}{2}k^i\left(i\langle\bar\partial X^j p\cdot X\rangle_{osc} + \frac{1}{2}\langle p\cdot\bar\psi\bar\psi^j\rangle_s\right) + \frac{1}{2}k^j\left(i\langle\partial X^i p\cdot X\rangle_{osc} - \frac{1}{2}\langle p\cdot\psi\psi^i\rangle_s\right)$$
$$\left.-\frac{1}{4}k^ik^j\right\}\ . \tag{4.84}$$

The proper time integrations in the amplitude (4.82) will eventually produce factors like $1/q^2$ or $1/k^2$ or both, corresponding to the denominators of the propagators of the massless particles emitted by the branes. Notice that the momentum integration could be explicitly carried out. Using Eqs. (4.74), one obtains

$$\int\frac{d^2\vec{k}_T}{(2\pi)^2}e^{i\vec{k}_T\cdot\vec{b}}e^{-\frac{q^2}{2}\tau}e^{-\frac{k^2}{2}l'} = \frac{1}{2\pi l}e^{-\frac{(\vec{b}+i\vec{p}_T\tau)^2}{2l}}e^{-\frac{p^{(1)2}\tau+p^{(2)2}l'}{2\sinh^2\pi\epsilon}}\ . \tag{4.85}$$

Because of the term $\exp\{-b^2/(2l)\}$, at fixed transverse distance b, world sheets with $l \ll b^2$ give an exponentially suppressed contribution. In particular, the large distance limit $b\to\infty$ implies $l\to\infty$, and selects the part of the amplitude where the fields are massless. Keeping in mind this information, it will nevertheless be convenient to work with the amplitude in its original form, before the momentum integration. Using the general properties and the definitions given in Appendix D, it is easy to show that when ξ_{ij} is antisymmetric, as for the axion, only the odd spin-structure can contribute to the amplitude, whereas vice versa, when ξ_{ij} is symmetric, as for the dilaton and the graviton, only the even spin-structure can contribute.

Consider first the case of the odd spin-structure. In order to get a non-vanishing result, it is necessary to soak up all the zero modes but those in the light-cone directions, which are twisted by the velocity. In particular, the internal partition function can be non-vanishing for example in the twisted sector of the $\mathbb{Z}_3$ orbifold compactification, since in that case there are no zero modes in the compact directions. For the non-compact directions, the z.m. part of the matrix element in the transverse (x^2, x^3) plane gives a vanishing result, unless two transverse fermions are inserted. Therefore, the effective operator to use for computing $\mathcal{M}_{odd}$ is obtained from Eq. (4.84) by factorizing in all possible ways the two transverse fermionic z.m. required in our definition of connected correlation in the $R-$ spin-structure. More precisely, the insertion of $\psi_0^2\psi_0^3$, $\tilde\psi_0^2\psi_0^3$, $\psi_0^2\tilde\psi_0^3$ and $\tilde\psi_0^2\tilde\psi_0^3$ gives the constant factors $i/2$, $1/2$, $1/2$ and $-i/2$. Consequently, only terms with at least two fermion fields will contribute. Splitting the bosons in left and right movers X^μ and $\bar X^\mu$, one finds

$$\mathcal{M}^{odd}_{R-} = \frac{1}{8}\xi_{[ij]}\left\{\epsilon^{ij}\langle p\cdot\psi p\cdot\bar\psi\rangle_{R-} + 2i\,\epsilon^{ik}p_kk^j\right. \tag{4.86}$$
$$\left.+4\,\epsilon^{ik}p_k\left(\langle\partial X^jp\cdot(X+\bar X)\rangle_{osc} - \frac{1}{2}\langle\psi^jp\cdot\bar\psi\rangle_{R-} - \frac{i}{2}\langle\psi^jp\cdot\psi\rangle_{R-}\right)\right\}\ .$$

where $\epsilon^ij = \epsilon^{ij1}$ is the Levi-Civita tensor in the transverse plane. To compute the amplitude, we need the partition functions of the bosonic oscillators and of the fermions in the R− spin-structure. The bosonic and fermionic oscillator cancel as usual by world-sheet supersymmetry, leaving only the contribution of the fermionic z.m.. Since there are no fermionic z.m. in the compact directions in the relevant $\mathbb{Z}_3$ twisted sector case, and the two transverse fermionic z.m. have been inserted, we are left with the constant contribution of the light-cone fermionic z.m., giving

$$Z_B Z^F_{R-} = 2 \sinh \pi\epsilon \ . \tag{4.87}$$

This just cancels the corresponding factor in the denominator of the amplitude coming from the bosonic z.m. in the light-cone directions.

Consider next the case of the even spin-structures. It will prove of great help in this case to integrate by parts the two-derivative bosonic term in the correlation (4.84). By using $\bar\partial = \frac{i}{2}\partial_\tau|_l = \frac{i}{2}(\partial_\tau|_{l'} - \partial_{l'}|_\tau)$, since $\bar\partial$ acts on a function of $z - \bar z = 2i\tau$, and observing that the partition function behaves like a constant with respect to the latter derivative since it depends only on $l = \tau + l'$, one gets

$$\begin{aligned} &\int_0^\infty d\tau \int_0^\infty dl' e^{-\frac{q^2}{2}\tau} e^{-\frac{k^2}{2}l'} \langle e^{ip\cdot X}\rangle_{osc} \langle \partial X^i \bar\partial \bar X^j\rangle_{osc} \\ &\quad = -\frac{i}{2}\int_0^\infty d\tau \int_0^\infty dl' \langle \partial X^i \bar X^j\rangle_{osc} (\partial_\tau - \partial_{l'}) \left\{ e^{-\frac{q^2}{2}\tau} e^{-\frac{k^2}{2}l'} \langle e^{ip\cdot X}\rangle_{osc} \right\} \\ &\quad = -\int_0^\infty d\tau \int_0^\infty dl' e^{-\frac{q^2}{2}\tau} e^{-\frac{k^2}{2}l'} \langle e^{ip\cdot X}\rangle_{osc} \langle \partial X^i \bar X^j\rangle_{osc} \left\{ \langle p\cdot\partial X p\cdot \bar X\rangle_{osc} + \frac{i}{4}(k^2 - q^2) \right\} \ . \end{aligned} \tag{4.88}$$

Taking into account that ξ_{ij} has in this case to be symmetric, and using the manipulation described above, the effective correlation $\mathcal{M}_s$ that one is left with is

$$\begin{aligned} \mathcal{M}_s^{even} = \xi_{(ij)} \Big\{ &-\langle \partial X^i \bar X^j\rangle_{osc} \langle p\cdot\partial X p\cdot\bar X\rangle_{osc} + \langle \partial X^i p\cdot(X+\bar X)\rangle_{osc} \langle \partial X^j p\cdot (X+\bar X)\rangle_{osc} \\ &+\frac{1}{4}\Big(\langle p\cdot\psi p\cdot\bar\psi\rangle_s \langle \psi^i \bar\psi^j\rangle_s - \langle p\cdot\psi\psi^i\rangle_s \langle p\cdot\bar\psi\bar\psi^j\rangle_s + \langle p\cdot\bar\psi\psi^i\rangle_s \langle p\cdot\psi\bar\psi^j\rangle_s \Big) \\ &+\frac{1}{2}\Big(i\langle \partial X^i p\cdot(X+\bar X)\rangle_{osc} + \frac{1}{2}k^i \Big)\Big(\langle p\cdot\psi\psi^j\rangle_s + \langle p\cdot\bar\psi\bar\psi^j\rangle^s \Big) \\ &+ik^i\langle \partial X^j p\cdot(X+\bar X)\rangle_{osc} - \frac{i}{4}(k^2-q^2)\langle \partial X^i \bar X^j\rangle_{osc} - \frac{1}{4}k^i k^j \Big\} \ . \end{aligned} \tag{4.89}$$

Due to the increased difficulty to handle exact expressions, we will limit in this case our analysis to the large distance limit $b \to \infty$, corresponding to $l \to \infty$, in which only the massless modes will contribute and we expect the low energy effective field theory to reproduce all the results. Since $l = \tau + l'$, in this limit at least one among τ and l' is large and thus a massless state is propagating between the two D-branes, which are far away from each other. If $\tau \to \infty$ and l' is finite, the particle is emitted near the first D-brane; if $l' \to \infty$ and τ is finite, it is emitted near the second D-brane. If both $\tau, l' \to \infty$, the particle is emitted far from both D-branes.

In order to compute the quantity $\mathcal{N}$ entering the amplitude, we need the behavior of the partition functions in the limit $l \to \infty$, which can be easily obtained from the results of the previous section. Due to a possible $e^{2\pi l}$ enhancement factor in the NS partition functions, one has to keep a sub-leading term in the corresponding contraction. Conversely, in the RR partition functions, no enhancement is possible and one can work at leading order. We shall therefore use the notation

$$\mathcal{M}_{R+} = \mathcal{M}_R \ , \ \ \mathcal{M}_{NS\pm} = \mathcal{M}_{NS}^{(1)} \pm e^{-2\pi l} \mathcal{M}_{NS}^{(2)} \ . \tag{4.90}$$

The appearance of a $\pm$ in the sub-leading term has been anticipated from the results obtained by explicit computation. On finds the following results for the various four-dimensional point-like configurations considered in the previous section.

Untwisted sector for D0-branes and D3-branes on T^6 or $T^2 \times T^4/\mathbb{Z}_2$

One finds

$$\begin{aligned} Z_B Z_F^{NS\pm} &\underset{l\to\infty}{\longrightarrow} e^{2\pi l} \pm 2\left(\cosh 2\pi\epsilon + 3\right) \ , \\ Z_B Z_F^{R+} &\underset{l\to\infty}{\longrightarrow} 16 \cosh \pi\epsilon \ , \end{aligned} \tag{4.91}$$

and therefore

$$\mathcal{N} = \left(\cosh 2\pi\epsilon + 3\right) \mathcal{M}_{NS}^{(1)} + \frac{1}{2} \mathcal{M}_{NS}^{(2)} - 4 \cosh \pi\epsilon \mathcal{M}_R \ . \tag{4.92}$$

Twisted sectors for D0-branes

In the orbifold twisted sectors, one has (counting a single twisted sector in the $\mathbb{Z}_2$ case and two identical ones in the $\mathbb{Z}_3$ case)

$$\begin{aligned} Z_B Z_F^{NS\pm} &\underset{l\to\infty}{\longrightarrow} \pm 2 \ , \\ Z_B Z_F^{R+} &\underset{l\to\infty}{\longrightarrow} 4 \cosh \pi\epsilon \ , \end{aligned} \tag{4.93}$$

and therefore

$$\mathcal{N} = \mathcal{M}_{NS}^{(1)} - \cosh \pi\epsilon \mathcal{M}_R \ . \tag{4.94}$$

D3-branes on $T^6/\mathbb{Z}_2$

One finds in this case

$$\begin{aligned} Z_B Z_F^{NS\pm} &\underset{l\to\infty}{\longrightarrow} e^{2\pi l} \pm 2 \cosh 2\pi\epsilon \ , \\ Z_B Z_F^{R+} &\underset{l\to\infty}{\longrightarrow} 4 \cosh \pi\epsilon \ , \end{aligned} \tag{4.95}$$

and therefore

$$\mathcal{N} = \cosh 2\pi\epsilon \mathcal{M}_{NS}^{(1)} + \frac{1}{2} \mathcal{M}_{NS}^{(2)} - \cosh \pi\epsilon \mathcal{M}_R \ . \tag{4.96}$$

In the large distance limit $l \to \infty$, the two-point functions entering $\mathcal{M}_s$ have constant parts as well as poles in τ and l'. Since $\mathcal{M}_s$ is quadratic in the two-point function, one gets in principle constant, simple pole and double pole behaviors. However, it is a matter of fact that the double poles always cancel between bosonic and fermionic contributions, and the simple poles appear only in the very particular form

$$f(\tau) = \frac{e^{-4\pi\tau}}{1 - e^{-4\pi\tau}} \ , \ \ f(l') = \frac{e^{-4\pi l'}}{1 - e^{-4\pi l'}} \ . \tag{4.97}$$

The bosonic exponential, whose explicit expression is reported in Eq. (D.63), reduces in the limit $l \to \infty$ to Eq. (D.64), that we repeat here for convenience

$$\langle e^{ip\cdot X}\rangle_{osc} = \left[1 - e^{-4\pi\tau}\right]^{-\frac{p^{(2)2}}{2\pi}} \left[1 - e^{-4\pi l'}\right]^{-\frac{p^{(1)2}}{2\pi}} \; . \tag{4.98}$$

Using this result, one can derive useful identities between various kind of terms in $\mathcal{M}_s$ by integrations by parts in the proper time integrals. For instance, since

$$\begin{aligned}
&\int_0^\infty d\tau e^{-\frac{q^2}{2}\tau}\left[1 - e^{-4\pi\tau}\right]^{-\frac{p^{(2)2}}{2\pi}} \left\{\frac{q^2}{4} + p^{(2)2}\frac{e^{-4\pi\tau}}{1 - e^{-4\pi\tau}}\right\} = \\
&\qquad = -\frac{1}{2}\int_0^\infty d\tau \partial_\tau \left\{e^{-\frac{q^2}{2}\tau}\left[1 - e^{-4\pi\tau}\right]^{-\frac{p^{(2)2}}{2\pi}}\right\} = 0 \; , \qquad (4.99) \\
&\int_0^\infty dl' e^{-\frac{k^2}{2}l'}\left[1 - e^{-4\pi l'}\right]^{-\frac{p^{(1)2}}{2\pi}} \left\{\frac{1}{4}k^2 + p^{(1)2}\frac{e^{-4\pi l'}}{1 - e^{-4\pi l'}}\right\} = \\
&\qquad = -\frac{1}{2}\int_0^\infty dl' \partial_{l'} \left\{e^{-\frac{k^2}{2}l'}\left[1 - e^{-4\pi l'}\right]^{-\frac{p^{(1)2}}{2\pi}}\right\} = 0 \; . \qquad (4.100)
\end{aligned}$$

One can establish the following rules in the amplitude $\mathcal{M}_s$

$$f(\tau) \doteq -\frac{1}{4}\frac{q^2}{p^{(2)2}} \; , \quad f(l') \doteq -\frac{1}{4}\frac{k^2}{p^{(1)2}} \; . \tag{4.101}$$

Using these relations, the contractions can be reduced to a function of the sole momenta, without any dependence on the proper times τ and l', and the amplitude (4.82) becomes

$$\mathcal{A} \underset{b\to\infty}{\longrightarrow} \frac{\hat{M}^2}{\sinh \pi\epsilon}\int \frac{d^2\vec{k}_T}{(2\pi)^2} e^{i\vec{k}_T\cdot\vec{b}} I_1(p,q) I_2(p,k) \mathcal{N}(p,k,q) \; . \tag{4.102}$$

The kinematical integrals $I_{1,2}$ over the two proper times can be explicitly carried out. One finds the usual dual structure with a double series of poles

$$\begin{aligned}
I_1(p,q) &= \frac{1}{2}\int_0^\infty d\tau e^{-\frac{q^2}{2}\tau}\left[1 - e^{-4\pi\tau}\right]^{-\frac{p^{(2)2}}{2\pi}} = \frac{1}{8\pi}\frac{\Gamma\left[\frac{q^2}{8\pi}\right]\Gamma\left[-\frac{p^{(2)2}}{2\pi}+1\right]}{\Gamma\left[\frac{q^2}{8\pi}-\frac{p^{(2)2}}{2\pi}+1\right]} \; , \\
I_2(p,k) &= \frac{1}{2}\int_0^\infty dl' e^{-\frac{k^2}{2}l'}\left[1 - e^{-4\pi l'}\right]^{-\frac{p^{(1)2}}{2\pi}} = \frac{1}{8\pi}\frac{\Gamma\left[\frac{k^2}{8\pi}\right]\Gamma\left[-\frac{p^{(1)2}}{2\pi}+1\right]}{\Gamma\left[\frac{k^2}{8\pi}-\frac{p^{(1)2}}{2\pi}+1\right]} \; .
\end{aligned} \tag{4.103}$$

These are the typical factors arising in the two-point functions on world-sheets with the disk topology [139, 140, 141, 142]. In the low energy limit $p \to 0$, these reduce to the usual propagator denominators of the particles emitted by the two D-branes,

$$I_1(p,q) \underset{p\to 0}{\longrightarrow} \frac{1}{q^2} \; , \quad I_2(p,k) \underset{p\to 0}{\longrightarrow} \frac{1}{k^2} \; . \tag{4.104}$$

We will show that the general structure of the quantity $\mathcal{N}$ is

$$\mathcal{N} = F^{(bulk)} + \sinh \pi\epsilon_1 \frac{k^2}{p^{(1)}} F^{(rad)} - \sinh \pi\epsilon_2 \frac{q^2}{p^{(2)}} F^{(rad)} \; . \tag{4.105}$$

Correspondingly, the amplitude in the low-energy limit becomes

$$\mathcal{A} \underset{b\to\infty}{\longrightarrow} \frac{\hat{M}^2}{\sinh \pi\epsilon} \int \frac{d^2\vec{k}_T}{(2\pi)^2} e^{i\vec{k}_T\cdot\vec{b}} \left\{ \frac{F^{(bulk)}}{k^2 q^2} + \sinh \pi\epsilon_1 \frac{F^{(rad)}}{p^{(1)} q^2} - \sinh \pi\epsilon_2 \frac{F^{(rad)}}{p^{(2)} k^2} \right\} . \qquad (4.106)$$

The k^2 and q^2 denominators correspond to the propagator of the particlea emitted by the D-branes. We shall see that the $p^{(1,2)}$ denominators correspond to the eikonal approximation of the propagator of a virtual D-brane which has been excited by absorbing a particle coming from the other D-brane and then emits the out-going particle in a bremsstrahlung process. Therefore, $F^{(bulk)}$ corresponds to the residue of a double-pole process in which two intermediate massless particles emitted by the two D-branes annihilate far away from the D-branes to produce the final out-going particle, as illustrated in Fig. 4.3. $F^{(rad)}$ corresponds instead to the residue of a single-pole process in which one massless state is emitted by one of the brane and is absorbed by the other which, after traveling some time in an excited state, re-decays by emitting the final state, as illustrated in Figs. 4.4 and 4.5.

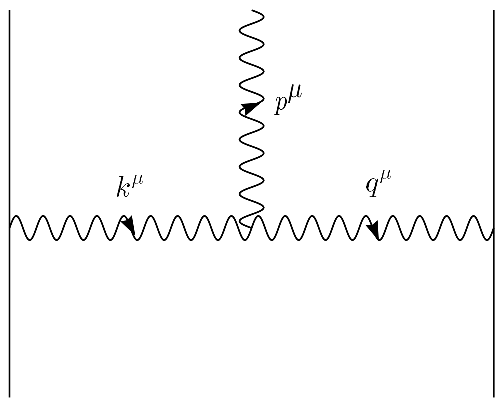


Figure 4.3: Bulk annihilation.

4.3.1 Axion

The axion is described by the antisymmetric polarization tensor $\xi^{(a)}_{ij} = 1/2\epsilon_{ijk}p^k/p$. As discussed above, only the odd spin-structure is relevant and can give a non-vanishing contribution only in the twisted sector of the $\mathbb{Z}_3$ orbifold compactification. We therefore start from Eq. (4.86), working exactly. Using Eqs. (D.79) relating correlations of periodic fermions and bosons by world-sheet supersymmetry, it is easy to see that in the (...) of Eq. (4.86), the oscillator part of the last two fermionic terms cancels against the two bosonic correlations, leaving in (...) only the z.m. part of the two fermionic correlations. These, as well as the z.m. part of the first correlation of fermions, can be easily evaluated using the results of Appendix D and, together with the constant term, give a function of the momenta and the rapidities. Using again world-sheet supersymmetry, Eqs. (D.79), the remaining oscillator part of the first fermionic correlation can be rewritten as the derivative of the oscillator part of a bosonic correlation. Using the explicit form of the polarization tensor, some straightforward algebra yields the very simple result

$$\mathcal{M}_{R-} = \frac{1}{8}\cos\theta \left[-\partial_\tau \langle p\cdot X p\cdot \bar{X}\rangle_{osc} + \frac{i}{2}\left(k^2 - q^2\right)\right] . \qquad (4.107)$$

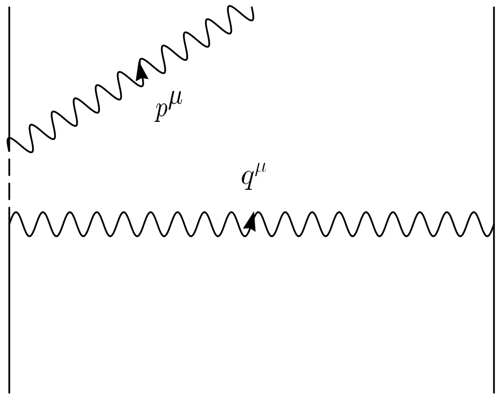


Figure 4.4: Radiation from the first D-brane.

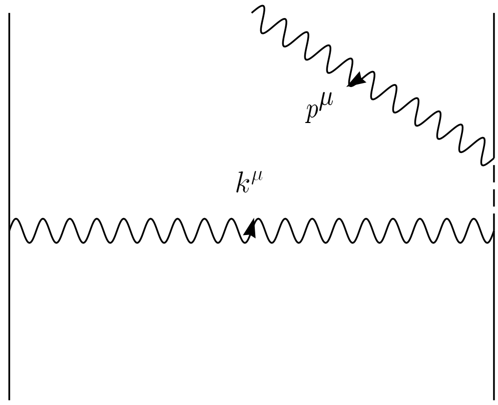


Figure 4.5: Radiation from the second D-brane.

Finally, using Eq. (4.87) and observing that $\partial_\tau|_l = \partial_\tau|_{l'} - \partial_{l'}|_\tau$, the amplitude for the emission of an axion can be written as

$$\mathcal{A}^{(a)} = \frac{i}{4}\cos\theta \int_0^\infty d\tau \int_0^\infty dl' \int \frac{d^2\vec{k}_T}{(2\pi)^2} e^{i\vec{k}\cdot\vec{b}} (\partial_\tau - \partial_{l'}) \left\{ e^{-\frac{q^2(\epsilon_2)}{2}\tau} e^{-\frac{k^2(\epsilon_1)}{2}l'} \langle e^{ip\cdot X}\rangle_{osc} \right\} . \quad (4.108)$$

Here, as in the following, possible surface terms at $\tau, l' = 0$ will be dropped by making an analytic continuation from $p^2 < 0$ of formula (D.64) for $\langle e^{ip\cdot X}\rangle_{osc}$. The amplitude is therefore a total derivative and vanishes identically

$$\mathcal{A}^{(a)} = 0 . \quad (4.109)$$

Thus, there is no on-shell axion emission during the interaction of two moving D-branes, even in the case of the $\mathbb{Z}_3$ orbifold compactification. This result is in qualitative agreement with [135], where the amplitude for axion production due to the interaction of an incoming graviton with two parallel D-branes at rest was computed. Indeed, the corresponding amplitude has no poles in the axion-graviton momentum transfer squared, and therefore no on-shell axion is produced.

4.3.2 Dilaton

The dilaton is described by the symmetric polarization tensor $\xi^{(\phi)}_{ij} = \delta_{ij} - p^i p^j/\vec{p}^2$. Therefore, as discussed above, only the even spin-structures will be relevant and we start from Eq. (4.89), working in the large distance limit. Using the explicit form for the polarization tensor and the notation defined in Appendix D, the correlation $\mathcal{M}_s$ in the three even spin-structures is found to be

$$\begin{aligned}\mathcal{M}_s = &\frac{p^2}{4}\sin^2\theta\Big\{(K_\epsilon - K)^2 - (F^s_\epsilon - F^s)^2 - L^2_\epsilon + G^{s2}_\epsilon - (U^s_\epsilon - W_\epsilon)^2 \\ &\qquad -2\,(U^s_\epsilon - W_\epsilon)\Big[L_\epsilon - \cos\theta\,(K_\epsilon - K)\Big]\Big\} \\ &+p^2\Big[(KK_\epsilon - F^s F^s_\epsilon) - \cos\theta\,(KL_\epsilon - F^s G^s_\epsilon)\Big] \\ &+\frac{1}{8}\Big(k^2 - q^2\Big)\Big[\sin^2\theta K_\epsilon + \Big(1+\cos^2\theta\Big)K\Big] \\ &+\frac{p}{2}h_{i1}k^i\Big[L_\epsilon - \cos\theta\,(K_\epsilon - K) + (U^s_\epsilon - W_\epsilon)\Big] - \frac{1}{4}h_{ij}k^i k^j\ . \end{aligned} \qquad (4.110)$$

In the limit $l \to \infty$, one can use Eqs. (D.91), (D.95) and (D.98) to obtain an explicit expression. Notice that the non exponential terms $-\pi(\epsilon_1 - \epsilon_2)/(2\pi l)$, present in both U^s_ϵ and W_ϵ, always cancel. In order to simplify the result, a crucial role is played by the following kinematic relation involving the dilaton polarization

$$\sinh 2\pi\epsilon\, p\xi^{(\phi)}_{i1}k^i = -\frac{1}{2}\cos\theta\sinh 2\pi\epsilon\Big(k^2 - q^2\Big) + p^2\sin^2\theta\sinh^2\pi\epsilon + p^{(1)2} + p^{(2)2}\ , \qquad (4.111)$$

$$\sinh\pi\epsilon_2\, p^{(2)}\xi^{(\phi)}_{i1}k^i = -\frac{1}{2}\cos\theta\sinh\pi\epsilon_2\Big(k^2 - q^2\Big)\frac{p^{(2)}}{p} + \left(\frac{k^0}{p} - 1\right)p^{(2)2}\ , \qquad (4.112)$$

$$\sinh\pi\epsilon_1\, p^{(1)}\xi^{(\phi)}_{i1}k^i = -\frac{1}{2}\cos\theta\sinh\pi\epsilon_1\Big(k^2 - q^2\Big)\frac{p^{(1)}}{p} + \left(\frac{q^0}{p} + 1\right)p^{(1)2}\ , \qquad (4.113)$$

$$\xi^{(\phi)}_{ij}k^i k^j = -\frac{1}{4p^2}(k^2 - q^2)^2 + \frac{k^0}{p}q^2 - \frac{q^0}{p}k^2\ . \qquad (4.114)$$

As discussed at the beginning of the section, one has to work to order $\mathcal{O}(e^{-2\pi l})$ in the R+ spin-structure and to order $\mathcal{O}(e^{-4\pi l})$ in the NS± ones. In the notation of Eq. (4.90), after heavy algebra one finds, in the notation (4.90)

$$\begin{aligned}\mathcal{M}_R = &\frac{1}{4p^2}\Big(k^2 - q^2\Big)\left\{\frac{1}{4}\Big(k^2 - q^2\Big) - p^{(2)2}f(\tau) + p^{(1)2}f(l')\right\} \\ &-\frac{k^0}{p}\left(\frac{q^2}{4} + p^{(2)2}f(\tau)\right) + \frac{q^0}{p}\left(\frac{k^2}{4} + p^{(1)2}f(l')\right) \\ &-\frac{1}{2}\cos\theta\tanh\pi\epsilon\left\{\frac{1}{4}\Big(k^2 - q^2\Big) - p^{(2)2}f(\tau) + p^{(1)2}f(l')\right\}\ , \end{aligned} \qquad (4.115)$$

$$\begin{aligned}\mathcal{M}^{(1)}_{NS} = &\frac{1}{4p^2}\Big[\Big(k^2 - q^2\Big)\Big]\left\{\frac{1}{4}\Big(k^2 - q^2\Big) - p^{(2)2}f(\tau) + p^{(1)2}f(l')\right\} \\ &-\frac{k^0}{p}\left(\frac{q^2}{4} + p^{(2)2}f(\tau)\right) + \frac{q^0}{p}\left(\frac{k^2}{4} + p^{(1)2}f(l')\right)\ , \end{aligned} \qquad (4.116)$$

$$\mathcal{M}^{(2)}_{NS} = -2\cos\theta\sinh 2\pi\epsilon\left\{\frac{1}{4}\Big(k^2 - q^2\Big) - p^{(2)2}f(\tau) + p^{(1)2}f(l')\right\}\ . \qquad (4.117)$$

The terms in (...) and {...} vanish by using the equivalence relations Eqs. (4.101), meaning that the amplitude for the emission of a dilaton is a total derivative in the large distance limit, and therefore vanishes

$$\mathcal{A}^{(\phi)} = 0 \ . \tag{4.118}$$

Thus, there is no on-shell dilaton emission during the interaction of two moving D-branes, in the large distance limit.

4.3.3 Graviton

The graviton is described by a symmetric and traceless polarization tensor $\xi^{(h)}_{ij} = h_{ij}$ which, in four dimensions, has two physical components. As discussed above, only the even spin-structures will be relevant and we start from Eq. (4.89), working in the large distance limit. Using the notation defined in Appendix D, the correlation $\mathcal{M}_s$ in the three even spin-structures is found to be

$$\begin{aligned}\mathcal{M}_s = &\frac{p^2}{4} h_{11} \Big\{ \left(K_\epsilon^2 - K^2 - L_\epsilon^2 \right) - \left(F_\epsilon^{s2} - F^{s2} - G_\epsilon^{s2} \right) - \left(U_\epsilon^s - W_\epsilon \right)^2 \\ &\quad -2 \left(U_\epsilon^s - W_\epsilon \right) \Big[L_\epsilon - \cos\theta \left(K_\epsilon - K \right) \Big] \Big\} \\ &+ \frac{1}{8} \left(k^2 - q^2 \right) h_{11} \left(K_\epsilon - K \right) \\ &+ \frac{p}{2} h_{i1} k^i \Big[L_\epsilon - \cos\theta \left(K_\epsilon - K \right) + \left(U_\epsilon^s - W_\epsilon \right) \Big] - \frac{1}{4} h_{ij} k^i k^j \ . \end{aligned} \tag{4.119}$$

In the limit $l \to \infty$, one can use Eqs. (D.91), (D.95) and (D.98). Again, the non-exponential terms $-\pi(\epsilon_1 - \epsilon_2)/(2\pi l)$, present in both U_ϵ^s and W_ϵ, always cancel. As before, one has to work to order $\mathcal{O}(e^{-2\pi l})$ in the R+ spin-structure and to order $\mathcal{O}(e^{-4\pi l})$ in the NS± ones. In the notation of Eq. (4.90), one finds

$$\begin{aligned}\mathcal{M}_R = &-\frac{1}{4} h_{ij} k^i k^j \\ &- \sinh \pi\epsilon_2 \left[p^{(2)} h_{i1} k^i + \frac{1}{4} \sinh \pi\epsilon_2 \left(k^2 - q^2 \right) h_{11} \right] f(\tau) \\ &+ \sinh \pi\epsilon_1 \left[p^{(1)} h_{i1} k^i + \frac{1}{4} \sinh \pi\epsilon_1 \left(k^2 - q^2 \right) h_{11} \right] f(l') \\ &+ \frac{p}{2} \tanh \pi\epsilon \left\{ \frac{1}{2} h_{i1} k^i + \sinh \pi\epsilon_2 \, p^{(2)} h_{11} f(\tau) - \sinh \pi\epsilon_1 \, p^{(1)} h_{11} f(l') \right\} \ , \end{aligned} \tag{4.120}$$

$$\begin{aligned}\mathcal{M}_{NS}^{(1)} = &-\frac{1}{4} h_{ij} k^i k^j \\ &- \sinh \pi\epsilon_2 \left[p^{(2)} h_{i1} k^i + \frac{1}{4} \sinh \pi\epsilon_2 \left(k^2 - q^2 \right) h_{11} \right] f(\tau) \\ &+ \sinh \pi\epsilon_1 \left[p^{(1)} h_{i1} k^i + \frac{1}{4} \sinh \pi\epsilon_1 \left(k^2 - q^2 \right) h_{11} \right] f(l') \ , \end{aligned} \tag{4.121}$$

$$\begin{aligned}\mathcal{M}_{NS}^{(2)} = &\, 2 \, p \sinh 2\pi\epsilon \left\{ \frac{1}{2} h_{i1} k^i + \sinh \pi\epsilon_2 \, p^{(2)} h_{11} f(\tau) - \sinh \pi\epsilon_1 \, p^{(1)} h_{11} f(l') \right. \\ &\quad \left. - \frac{1}{4} \tanh \pi\epsilon \, p \, h_{11} \right\} \ . \end{aligned} \tag{4.122}$$

Using the equivalence relations Eqs. (4.101), one can trade the $f(\tau)$ and $f(l')$ poles for momenta. The terms which have both of momentum squared, k^2 or q^2, and a pole, $f(\tau)$ or $f(l')$, can be neglected since they are less singular. The results for the functions $F^{(bulk)}$ and $F^{(1,2)}$ for the various D-brane configurations we are considering are conveniently organized by splitting the contributions of the RR and NSNS sectors, which come below with a $-$ and a $+$ sign respectively outside the [...].

Untwisted sector for D0-branes and D3-branes on T^6 or $T^2 \times T^4/\mathbb{Z}_2$

$$
\begin{aligned}
F^{(bulk)} &= \frac{3}{4}\left[h_{ij}k^ik^j\right] + \frac{1}{4}\left[\cosh 2\pi\epsilon h_{ij}k^ik^j - 2p\sinh 2\pi\epsilon h_{i1}k^i + 2p^2\sinh^2\pi\epsilon h_{11}\right] \\
&\quad - \left[\cosh\pi\epsilon h_{ij}k^ik^j - p\sinh\pi\epsilon h_{i1}k^i\right] \ , \qquad (4.123) \\
F^{(rad)} &= \frac{3}{4}\left[h_{i1}k^i\right] + \frac{1}{4}\left[\cosh 2\pi\epsilon h_{i1}k^i - p\sinh 2\pi\epsilon h_{i1}k^i\right] \\
&\quad - \left[\cosh\pi\epsilon h_{i1}k^i - \frac{p}{2}\sinh\pi\epsilon h_{i1}k^i\right] \ . \qquad (4.124)
\end{aligned}
$$

Twisted sectors for D0-branes

$$
\begin{aligned}
F^{(bulk)} &= \frac{1}{4}\left[h_{ij}k^ik^j\right] - \frac{1}{4}\left[\cosh\pi\epsilon h_{ij}k^ik^j - p\sinh\pi\epsilon h_{i1}k^i\right] \ , \qquad (4.125) \\
F^{(rad)} &= \frac{1}{4}\left[h_{i1}k^i\right] - \frac{1}{4}\left[\cosh\pi\epsilon h_{i1}k^i - \frac{p}{2}\sinh\pi\epsilon h_{i1}k^i\right] \ . \qquad (4.126)
\end{aligned}
$$

D3-branes on $T^6/\mathbb{Z}_2$

$$
\begin{aligned}
F^{(bulk)} &= \frac{1}{4}\left[\cosh 2\pi\epsilon h_{ij}k^ik^j - 2p\sinh 2\pi\epsilon h_{i1}k^i + 2p^2\sinh^2\pi\epsilon h_{11}\right] \\
&\quad -\frac{1}{4}\left[\cosh\pi\epsilon h_{ij}k^ik^j - p\sinh\pi\epsilon h_{i1}k^i\right] \ , \qquad (4.127) \\
F^{(rad)} &= \frac{1}{4}\left[\cosh 2\pi\epsilon h_{i1}k^i - p\sinh 2\pi\epsilon h_{i1}k^i\right] \\
&\quad -\frac{1}{4}\left[\cosh\pi\epsilon h_{i1}k^i - \frac{p}{2}\sinh\pi\epsilon h_{i1}k^i\right] \ . \qquad (4.128)
\end{aligned}
$$

For collinear emission at $\theta = 0$, the results simplify a lot since then $h_{i1} = h_{11} = 0$. Indeed, in this simple case the correlation is independent of the spin structure, $\mathcal{M}_s = -1/4 h_{ij}k^ik^j$, and therefore the partition function factorizes. There are no poles and therefore no bremstrahlung terms, and one finds simply

$$
F^{(bulk)}(\theta = 0) = \frac{1}{16}h_{ij}k^ik^j Z(\epsilon) \ , \quad F^{(rad)}(\theta = 0) = 0 \ . \qquad (4.129)
$$

where $Z(\epsilon)$ is the appropriate total partition function.

4.3.4 Non-relativistic behavior versus supersymmetry

As in the case of the pure phase-shift, the non-relativistic behavior of the emission amplitude is intimately related to the supersymmetry preserved by the configuration of D-branes in interaction. Again, to get a non-vanishing result in a Green-Schwarz path-integral treatment, one has to soak the n fermionic z.m. corresponding to the 2n preserved supersymmetries. In the present case, this can happen either by means of the velocity or through the vertex operator of the emitted particle. As before, each power of the velocity is accompanied by two fermionic z.m.. In the vertex operator of the emitted NSNS particles, one can instead have three kind of terms, with 0, 2 and 4 fermionic fields. When taking these to be anticommuting z.m., each term will correspond to particular restrictions of the polarization. For example, in the case of the graviton, the terms with 0, 2 and 4 fermionic z.m. should correspond to h_{ij}, h_{i1} and h_{11}. Discarding the $\sinh \pi\epsilon$ factor in the amplitude, and therefore concentrating on the functions F, one finds a result which vanishes schematically at least like $F \sim v^{n/2} h_{ij} + v^{n/2-1} h_{i1} + v^{n/2-2} h_{11}$. Indeed, one finds the following behaviors (we omit the numerical coefficients we are here not interesting)

Untwisted sector for D0-branes and D3-branes on T^6 or $T^2 \times T^4/\mathbb{Z}_2$

$$F^{(bulk)} \sim v^4 h_{ij} k^i k^j + v^3 p h_{i1} k^i + v^2 p^2 h_{11} \ , \qquad (4.130)$$
$$F^{(rad)} \sim v^4 h_{i1} k^i + v^3 p h_{11} \ . \qquad (4.131)$$

Twisted sectors for D0-branes

$$F^{(bulk)} \sim v^2 h_{ij} k^i k^j + v p h_{i1} k^i \ , \qquad (4.132)$$
$$F^{(rad)} \sim v^2 h_{i1} k^i + v p h_{11} \ . \qquad (4.133)$$

D3-branes on $T^6/\mathbb{Z}_2$

$$F^{(bulk)} \sim v^2 h_{ij} k^i k^j + v p h_{i1} k^i + v^2 p^2 h_{11} \ , \qquad (4.134)$$
$$F^{(rad)} \sim v^2 h_{i1} k^i + v p h_{11} \ . \qquad (4.135)$$

Notice that the radiation terms are suppressed by an additional power of the velocity with respect to the bulk terms. For collinear emission one has

$$F^{(bulk)}(\theta = 0) \sim v^m h_{ij} k^i k^j \ , \qquad (4.136)$$
$$F^{(rad)}(\theta = 0) \sim 0 \ . \qquad (4.137)$$

with m=2 or 4 depending on the case at hand. In this case, it is straightforward to estimated the total radiated energy. Carrying out the momentum integration by keeping only the most singular terms, and assuming a typical size of the compact part of spacetime equal to the string length, $L \sim l_s$, one finds $\mathcal{A} \sim g_s l_s v^{m-1} f(\vec{p} \cdot \vec{b}/v) \exp\{-\vec{p} \cdot \vec{b}/v\}$, where f is a slowly varying function. The probability that the two interacting branes radiate a graviton, at fixed impact parameter $\vec{b}_T$, is $dP = |\mathcal{A}|^2 d^3p/p$, and the total radiated energy is therefore $\langle E \rangle = \int d^3p|\mathcal{A}|^2$. A simple estimate gives

$$\langle E \rangle \sim g_s^2 l_s^2 \frac{v^{1+2n}}{b^3} . \tag{4.138}$$

4.3.5 Field theory interpretation

In order to give a field theory interpretation of the emission amplitudes that we have obtained, we proceed as in previous section. Consider first the radiation terms. In the eikonal approximation, bremsstrahlung processes like these suffer from a certain ambiguity in their definition. Moreover, in order to compute the true single-pole part, one should also compute the diagrams as in Fig. 4.3 contributing to the bulk part exactly, beside those corresponding really to radiation as in Figs 4.4 and 4.5, and extract the single-pole part. For these reasons, we shall not analyze the radiation term in detail. Notice only that the structure of their denominator is correct. Indeed, the propagator of an excited D-brane with an excess of momentum p^μ to be eventually radiated, and mass set to one, is $1/[(B_{1,2}+p)^2+1]$. In the limit $p \to 0$ required by the eikonal approximation, this reduces to $1/(2p^{(1,2)})$. Therefore, the denominators of the radiation processes on the first and second D-brane are $1/(p^{(1)}q^2)$ and $1/(p^{(2)}k^2)$, as anticipated. Let us now concentrate on the double-pole bulk term, whose denominator is $1/(q^2k^2)$. In field theory, this corresponds to all possible diagrams in which two particles are emitted from the two D-branes and annihilate into the final particle in the bulk, far away from both D-branes, as in Fig. 4.3. The axion and dilaton emission aplitudes are vanishing since there are no SUGRA vertices with a single axion or dilaton and two other particles. For the graviton, there are couplings of a single graviton to pairs of particles, through the effective energy momentum tensor. Since point-like D-branes only couple scalars, vectors and gravitons, there are three possible diagrams, with two scalars, two vectors or two gravitons emitted by the two D-branes and annihilating into the final graviton.

The fields corresponding to the intermediate particles emitted by the two D-branes are given by Eqs. (4.46), but now the kinematics is different and momentum conservation implies $k^\mu - q^\mu = p^\mu$, and translates into the relations (4.71) and (4.72). The emission amplitude is obtained by introducing the fields emitted by the two D-branes in the effective Lagrangian describing the three-particle couplings of the latter two with the out-going graviton. One keeps only terms with the leading double-pole singularity, neglecting eventual single-pole contact degenerations. The contribution from scalars is encoded in

$$\mathcal{L}_{(h\phi\phi)} = -\frac{1}{2}h_{\mu\nu}\partial^\mu\phi\partial^\nu\phi , \tag{4.139}$$

and one finds

$$F_{(\phi)}^{(bulk)} = \hat{a}^2 h_{ij}k^ik^j . \tag{4.140}$$

Similarly, the contribution from vectors is encoded in

$$\mathcal{L}_{(hAA)} = -\frac{1}{2}h_{\mu\nu}\left(F^{\mu\alpha}F^\nu{}_\alpha - \frac{1}{4}\eta^{\mu\nu}F^\alpha{}_\alpha F^\beta{}_\beta\right) \tag{4.141}$$

and one finds

$$F_{(A)}^{(bulk)} = -\hat{e}^2\left[\cosh\pi\epsilon\, h_{ij}k^ik^j - \sinh\pi\epsilon\, ph_{i1}k^i\right] . \tag{4.142}$$

Finally, the contribution from graviton exchange is encoded in (see e.g. [143, 144])

$$\mathcal{L}_{(hhh)} = \partial^\mu h^\alpha_\beta h^\nu_\alpha \partial_\nu h^\beta_\mu - \frac{1}{2} h^\alpha_\beta \partial^\mu h^\beta_\nu \partial_\mu h^\nu_\alpha + \frac{1}{2} h^\alpha_\beta \partial^\nu h^\mu_\alpha \partial_\mu h^\beta_\nu + \frac{1}{2} h^\alpha_\beta \partial_\mu h^\beta_\alpha \partial^\mu h$$
$$+ \frac{1}{4} \partial_\mu h^\alpha_\beta \partial^\nu h^\beta_\alpha h^\mu_\nu + \frac{1}{2} \partial_\nu \partial^\mu h h^\beta_\mu h^\nu_\beta - \frac{1}{4} h \partial_\nu h^\alpha_\beta \partial^\beta h^\nu_\alpha + \frac{1}{8} h \partial^\beta h^\mu_\nu \partial_\beta h^\nu_\mu$$
$$- \frac{1}{8} h \partial^\mu h \partial_\mu h - \frac{1}{4} h \partial^\nu \partial_\beta h h^\beta_\nu + \frac{1}{2} h^\mu_\nu \partial^\alpha \partial_\beta h^\nu_\mu h^\beta_\alpha \, . \tag{4.143}$$

One has to choose in all possible ways one of the gravitons to be the on-shell out-going one, and the other two to be the off-shell gravitons coming from the two branes. After very heavy algebra, and neglecting single pole contact terms, one obtains

$$F^{(bulk)}_{(h)} = \frac{\hat{m}^2}{4} \left[\cosh 2\pi\epsilon h_{ij} k^i k^j - 2p \sinh 2\pi\epsilon h_{1i} k^i + 2p^2 \sinh^2 \pi\epsilon h_{11} \right] \, . \tag{4.144}$$

Therefore, the total contribution to $F^{(bulk)}$ is

$$F^{(bulk)} = \hat{a}^2 \left[h_{ij} k^i k^j \right] + \frac{\hat{m}}{4} \left[\cosh 2\pi\epsilon h_{ij} k^i k^j - 2p \sinh 2\pi\epsilon h_{i1} k^i + 2p^2 \sinh^2 \pi\epsilon h_{11} \right]$$
$$- \hat{e}^2 \left[\cosh \pi\epsilon h_{ij} k^i k^j - p \sinh \pi\epsilon h_{i1} k^i \right] \, . \tag{4.145}$$

Comparing with the results obtained in the large distance limit, one can determine the coupling $\hat{a}$, $\hat{e}$ and $\hat{m}$ for the various configurations, finding perfect agreement with those extracted in previous section from the phase-shift.

Chapter 5

Point-like D-branes as black holes

In this chapter, we further investigate the nature of the four-dimensional configurations discussed in Chapter 4. We present a general description of electromagnetic RR interactions between pairs of magnetically dual Dp and D(6−p)-branes, showing that the electric-electric and/or magnetic-magnetic interaction is encoded in the RR even spin structure and the electric-magnetic interaction in the RR odd spin structure. We then discuss in detail the case of the self-dual D3-brane wrapped on T^6 and $T^6/\mathbb{Z}_3$, and related its electric and magnetic charges to the orientation of the original ten-dimensional D3-brane. We then discuss an explicit construction of a SUGRA solution corresponding to a 3-brane wrapped on a generic Calabi-Yau threefold, reproducing the right structure of four-dimensional charges in the orbifold case. Final evidence for the identification of this wrapped D3-brane with a dyonic R-N black hole is obtained by computing one-point functions of the four-dimensional SUGRA fields. We follow [101] and [102]. See also [145].

5.1 RR interaction for dual Dp-D(6−p)-branes

As already discussed, the RR sector of closed strings contains gauge forms which couple to D-branes. A Dp-brane is electrically charged with respect to the (p+1)-form $C_{(p+1)}$, and magnetically charged with respect to the (7−p)-form $C_{(7-p)}$, with elementary charge μ_p. Similarly, a D(6−p) is electrically charged with respect to the (7−p)-form $C_{(7-p)}$, and magnetically charged with respect to the (p+1)-form $C_{(7-p)}$, with elementary charge μ_{6-p}. Dp-branes and D(6−p)-branes can therefore have both an electric-electric and magnetic-magnetic interaction among themselves, and an electric-magnetic and magnetic-electric interaction between each other.

More in general, consider generic dyonic objects [146, 147, 148] carrying both an electric and a magnetic charge with respect to some gauge fields. Their electric-electric and magnetic-magnetic interaction, to which we shall refer as *diagonal*, can be defined in the usual way through potentials, whereas their electric-magnetic and magnetic-electric interaction, to which we shall refer as *off-diagonal*, is more difficult to defined since the presence of both electric and magnetic charges does not allow for globally defined potentials. A general theoretical framework for describing in a unified way both the diagonal and the off-diagonal interaction has been developed long ago in ref. [149, 150]. We will review shortly this general framework, which is in fact very well suited for discussing D-brane RR interactions, showing that some recently derived results for dyons in various dimensions [151, 152] are naturally obtained within this scheme.

5.1.1 Interactions of charges, monopoles and dyons

As well known, the electromagnetic potential generated by a magnetic monopole cannot be defined everywhere; in the case of a p-extended object in D spacetime dimensions, there exists a Dirac hyper-string on which the potential is singular. As a consequence, the phase-shift of another electrically charged q-extended object along a closed trajectory in this monopole background, which would be a gauge-invariant quantity if the potential were well defined, suffers from an ambiguity. In fact, the requirement that the phase-shift should remain unchanged mod 2π leads to the famous Dirac quantization condition $eg = 2\pi n$.

It is possible to define a mod 2π gauge-invariant phase shift also for open trajectories by considering a pair of charge and anti-charge instead of a single charge. Since an anti-charge traveling forward in time is equivalent to a charge traveling backward, this system can in fact be considered as a single charge describing a closed trajectory [1]. The phase-shift for such a configuration in the monopole background is then a gauge-invariant quantity (provided Dirac's quantization condition holds). Actually, this is the setting that can be most easily analyzed in the string theory framework, since it corresponds to D-branes moving with constant velocities. Indeed the available techniques for computing explicitly D-brane interactions allows to deal only with rectilinear trajectories, more in general with hyperplanes as world-surfaces.

The phase-shift for a system of a charge and an anti-charge moving along two parallel straight trajectories in a monopole background is a special case of the general analysis carried out in ref. [149, 150] that we shall briefly review. We will consider dual pairs of branes, namely p-branes and (D−4−p)-branes (with D being the dimension of the corresponding spacetime). It is convenient to describe the interactions formally in the Euclidean signature (which can be then continued to the Lorentz one). With such a metric one can consider closed world-surfaces of the branes, as they would correspond, in Lorentz spacetime, to brane-antibrane pairs, as explained above.

The world-surface $\Sigma_{(p+1)}$ of the p-brane is (p+1)-dimensional and it couples to the (p+1)-form gauge potential $A_{(p+1)}$. We introduce the notation:

$$\int_{\Sigma_{(p+1)}} A_{(p+1)} = \Sigma_{(p+1)} \cdot A_{(p+1)} \ . \tag{5.1}$$

This can be rewritten as

$$\Sigma_{(p+1)} \cdot A_{(p+1)} = \Sigma_{(p+2)} \cdot F_{(p+2)} \ , \tag{5.2}$$

where F is the field strength $F_{(p+2)} = \nabla A_{(p+1)}$ and $\Sigma_{(p+2)}$ is an arbitrary (p+2)-dimensional surface whose boundary $\partial\Sigma_{(p+2)}$ is $\Sigma_{(p+1)}$. In formulæ :

$$\Sigma_{(p+2)} \cdot \nabla A_{(p+1)} = \partial\Sigma_{(p+2)} \cdot A_{(p+1)} = \Sigma_{(p+1)} \cdot A_{(p+1)} \ . \tag{5.3}$$

[1]If one consider only the usual electric-electric part of the interaction, one can even consider a single infinite straight trajectory; the corresponding phase-shift is gauge-invariant provided we require any gauge transformation to vanish at infinity.

The diagonal (electric-electric and/or magnetic-magnetic) interaction of two p-branes, whose world surfaces are $\Sigma'_{(p+1)}$ and $\Sigma_{(p+1)}$ respectively, can be written as

$$I_{diag} = (e'e + g'g)\,\Sigma'_{(p+2)} \cdot P\Sigma_{(p+2)} = (e'e + g'g)\,\Sigma'_{(p+1)} \cdot D\Sigma_{(p+1)} \;, \tag{5.4}$$

where e, e' (g, g') are the electric (magnetic) charges carried by the two branes, D is the propagator, that is the inverse of the Laplace-Beltrami operator $\Delta = \partial\nabla + \nabla\partial$, i.e. $\Delta D = 1$, and $P = \nabla D \partial$. In the Euclidean path-integral, this interaction appears at the exponent, the integrand being $\exp\{-I_{diag}\}$.

The off-diagonal interaction of two mutually dual branes, a p-brane and a (D−4−p)-brane, in D = 2(q+1) dimensions (the case p = q−1 is self-dual) is instead given by

$$I_{off} = eg'\Sigma'_{(D-2-p)} \cdot {}^*P\Sigma_{(p+2)} + e'g\Sigma_{(p+2)} \cdot {}^*P\Sigma'_{(D-2-p)} \;. \tag{5.5}$$

Here ${}^*F = \epsilon/2^q F$ means the Hodge dual of the form F, obtained by contracting its components with the antisymmetric tensor. It is crucial to observe that the Hodge duality operation depends on the dimension D=2(q+1) of spacetime (that we shall suppose to be even in any case). In fact, the ϵ tensor satisfies $(\epsilon/2^q)^2 = (-1)^{q+1}\,\mathbb{1}$ and $\epsilon^T = (-1)^{q+1}\epsilon$. Using these properties, one can see that $P + (-1)^{q+1*}P^* = \mathbb{1}$ in the space of antisymmetric tensors, as it is equivalent to the Hodge decomposition. Therefore ${}^*P + P^* = {}^*\mathbb{1}$. Now, the insertion of the ${}^*\mathbb{1}$ between $\Sigma'_{(D-2-p)}$ and $\Sigma_{(p+2)}$ yields a contact term given by their intersection number. Assuming by a *Dirac veto* that this number is zero, we get ${}^*P \doteq -P^*$. Finally, transposing the second term in Eq. (5.5) and using the above properties, we get finally

$$\begin{aligned} I_{off} &= (eg' + (-1)^q e'g)\,\Sigma'_{(D-2-p)} \cdot {}^*P\Sigma_{(p+2)} \\ &= \frac{1}{2}\,(eg' + (-1)^q e'g)\left(\Sigma'_{(D-2-p)} \cdot {}^*P\Sigma_{(p+2)} + (-1)^q\Sigma_{(p+2)} \cdot {}^*P\Sigma'_{(D-2-p)}\right) \;. \end{aligned} \tag{5.6}$$

In order for the path integral over $\exp\{iI_{off}\}$ to be well defined, it is necessary to impose the Dirac quantization condition [151]

$$(eg' + (-1)^q e'g) = 2\pi n \;. \tag{5.7}$$

The point is that I_{off} depends on the (supposed irrelevant) choice of the unphysical $\Sigma'_{(D-2-p)}$, which is only constrained to have the physical brane world-surface $\Sigma'_{(D-3-p)}$ as its boundary: $\partial\Sigma'_{(D-2-p)} = \Sigma'_{(D-3-p)}$. However, the path-integral integrand is in this case $\exp\{iI_{off}\}$ and this has no ambiguity. Indeed,

$$I_{off} = (2\pi n)\Sigma'_{(D-2-p)} \cdot {}^*\nabla D\Sigma_{(p+1)} \;. \tag{5.8}$$

Now, if we change $\Sigma'_{(D-2-p)}$ keeping its boundary fixed, the ensuing change of I_{off} can be written as $\delta I_{off} = (2\pi n)\partial\mathcal{V}_{(D-1-p)} \cdot {}^*\nabla D\Sigma_{(p+1)}$, where the boundary of $\mathcal{V}_{(D-1-p)}$ is the union of the old $\Sigma'_{(D-2-p)}$ and the new one. By integrating by parts, using $\nabla^* = {}^*\partial$ and $\partial\Sigma_{(p+1)} = 0$ since we consider closed world surfaces, we get

$$\delta I_{off} = (2\pi n)\mathcal{V}_{(D-1-p)} \cdot {}^*\Sigma_{(p+1)} = 2\pi(\text{integer}) \;, \tag{5.9}$$

since $\mathcal{V}_{(D-1-p)} \cdot {}^*\Sigma_{(p+1)}$ is the intersection number of the closed hypersurface $\Sigma_{(p+1)}$ and the hypervolume $\mathcal{V}_{(D-1-p)}$ and is therefore an integer. Notice that relaxing the Dirac veto, Eq. (5.6) is a consistent expression provided $eg' + (-1)^q e'g = 4\pi n$.

5.1.2 Compactification

The above properties remain valid also when we compactify some of the D dimensions, in particular compactifying six of the ten dimensions of string theory. Objects whose extended dimensions are wrapped in the compact directions will appear point-like in four-dimensional spacetime. In particular, we will be interested in the case of the D3-brane wrapped on T^6 and $T^6/\mathbb{Z}_3$. The D3-brane of Type IIB is a special case since it is both electrically and magnetically charged with respect to the self-dual RR 4-form; this peculiarity will be relevant in our study giving rise, both before and after the compactification, to dyonic states. From the four-dimensional spacetime point of view, these will look like dyons with electric and magnetic charges determined by the D3-brane's different orientations in the compact directions. For instance, if two interacting D-branes are parallel in the compact directions, then it is easy to see (we will be explicit in the following) that $I_{off} = (2\pi n)\Sigma'_{(D-2-p)} \cdot {}^*\nabla D\Sigma_{(p+1)} = 0$ and this will be interpreted in four dimensions by saying that there is no off-diagonal interaction between to "parallel" dyons, that is having the same ratio (magnetic charge)/(electric charge). In fact, two such dyons behave with respect to each other as purely electrically charged particles. It is amusing to notice that although the Dirac quantization condition is automatically implemented, as we said, once the off-diagonal interaction is correctly normalized in ten dimensions, it might look somewhat non-obvious at first sight in four dimensions. We will explore the ensuing pattern of charge quantization in the following subsections.

In the following, we are going to consider the off-diagonal interaction of two pairs of D3-branes-antibranes, wrapped on the compact part of spacetime and moving linearly in the non-compact part of spacetime (the brane's parameters will be labeled by B, the antibrane's ones by A and the index $i = 1, 2$ labels the two pairs). We will take the trajectories in spacetime to describe a line in the (t, x) plane. In each of the two pairs, the brane and the antibrane are parallel to each other. This means that each pair is described by two parallel four-dimensional hyperplanes. The directions $\vec{\alpha}^{(i)}$ in the three compact planes (x^a, x^{a+1}) are specified by the angles $\theta_a^{(i)}$ $(a = 4, 6, 8)$, common to the brane and the antibrane, so that $\alpha_a^{(i)} = \cos\theta^{(i)}$ and $\alpha_{a+1}^{(i)} = \sin\theta^{(i)}$. In the (t, x) plane, the direction $\vec{w}^{(i)}$ of each pair is specified by the rapidity $\epsilon^{(i)}$, so that $w_t^{(i)} = \sinh\pi\epsilon^{(i)}$ and $w_x^{(i)} = \cosh\pi\epsilon^{(i)}$. The (t, x) trajectory of the D-branes of the pair i is taken in the positive t-direction and is located at position $y_B^{(i)}, z_B^{(i)}$ in the transverse (y, z) plane, while the trajectory of the antibrane is taken in the negative t-direction and is located at position $y_A^{(i)}, z_A^{(i)}$. It is convenient to introduce a complex variable $\xi = y + iz$. The positions of the brane and the antibrane of the two pairs in the transverse (y, z) plane is depicted in Fig. 5.1.

According to the general construction, the diagonal and off-diagonal interactions I_{diag} and I_{off} are given by Eqs. (5.4) and (5.6) respectively. In order to integrate along the hypersurfaces, let us suppose first that the angles $\theta_a^{(2)}$ are different from the angles $\theta_a^{(1)}$. Consider the propagator D, that we shall from now write as $\Delta_{(D)}(r) = \int d^Dk/(2\pi)^D\tilde{\Delta}(k)e^{ikr}$ with $\tilde{\Delta}(k) = 1/k^2 = \int_0^\infty dl e^{-lk^2}$. The integration along the planes in the compact space and along the (t, x) plane will result in putting to zero all the compact and the (t, x) components of the momentum k. Hence, after those integrations, the propagator D will be reduced to the Fourier transform of $\tilde{\Delta}_(k)$ where only k_y, k_z are different from zero, that is the two dimensional propagator $\Delta_{(2)}(r)$ in the plane (y, z). Thus, the only possible derivatives occurring in the previous equation will be in the (y, z) plane. Actually, by doing the integration over

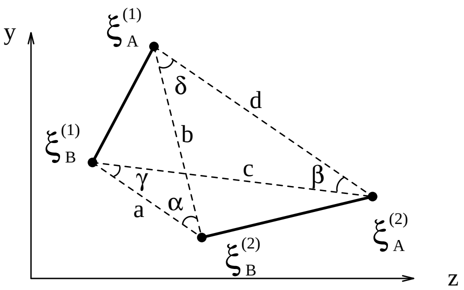


Figure 5.1:

l as the last one, the other integrations factorize into the product of integrations along the planes (t,x), (y,z) and the three compact planes (x^a, x^{a+1}) respectively. In the following, it will be convenient to use the two-dimensional *complex* propagator (λ is an infrared cut-off)

$$\mathcal{D}_2(\xi,\xi') = \frac{1}{2\pi}\ln\frac{\xi-\xi'}{\lambda} \ . \tag{5.10}$$

whose real part is $\Delta_{(2)}(\xi,\xi') = \mathrm{Re}\mathcal{D}_2(\xi,\xi')$.

In the diagonal case, the integration in the (t,x) plane gives

$$\begin{aligned} I^{(t,x)}_{diag} &= (\vec{w}^{(1)}\cdot\vec{w}^{(2)})\int dt^{(1)}\int dt^{(2)}\int\frac{d^2\vec{k}_{t,x}}{(2\pi)^2}e^{i(t^{(1)}\vec{w}^{(1)}-t^{(2)}\vec{w}^{(2)})\cdot\vec{k}_{t,x}}e^{-l\vec{k}^2_{t,x}} \\ &= \frac{\vec{w}^{(1)}\cdot\vec{w}^{(2)}}{|\vec{w}^{(1)}\wedge\vec{w}^{(2)}|} = \coth\pi\left(\epsilon^{(1)}-\epsilon^{(2)}\right) \ . \end{aligned} \tag{5.11}$$

Similarly, the integrations in the three (x^a, x^{a+1}) planes give

$$I^{(comp)}_{diag} = \frac{\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)}}{|\vec{\alpha}^{(1)}\wedge\vec{\alpha}^{(2)}|} = \frac{V^{(1)}V^{(2)}}{\mathrm{Vol}(T^6/\mathbb{Z}_3)}\prod_a\cos\left(\theta^{(1)}_a-\theta^{(2)}_a\right) \ . \tag{5.12}$$

where $V^{(1,2)}$ are the volumes of the wrapped 3-branes. This factor turns the ten-dimensional charges $e'e+g'g$ into the four-dimensional dyon charge combination $e^{(1)}e^{(2)}+g^{(1)}g^{(2)}$. The remaining integrations in the (y,z) plane are over the straight lines joining the brane in $\xi^{(i)}_B$ and the antibrane in $\xi^{(i)}_A$ for each of the two pairs $i=1,2$, and give,

$$\begin{aligned} I^{(y,z)}_{diag} &= \int_{\xi^{(1)}_B}^{\xi^{(1)}_A} d\xi^{(1)}\cdot\partial_{\xi^{(1)}}\int_{\xi^{(2)}_B}^{\xi^{(2)}_A} d\xi^{(2)}\cdot\partial_{\xi^{(2)}}\mathrm{Re}\mathcal{D}_2(\xi^{(1)},\xi^{(2)}) \\ &= \frac{1}{2\pi}\mathrm{Re}\ln\left(\frac{\xi^{(1)}_A-\xi^{(2)}_A}{\xi^{(1)}_B-\xi^{(2)}_A}\cdot\frac{\xi^{(1)}_B-\xi^{(2)}_B}{\xi^{(1)}_A-\xi^{(2)}_B}\right) \\ &= \frac{1}{2\pi}\ln\frac{ad}{bc} \ . \end{aligned} \tag{5.13}$$

In the off-diagonal case, the integration in the (t,x) plane gives

$$\begin{aligned} I^{(t,x)}_{off} &= (\vec{w}^{(1)}\wedge\vec{w}^{(2)})\int dt^{(1)}\int dt^{(2)}\int\frac{d^2\vec{k}_{t,x}}{(2\pi)^2}e^{i(t^{(1)}\vec{w}^{(1)}-t^{(2)}\vec{w}^{(2)})\cdot\vec{k}}e^{-l(\vec{k}^2_{t,x})} \\ &= \frac{\vec{w}^{(1)}\wedge\vec{w}^{(2)}}{|\vec{w}^{(1)}\wedge\vec{w}^{(2)}|} = \pm 1 \ . \end{aligned} \tag{5.14}$$

The result is therefore ± 1 (the degenerate case where the trajectories (1) and (2) are parallel should be taken to be zero). The integrations in the (x^a, x^{a+1}) planes give instead

$$I_{diag}^{(comp)} = \frac{\vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)}}{|\vec{\alpha}^{(1)} \wedge \vec{\alpha}^{(2)}|} = \frac{V^{(1)} V^{(2)}}{\mathrm{Vol}(T^6/\mathbb{Z}_3)} \prod_a \sin\left(\theta_a^{(1)} - \theta_a^{(2)}\right) . \tag{5.15}$$

This factor turns the ten-dimensional charges $eg' + e'g$ into the four-dimensional dyon charge combination $e^{(1)} g^{(2)} - g^{(1)} e^{(2)} = 2\pi n$. The remaining integrations in the (y, z) plane give in this case

$$\begin{aligned} I_{off}^{(y,z)} &= \int_{\xi_B^{(1)}}^{\xi_A^{(1)}} d\xi^{(1)} \wedge \partial_{\xi^{(1)}} \int_{\xi_B^{(2)}}^{\xi_A^{(2)}} d\xi^{(2)} \cdot \partial_{\xi^{(2)}} \mathrm{Re}\mathcal{D}_2(\xi^{(1)}, \xi^{(2)}) \\ &= \frac{1}{2\pi} \mathrm{Im} \ln \left(\frac{\xi_A^{(1)} - \xi_A^{(2)}}{\xi_B^{(1)} - \xi_A^{(2)}} \cdot \frac{\xi_B^{(1)} - \xi_B^{(2)}}{\xi_A^{(1)} - \xi_B^{(2)}} \right) \\ &= \frac{\beta - \alpha}{2\pi} = \frac{\delta - \gamma}{2\pi} . \end{aligned} \tag{5.16}$$

There are here two important observations that we can make. First, considering pairs of branes-antibranes automatically eliminates any infrared divergence. Second, since the combination of charges contributes 2π times an integer, the off-diagonal interaction is given, apart from this integer, by the difference of the angles by which any curve joining $\xi_B^{(1)}$ and $\xi_A^{(1)}$ is seen from $\xi_B^{(1)}$ and $\xi_A^{(1)}$, or vice versa. We thus see explicitly that I_{off} is defined modulo 2π. Concluding, the total diagonal and off-diagonal interactions are given by

$$I_{diag} = \frac{\left(e^{(1)} e^{(2)} + g^{(1)} g^{(2)}\right)}{\tanh \pi(\epsilon^{(1)} - \epsilon^{(2)})} \mathrm{Re}\mathcal{D}_2 \ , \tag{5.17}$$

$$I_{off} = \pm \left(e^{(1)} g^{(2)} - g^{(1)} e^{(2)}\right) \mathrm{Im}\mathcal{D}_2 \ , \tag{5.18}$$

with

$$\mathcal{D}_2 = \ln \left(\frac{\xi_A^{(1)} - \xi_A^{(2)}}{\xi_B^{(1)} - \xi_A^{(2)}} \cdot \frac{\xi_B^{(1)} - \xi_B^{(2)}}{\xi_A^{(1)} - \xi_B^{(2)}} \right) . \tag{5.19}$$

Notice the interesting fact that in D=2(q+1)=10, where the gauge field is a q=4 even form, the 3-brane is a dyon in the sense that it has $e = g = \mu_3 = \sqrt{2\pi}$ and that it has both a diagonal and an off-diagonal interaction with itself. In fact, the off-diagonal interaction is in this case proportional to $e^{(1)} g^{(2)} + e^{(2)} g^{(1)}$ (whereas for q odd it is proportional to $e^{(1)} g^{(2)} - e^{(2)} g^{(1)}$) and different from zero also for $e^{(1)} = e^{(2)}$, $g^{(1)} = g^{(2)}$. On the contrary, for D=2(q+1)=4, where the gauge field is a q=1 odd form, two "parallel" dyons having equal charges $e^{(1)} = e^{(2)}$ and $g^{(1)} = g^{(2)}$ do not have any off-diagonal interaction, the latter being proportional to $e^{(1)} g^{(2)} - e^{(2)} g^{(1)}$.

It turns out from our analysis that the D=10 off-diagonal interaction, proportional to eg, becomes automatically proportional to $e^{(1)} g^{(2)} - e^{(2)} g^{(1)}$ upon compactification down to D=4. This happens because the off-diagonal interaction is proportional to the factor $\prod_a \sin(\theta_a^{(1)} - \theta_a^{(2)})$, which is zero when the branes (1) and (2) are seen by a non-compact observer to be parallel in the sense that $e^{(1)} = e^{(2)}$ and $g^{(1)} = g^{(2)}$. More in general, notice that the off-diagonal interaction between two dyons (1) and (2) is symmetric both for q even and for q odd, under the exchange of every quantum number, $(1) \leftrightarrow (2)$. In fact, the transverse (y, z) contribution to the amplitude, that is $\mathcal{D}_2$, is symmetric, $\mathcal{D}_2(1,2) = \mathcal{D}_2(2,1)$, whereas each pair of the remaining non-transverse directions (t, x) and (x^a, x^{a+1}) gives an antisymmetric contribution; therefore, since $e^{(1)}g^{(2)} + (-1)^q e^{(2)}g^{(1)}$ is symmetric for q even and antisymmetric for q odd, the total amplitude turns out to be symmetric in both cases (see Eq. (5.6)).

5.1.3 The interactions in string theory

As already noticed, the diagonal electric-electric and/or magnetic-magnetic interaction between two Dp-branes is a well defined quantity also for open trajectories. In this case, in fact, there is no strict necessity of considering interactions among pairs of Dp-brane-antibrane (although this is advisable to avoid infrared problems). In string theory, the diagonal RR interaction of just one Dp-brane at $\xi^{(1)}$ and another Dp-brane at $\xi^{(2)}$ is encoded in the RR+ spin-structure cylinder amplitude

$$\mathcal{A}_{diag} = \frac{\hat{\mu}_p^2}{2^4} \int_0^\infty dl \langle B_p^{(1)}, \xi^{(1)} | e^{-lH} | B_p^{(2)}, \xi^{(2)} \rangle_{RR+} \; , \tag{5.20}$$

For convenience, we have rescaled the modulus l by a factor 2 in order to get a factor 2 in the definition of H, in order to have p^2 for the z.m. part, rather that $p^2/2$. This gives an overall factor of 2 which has changed the normalization, and a torus modulus equal to $4il$ rather than $2il$.

Also the off-diagonal RR interaction can be expressed in string theory within the boundary state formalism. Intuitively, it is quite obvious that the off-diagonal interaction must be encoded in the RR− spin-structure, which indeed produces the correct topological structure of the interaction, and gives a potentially non-vanishing result for dual pair of a Dp-brane and a D(6−p)-brane, as we shall see. More precisely, the situation for the odd spin-structure cylinder amplitude for this configuration of D-branes is the following. The Dp-D(6−p) system can have a maximum of 6 ND directions, when the Dp and the D(6−p)-branes are taken to be completely orthogonal. In these directions there are no true z.m. and therefore the contribution of the fields along these directions the odd spin-structure partition function is non-vanishing. More in general, the same remains true as long as one keeps non-zero relative angles or fluxes in these directions. There are then the two light-cone directions t and x which are tilted by the velocity and therefore the corresponding bosonic and fermionic pairs of fields again have no true z.m. and give a non-vanishing contribution to the partition function. Finally, there always remains a pair of DD transverse directions, y and z, in which there are true z.m., in particular fermionic ones which give a vanishing result. It is therefore clear that some modification of the simple cylinder amplitude is required in order to obtain a sensitive result. This is related to the already discussed necessity of considering the more complex system of a D-brane-antibrane pair, say located at $\xi^{(1)}_{B,A}$ in the transverse plane, with one D(6−p)-brane (or antibrane) located at $\xi^{(2)}$ in the transverse plane. According to the general description developed in previous section, this interaction is expressed by an integral over a Dirac string joining $\xi^{(1)}_B$ and $\xi^{(1)}_A$, which we represent parametrically by $\xi^{(1)}(s)$, $s = (0, 1)$. We shall propose the following string theory expression for the phase-shift

$$\mathcal{A}_{off} = \frac{\hat{\mu}_p \hat{\mu}_{6-p}}{2^4} \int_0^\infty dl \int_0^1 ds \, \langle B_p^{(1)}, \epsilon^{(1)}, \xi^{(1)}(s) | J(s) \bar{J}(s) e^{-lH} | B_{6-p}^{(2)}, \epsilon^{(2)}, \xi^{(2)} \rangle_{RR-} \; , \tag{5.21}$$

where J and $\bar{J}$ are the left and right moving supercurrents, whose matter part is $J = \partial X^\mu \psi_\mu$ and $\bar{J} = \bar{\partial} X^\mu \tilde{\psi}_\mu$. Along the Dirac string, $\partial, \bar{\partial} = \partial_s \mp i\partial_\tau$, where ∂_τ is the normal derivative, that is along the direction τ orthogonal to the Dirac string; τ is therefore the (Euclidean) world-sheet evolution time of the closed superstring.

The odd spin-structure correlation is non-vanishing due to the supercurrent insertion. Only the z.m. of the matter part contributes, providing the transverse fermionic z.m. insertion $\psi_0^y \tilde{\psi}_0^z$ (or z, y interchanged) required to get a non-vanishing result. Since the fermionic correlation gives an antisymmetric result, one is left with an antisymmetric bosonic correlation which is zero except for the z.m. part

$$\langle B_p^{(1)} | J(s) \bar{J}(s) e^{-lH} | B_{(6-p)}^{(2)} \rangle_{RR-} = 2i \langle B_p^{(1)} | (\partial_s y \partial_\tau z - \partial_s z \partial_\tau y) \psi_0^y \tilde{\psi}_0^z e^{-lH} | B_{6-p}^{(2)} \rangle_{RR-} \; . \quad (5.22)$$

Recall that in the the odd spin-structure, the contribution to the partition function of the bosonic and fermionic oscillator modes cancel by world-sheet supersymmetry. With our normalization, the fermionic z.m. insertion gives $\langle \psi_0^y \tilde{\psi}_0^z \rangle = -\langle \psi_0^z \tilde{\psi}_0^y \rangle = 1/2$. The (y, z) bosonic z.m. give instead the correct position dependence of the amplitude. Indeed, notice that $ds\,(\partial_s y, \partial_s z) = (dy, dz)$ along the integration line, and that as an operator $(\partial_\tau y, \partial_\tau z) = -(\partial_y, \partial_z)$, since the ∂_τ derivatives of the coordinates are canonical momenta acting as derivatives on the corresponding coordinate. Therefore, it follows that $ds\,(\partial_s y \partial_\tau z - \partial_s z \partial_\tau y) = dy \partial_z - dz \partial_y = d\xi \wedge \partial_\xi$. Moreover, for the transverse bosonic modes $\int_0^\infty dl \langle \xi^{(1)}(s) | e^{-lH} | \xi^{(2)} \rangle = \Delta_{(2)}(\xi^{(1)}(s), \xi^{(2)})$. Finally, one obtains

$$\int_0^\infty dl \int_0^1 ds\, \langle B_p^{(1)} | J(s) \bar{J}(s) e^{-lH} | B_{(6-p)}^{(2)} \rangle_{RR-}^{(y,z)} = \int_{\xi_B^{(1)}}^{\xi_A^{(1)}} d\xi^{(1)} \wedge \partial_{\xi^{(1)}} \Delta_{(2)}(\xi^{(1)}, \xi^{(2)}) \; , \quad (5.23)$$

which reproduces precisely the expected result for the off-diagonal interaction. In the case of the self-dual D3-brane wrapped on the compact part of spacetime, the details of the computation of off-diagonal interaction follows closely the general pattern described in previous subsection. The fermionic z.m. in the light-cone and compact directions give a non-vanishing result due to the non-vanishing relative rapidity $\epsilon^{(1)} - \epsilon^{(2)}$ and relative angles $\theta_a^{(1)} - \theta_a^{(2)}$ which, together with the constant contribution of the bosonic z.m., changes the ten-dimensional coupling into the four-dimensional one.

A comment is in order about the subtle treatment of the fermionic and superghost z.m. in the odd spin-structure. As already discussed in Chapter 3, two different approaches are possible for the odd spin-structure cylinder amplitude. In the path-integral approach to the superstring, it is known [155] that the integral over the supermoduli produces supercurrent insertions. Actually, in the cylinder case there is only one modulus, the previously introduced l, and correspondingly one has only one supermodulus and one supercurrent insertion (the sum $J + \bar{J}$). In the case at hand, however, one is forced to consider simultaneously the interaction of a D-brane-antibrane pair with a given D-brane (or antibrane). It is therefore not so surprising to see the occurrence of the pair of supercurrents J and $\bar{J}$ as if the interaction would correspond to some extent to the torus topology, rather than cylinder one. Another suggestive observation in this directions is that string world-sheets, or in the low-energy limit particle world-lines, are associated in the some sense to the flux lines of the interactions they mediate. The diagonal electric interaction between two D-branes is described by flux lines starting from one D-brane and ending on the other D-brane, whereas the off-diagonal interaction between a pair of D-brane-antibrane with another D-brane is described by flux lines closing through the Dirac string stretched between the D-brane-antibrane pair. In any case, it is a fact that the boundary state amplitude Eq. (5.21) reproduces exactly the correct result for the off-diagonal electric-magnetic interaction.

Another approach consists in canceling the transverse fermionic z.m. with the corresponding superghost z.m., as described in [131, 134]. By doing so, the problem of the fermionic z.m. is cured in a very simple way, and one obtains a non-vanishing result for the odd spin-structure cylinder amplitude. However, this cannot be interpreted in any sensitive way as a phase-shift. Probably, the naive result that one obtains in that way corresponds to integrating the modulus of the Lorentz force over the trajectory, without considering the wedge product with actually prevents magnetic interactions to change the energy and is the source of the complications discussed in this section.

5.2 Wrapped D3-branes as dyons

In this section, we will apply the formalism developed in previous section to various configurations obtained from the D3-brane. We will first study the diagonal and off-diagonal interactions of the self-dual D3-brane in ten dimensions, and then turn to the point-like objects studied in Chapter 4 which can be obtained by wrapping the D3-brane on T^6 and $T^6/\mathbb{Z}_3$.

5.2.1 D3-branes in ten dimensions

Let us start from a D3-brane configuration with N. b.c. in the directions $x^0 = t$ and x^a, and D in $x^1 = x$, $x^2 = y$, $x^3 = z$ and x^{a+1}, with $a = 4, 6, 8$. The directions x^a, x^{a+1} will eventually become compact. Consider then two of these D3-branes moving with velocities $v^{(1,2)} = \tanh \pi\epsilon^{(1,2)}$ along the x^1 direction, at transverse positions $\vec{Y}^{(1,2)}$, and tilted in the (x^a, x^{a+1}) planes with generic angles $\theta_a^{(1,2)}$. The cylinder amplitude reads

$$\mathcal{A} = \frac{\hat{\mu}_3^2}{2^4} \int_0^\infty dl \sum_\alpha (\pm) \langle B^{(1)}, \epsilon^{(1)}, \theta_a^{(1)}, \vec{Y}^{(1)} | e^{-lH} | B^{(2)}, \epsilon^{(2)}, \theta_a^{(2)}, \vec{Y}^{(2)} \rangle_\alpha \ . \tag{5.24}$$

The bosonic z.m. part of the boundary state is

$$|B_0, \epsilon, \theta_a, \vec{Y}\rangle_B = \int \frac{d^6\vec{k}}{(2\pi)^6} e^{i\vec{k}\cdot\vec{Y}} |k^\mu(\epsilon, \theta)\rangle \ , \tag{5.25}$$

with $k^\mu(\epsilon, \theta) = (\sinh \pi\epsilon k^1, \cosh \epsilon k^1, k^2, k^3, \cos\theta_a k^a, \sin\theta_a k^a)$. Integrating over the momenta and taking into account momentum conservation which for non-vanishing tilts denoted by $\epsilon = \epsilon^{(1)} - \epsilon^{(2)}$ and $\theta_a = \theta_a^{(1)} - \theta_a^{(2)}$, forces all the D momenta but k^2, k^3 to be zero, the amplitude factorizes as usual into a bosonic and a fermionic partition functions

$$\mathcal{A} = \frac{\hat{\mu}_3^2}{16 \sinh|\pi\epsilon| \prod_a \sin|\theta_a|} \int_0^\infty \frac{dl}{4\pi l} e^{-\frac{b^2}{4l}} \sum_\alpha Z_B Z_F^\alpha \ , \tag{5.26}$$

where $\mu_3 = \sqrt{2\pi}$ is the D3-brane tension, $\vec{b} = \vec{Y}_T^{(1)} - \vec{Y}_T^{(2)}$ ($b = |\xi^{(1)} - \xi^{(2)}|$) is the transverse impact parameter and

$$Z_{B,F}^\alpha = \langle B^{(1)}, \epsilon^{(1)}, \theta_a^{(1)} | e^{-lH} | B^{(2)}, \epsilon^{(2)}, \theta_a^{(2)} \rangle_{B,F}^\alpha \ . \tag{5.27}$$

One has only oscillator modes in the bosonic case, since the z.m. have been already taken into account. Also, according to the discussion of previous section, we will imagine that in the odd spin-structure the two transverse fermionic z.m. are soaked up due to the supercurrent insertions (but we will omit to write explicitly the integral over the Dirac string in these intermediate steps). The amplitude $\mathcal{A}$ can be written as a world-sheet integral

$$\mathcal{A} = \hat{\mu}_3^2 \int_{-\infty}^{\infty} d\tau \prod_a \int_{-\infty}^{\infty} d\xi_a \int_0^{\infty} \frac{dl}{(4\pi l)^3}\, e^{-\frac{r^2}{4l}} \frac{1}{16} \sum_\alpha Z_B Z_F^\alpha \,, \tag{5.28}$$

in terms of the true distance $r = \sqrt{\vec{b}^2 + \sinh^2 \pi\epsilon\, \tau^2 + \sum_a \sin^2\theta_a\, \xi_a^2}$. In the limit $\epsilon, \theta_a \to 0$, translational invariance along the directions x^1, x^a is restored and the integral over the world-sheet produces simply the volume V_{3+1} of the D3-branes. The remaining part of the boundary state is the same as that constructed in Appendix D. The total partition functions are

$$Z_B = \eta(2il)^4 \frac{2i\sinh \pi\epsilon}{\vartheta_1(i\epsilon|4il)} \prod_a \frac{2\sin\theta_a}{\vartheta_1\left(\frac{\theta_a}{\pi}|4il\right)} \,, \tag{5.29}$$

$$Z_F^{even} = \eta(4il)^{-4} \sum_{\alpha=2,3,4} (-1)^{1+\alpha} \vartheta_\alpha(i\epsilon|4il) \prod_a \vartheta_\alpha\left(\frac{\theta_a}{\pi}|4il\right) \,, \tag{5.30}$$

$$Z_F^{odd} = \eta(4il)^{-4} \vartheta_1(i\epsilon|4il) \prod_a \vartheta_1\left(\frac{\theta_a}{\pi}|4il\right) \,. \tag{5.31}$$

For the even part, the $l \to \infty$ relevant in the large distance limit $b \to \infty$, is

$$Z_B Z_F^{even} \underset{l\to\infty}{\longrightarrow} 16\cosh\pi\epsilon \prod_a \cos\theta_a - 4\left(\cosh 2\pi\epsilon + \sum_a \cos 2\theta_a\right) \,. \tag{5.32}$$

In the odd part, instead, there is the usual cancellation between bosonic and fermionic oscillators and one has simply

$$Z_B Z_F^{odd} = 16 i \sinh\pi\epsilon \prod_a \sin\theta_a \,.$$

Recall finally that the bosonic fields present in the supercurrents alter the z.m. part of the amplitude precisely in the right way to allow the interpretation of the previous section.

Summarizing, the diagonal interaction between two D3-branes at positions $\xi^{(1)}$ and $\xi^{(2)}$ in the transverse plane is, at large distances,

$$I_{diag} = \hat{\mu}_3^2 \coth\pi\epsilon \prod_a \cot\theta_a \Delta_{(2)}|\xi^{(1)} - \xi^{(2)}| \,, \tag{5.33}$$

The off-diagonal interaction between a D3-brane at transverse position $\xi^{(2)}$ and a pair of D3-brane and D3-antibrane at $\xi_B^{(1)}$ and $\xi_A^{(1)}$ is instead the same all distances and given by

$$I_{off} = \pm\hat{\mu}_3^2 \int_{\xi_B^{(1)}}^{\xi_A^{(1)}} d\xi^{(1)} \wedge \partial_{\xi^{(1)}} \Delta_{(2)}|\xi^{(1)} - \xi^{(2)}| \,. \tag{5.34}$$

Here $\Delta_{(d)}(r)$ is the Green function in d dimensions

$$\Delta_{(d)}(r) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2} = \int_0^\infty \frac{dl}{(4\pi l)^{d/2}}\, e^{-\frac{r^2}{4l}} \,. \tag{5.35}$$

5.2.2 D3-branes on T^6 and $T^6/\mathbb{Z}_3$

In this section we shall apply the general construction that we have introduced to the case of the Type IIB D3-brane wrapped on the orbifold $T^6/\mathbb{Z}_3$. Compactifying the directions x^a, x^{a+1}, $a = 4, 6, 8$ on T^6, one gets N=8 supersymmetry, which is further broken down to N=2 by the $\mathbb{Z}_3$ identification. The orbifold $T^6/\mathbb{Z}_3$ is a singular limit of a CY manifold with Hodge numbers $h^{(1,1)} = 9$ and $h^{(1,2)} = 0$. The standard counting of hyper and vector multiplets for Type IIB compactifications then yields $n_V = h_{1,2}$ and $n_H = h^{(1,1)} + 1$ [19, 20] and the LEEA is therefore D=4 N=2 SUGRA coupled to 10 hypermultiplets and 0 vector multiplets (see [153, 154] and references therein). In particular, the only vector field arising in the compactification, namely the graviphoton, comes from the self-dual RR 4-form $C_{\mu\nu\rho\sigma}$ under which the D3-brane is already charged in 10 dimensions. We have seen in Chapter 4 that the wrapped D3-brane configuration corresponds to a solution which does not couple to any scalar, but only to the graviton and the graviphoton of the N=2 gravitational multiplet.

We shall generalize here the phase-shift computation of Chapter 4 by considering D3-branes wrapping with arbitrary angles on the compact directions. The boundary states describing these D3-brane differ from the one constructed for the non-compact D3-brane essentially through the usual quantization of the momentum along a compact direction. There are only minor changes with respect to the construction described in Appendix D, for the compact part of the boundary state.

Let us start concentrating on a single T^2 factor, then. The only lattice compatible with the eventual $\mathbb{Z}_3$ gauging is the triangular one, with modulus $\tau = R \exp\{i\frac{\pi}{3}\}$. The lattice of windings $\bar{L} = L_x + iL_y$ is given by $\bar{L} = m\tau + nR = (R/2)(2n + m) + i(\sqrt{3}/2)Rm$, with m, n integers, that is

$$L_x = \frac{R}{2} N_x \ , \quad L_y = \frac{\sqrt{3}}{2} R N_y \ , \tag{5.36}$$

where N_x, N_y are integers of the same parity. The lattice of momenta is as usual determined by the requirement that the plane wave $\exp\{ip \cdot X\}$ is well defined when X is shifted by a vector belonging to the winding lattice, and one finds

$$p_x = \frac{2\pi}{R} n_x \ , \quad p_y = \frac{2\pi}{\sqrt{3}R} n_y \ , \tag{5.37}$$

where n_x, n_y are again integers of the same parity.

We choose in each T^2 an arbitrary D direction x' at angle θ with the x direction and an orthogonal N direction y' forming an angle $\Omega = \theta + \pi/2$ with the x direction, and fix its length. This amounts to choose an arbitrary vector $\bar{L}$ in the winding lattice, which is identified by the pair (N_x, N_y) or, more conveniently for the following, by the orthogonal pair $(\bar{n}_y, -\bar{n}_x)$, which corresponds to the orthogonal direction of allowed momenta (see Fig. 5.3). In this way

$$L_x = -L \sin\theta \ , \quad L_y = L \cos\theta \ , \tag{5.38}$$

$$\cos\theta = -\frac{\sqrt{3}R}{2L} \bar{n}_x \ , \quad \sin\theta = -\frac{R}{2L} \bar{n}_y \ , \tag{5.39}$$

where

$$L = |\bar{L}| = \frac{R}{2} \sqrt{\bar{n}_y^2 + 3\bar{n}_x^2} \ . \tag{5.40}$$

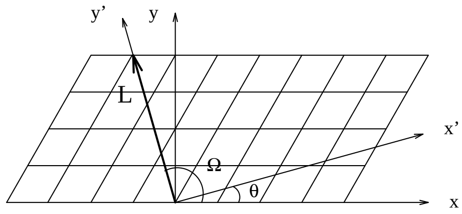


Figure 5.2:

We are now interested in the bosonic z.m. contribution. For simplicity, let us recall the result for the non-compact case. The boundary state for the bosonic z.m. in a given (x^a, x^{a+1}) plane is

$$
\begin{aligned}
|B_0, \theta, \vec{Y}\rangle_B &= \delta\left(x'_0 - Y'\right)|0\rangle \\
&= \iint \frac{dp_x dp_y}{(2\pi)} e^{-i(p_x \cdot Y_x + p_y \cdot Y_y)} \delta\left(\cos\theta\, p_y - \sin\theta\, p_x\right) |p_x, p_y\rangle\,. \qquad (5.41)
\end{aligned}
$$

The δ-function selects momenta parallel to the D direction we have chosen. Indeed if ω is the direction of the generic $\vec{p}$ momentum, the argument of the δ-function becomes proportional to $\sin(\theta - \omega)$. Using of the normalization $\langle p_x, p_y | q_x, q_y\rangle = (2\pi)^2 \delta(p_x - q_x)\delta(p_y - q_y)$ one recovers the following vacuum amplitude

$$
\begin{aligned}
&\langle B_0^{(1)}, \theta^{(1)}, \vec{Y}^{(1)} | e^{-lH} | B_0^{(2)}, \theta^{(2)}, \vec{Y}^{(2)}\rangle_B = \\
&= \iint dp_x dp_y e^{-i(p_x \cdot \Delta Y_x + p_y \cdot \Delta Y_y)} \delta\left(\cos\theta^{(1)} p_y - \sin\theta^{(1)} p_x\right) \delta\left(\cos\theta^{(2)} p_y - \sin\theta^{(2)} p_x\right) \\
&= \frac{1}{\sin|\theta^{(1)} - \theta^{(2)}|}\,. \qquad (5.42)
\end{aligned}
$$

In discretizing this result we adopt the following strategy. Let us begin by supposing $\theta^{(1)} \neq \theta^{(2)}$. First we substitute in Eq. (5.42) the previously derived expressions for the discretized quantities $\vec{p}$ and θ and extract some jacobians from the Dirac δ-functions, obtaining

$$
\langle B_0^{(1)}, \theta^{(1)} | e^{-lH} | B_0^{(2)}, \theta^{(2)}\rangle_B = \frac{L(\theta^{(1)}) L(\theta^{(2)})}{(\sqrt{3}/4) R^2} \sum_{n_x, n_y}^{s.p.} \delta\left(\bar{n}_x^{(1)} n_y - \bar{n}_y^{(1)} n_x\right) \delta\left(\bar{n}_x^{(2)} n_y - \bar{n}_y^{(2)} n_x\right)\,.
$$

Since in this case the solution of the condition enforced by the δ-functions is $n_x = n_y = 0$, all the momenta are zero and the exponential drops as in the continuum case. The Dirac δ-function containing only integers can now be turned to a Kronecker one. However, since the latter is insensitive to an integer rescaling whereas the former transforms with an integer jacobian, we shall keep an arbitrary integer constant in this step:

$$
\delta\left(\bar{n}_x^{(1)} n_y - \bar{n}_y^{(1)} n_x\right) \delta\left(\bar{n}_x^{(2)} n_y - \bar{n}_y^{(2)} n_x\right) = N \delta_{\bar{n}_x^{(1)} n_y, \bar{n}_y^{(1)} n_x} \delta_{\bar{n}_x^{(2)} n_y, \bar{n}_y^{(2)} n_x} = N \delta_{n_x,0} \delta_{n_y,0}\,. \qquad (5.43)
$$

Therefore

$$
\langle B_0^{(1)}, \theta^{(1)} | e^{-lH} | B_0^{(2)}, \theta^{(2)}\rangle_B = N \frac{L(\theta^{(1)}) L(\theta^{(2)})}{\mathrm{Vol}(T^2)}. \qquad (5.44)
$$

with $\mathrm{Vol}(T^2) = (\sqrt{3}/2)R^2$. The integer N is fixed to 1 by the requirement that for $\theta^{(1)} = \theta^{(2)}$ the amplitude reduces to the "winding" $L^2/\mathrm{Vol}(T^2)$. Actually, in order to achieve the above limit, an infinite $L(\theta)$ is in general required because of the discreteness of the allowed angles, even if in the strictly parallel case finite $L(\theta)$'s are possible. Indeed, one can check that $L(\theta^{(1)})L(\theta^{(2)}) \sin|\theta^{(1)} - \theta^{(2)}| = |\bar{n}_x^{(1)}\bar{n}_y^{(2)} - \bar{n}_y^{(1)}\bar{n}_x^{(2)}|\mathrm{Vol}(T^2)$. In this way the continuum and discrete results differ by the integer jacobian $|\bar{n}_x^{(1)}\bar{n}_y^{(2)} - \bar{n}_y^{(1)}\bar{n}_x^{(2)}|$ (which vanishes for $\theta^{(1)} = \theta^{(2)}$). The final result is then

$$\langle B_0^{(1)}, \theta^{(1)}|e^{-lH}|B_0^{(2)}, \theta^{(2)}\rangle_B = \frac{L(\theta^{(1)})L(\theta^{(2)})}{\mathrm{Vol}(T^2)} = \frac{|\bar{n}_x^{(1)}\bar{n}_y^{(2)} - \bar{n}_y^{(1)}\bar{n}_x^{(2)}|}{\sin|\theta^{(1)} - \theta^{(2)}|} \; . \tag{5.45}$$

The above result could have been obtained starting directly from the compact boundary state, that is, by first discretizing the continuum boundary state (5.41) and then computing the amplitude. The correct discrete boundary state turns out to be

$$|B_0, \theta, \vec{Y}\rangle_B = L(\theta) \sum_{n_x, n_y}^{s.p.} \frac{1}{(\sqrt{3}/2)R^2} e^{-\frac{2\pi}{R} i(n_x Y_x + n_y Y_y/\sqrt{3})} \delta\left(\bar{n}_x n_y - \bar{n}_y n_x\right) |n_x, n_y\rangle \; , \tag{5.46}$$

and reproduces correctly Eq. (5.45) with the definition $\langle n_x, n_y | m_x, m_y\rangle = \sqrt{3}R^2 \delta_{n_x, m_x}\delta_{n_y, m_y}$.

T^6 case

Postponing for the moment the $\mathbb{Z}_3$ identification, let us now consider as an instructive intermediate result the case of T^6. The result Eq. (5.45) can be generalized in a straightforward way giving for the total contribution from the compact part of the bosonic z.m.

$$\langle B_0^{(1)}, \theta_a^{(1)}|e^{-lH}|B_0^{(2)}, \theta_a^{(2)}\rangle_B = \frac{V^{(1)}V^{(2)}}{\mathrm{Vol}(T^6)} \; , \tag{5.47}$$

where $V^{(1,2)}$ are the volumes of the two D3-branes. This factor is reabsorbed in the definition of a four-dimensional mass $\hat{M}$ ($\theta_a = \theta_a^{(1)} - \theta_a^{(2)}$)

$$\hat{M}^2 = \hat{\mu}_3^2 \frac{V^{(1)}V^{(2)}}{\mathrm{Vol}(T^6)} = 2\pi \prod_a \frac{|\bar{n}_a^{(1)}\bar{n}_{a+1}^{(2)} - \bar{n}_{a+1}^{(1)}\bar{n}_a^{(2)}|}{\sin|\theta_a|} \; . \tag{5.48}$$

The contribution of the fermions does not change during the compactification and the amplitude (5.26) becomes in this case

$$\mathcal{A} = \frac{\hat{M}^2}{\sinh|\pi\epsilon|} \int_0^\infty \frac{dl}{4\pi l} e^{-\frac{b^2}{4l}} \frac{1}{16} \sum_s Z_B Z_F^s \; , \tag{5.49}$$

and can be rewritten this time as a one-dimensional world-sheet integral

$$\mathcal{A} = \hat{M}^2 \int_{-\infty}^{\infty} d\tau \int \frac{dl}{(4\pi l)^{3/2}} e^{-\frac{r^2}{4l}} \frac{1}{16} \sum_s Z_B Z_F^s \; , \tag{5.50}$$

in terms of the four-dimensional distance $r = \sqrt{\vec{b}^2 + \sinh^2 \pi\epsilon\, \tau^2}$.

Eqs. (5.33) for the large distance diagonal interaction between two D-branes at positions $\xi^{(1)}$ and $\xi^{(2)}$, and (5.34) for the scale-independent off-diagonal interaction between a D-brane at transverse position $\xi^{(2)}$ and a pair of D-brane and D-antibrane at $\xi^{(1)}_B$ and $\xi^{(1)}_A$, modify to

$$I_{diag} = \alpha_{even} \coth \pi\epsilon \Delta_{(2)} |\xi^{(1)} - \xi^{(2)}| \,, \tag{5.51}$$

$$I_{off} = \pm\alpha_{odd} \int_{\xi^{(1)}_B}^{\xi^{(1)}_A} d\xi^{(1)} \wedge \partial_{\xi^{(1)}} \Delta_{(2)} |\xi^{(1)} - \xi^{(2)}| \,, \tag{5.52}$$

with

$$\alpha_{even} = \hat{M}^2 \prod_a \cos\theta_a \ , \ \ \alpha_{odd} = \hat{M}^2 \prod_a \sin\theta_a \ . \tag{5.53}$$

Recalling (5.48) and noticing that

$$\cot\theta_a = \sqrt{3} \frac{3\bar{n}^{(1)}_a \bar{n}^{(2)}_a + \bar{n}^{(1)}_{a+1} \bar{n}^{(2)}_{a+1}}{\bar{n}^{(1)}_a \bar{n}^{(2)}_{a+1} - \bar{n}^{(1)}_{a+1} \bar{n}^{(2)}_a} \,, \tag{5.54}$$

the two couplings can also be written as

$$\begin{aligned} \alpha_{even} &= 2\pi \prod_a \sqrt{3} \left(3\bar{n}^{(1)}_a \bar{n}^{(2)}_a + \bar{n}^{(1)}_{a+1} \bar{n}^{(2)}_{a+1} \right) \,, \\ \alpha_{odd} &= 2\pi \prod_a \left(\bar{n}^{(1)}_a \bar{n}^{(2)}_{a+1} - \bar{n}^{(1)}_{a+1} \bar{n}^{(2)}_a \right) \,. \end{aligned} \tag{5.55}$$

As expected, the orientation of the D3-branes in ten dimensions affects the effective electric and magnetic couplings of the corresponding point-like objects in four dimensions. Notice that the Dirac quantization condition for the off-diagonal coupling α_{odd}, which is satisfied in ten dimensions with the minimal allowed charges, remains satisfied in four with an integer which depends on the D-branes' orientation. This result can also be understood in terms of the relevant N=8 supergravity. Notice in fact that

$$\prod_a \cos\theta_a = \frac{1}{4} \sum_{i=1}^{4} \cos\phi_i \ , \ \ \prod_a \sin\theta_a = -\frac{1}{4} \sum_{i=1}^{4} \sin\phi_i \,, \tag{5.56}$$

with $\phi_i = \phi^{(1)}_i - \phi^{(2)}_i$ and

$$\begin{aligned} \phi^{(1,2)}_1 &= \theta^{(1,2)}_4 + \theta^{(1,2)}_6 + \theta^{(1,2)}_8 \ , \ \ \phi^{(1,2)}_2 = -\theta^{(1,2)}_4 - \theta^{(1,2)}_6 + \theta^{(1,2)}_8 \,, \\ \phi^{(1,2)}_3 &= \theta^{(1,2)}_4 - \theta^{(1,2)}_6 - \theta^{(1,2)}_8 \ , \ \ \phi^{(1,2)}_4 = -\theta^{(1,2)}_4 + \theta^{(1,2)}_6 - \theta^{(1,2)}_8 \,. \end{aligned} \tag{5.57}$$

The effective couplings can thus be rewritten as

$$\alpha_{even} = \sum_{i=1}^{4} \left(\hat{e}^{(1)}_i \hat{e}^{(2)}_i + \hat{g}^{(1)}_i \hat{g}^{(2)}_i \right) \ , \ \ \alpha_{odd} = \sum_{i=1}^{4} \left(\hat{e}^{(1)}_i \hat{g}^{(2)}_i - \hat{g}^{(1)}_i \hat{e}^{(2)}_i \right) \,, \tag{5.58}$$

with

$$\hat{e}^{(1)}_i = \frac{\hat{M}}{2} \cos\phi^{(1)}_i \ , \ \ \hat{e}^{(2)}_i = \frac{\hat{M}}{2} \cos\phi^{(2)}_i \,, \tag{5.59}$$

$$\hat{g}^{(1)}_i = \frac{\hat{M}}{2} \sin\phi^{(1)}_i \ , \ \ \hat{g}^{(2)}_i = \frac{\hat{M}}{2} \sin\phi^{(2)}_i \,. \tag{5.60}$$

This second consideration allows to keep track of the coupling to the various vector fields. In fact, it happens that the ten vectors fields arising from dimensional reduction of the RR 4-form, couple to the wrapped D3-brane only through four independent combinations of fields, with electric and magnetic charges parameterized by the four angles $\phi_i^{(1,2)}$. Since the electric and magnetic charges corresponding to a given $\phi_i^{(1,2)}$ cannot vanish simultaneously, the 3-brane cannot decouple from any of the four effective gauge fields, in agreement with a pure SUGRA argument achieved in ref. [156].

Therefore, wrapping a D3-brane on T^6 on obtains a four-parameter family of inequivalent four-dimensional dyons, whose effective couplings depend on the orientation of the D3-brane in the compact part of spacetime. Notice finally that when two of these branes have equal $\phi_i^{(1,2)}$'s (yielding vanishing ϕ_i's) their diagonal coupling no longer depends on the angles and the off-diagonal one vanish, as appropriate for identical dyons in $D=4$ dimensions.

$T^6/\mathbb{Z}_3$ case

Let us discuss finally the orbifold case. As explained in Chapter 4, the only effect of the $\mathbb{Z}_3$ identification is to project the boundary state obtained in the T^6 case onto its $\mathbb{Z}_3$-invariant part. This projection can be easily performed by first computing the amplitude on T^6 with a relative twist w_a in the orientations, $\theta_a \to \theta_a + 2\pi w_a$, and then averaging on all the possible w_a's. Recall moreover that the twists w_a in the three (x^a, x^{a+1}) planes satisfy $\sum_a w_a = 2\pi n$ in order to preserve at least one supersymmetry.

Since the bosonic z.m. contribution (5.47) does not depend explicitly on the angles, the only modification introduced by the $\mathbb{Z}_3$ identification is in the volume: $\mathrm{Vol}(T^6/\mathbb{Z}_3) = \mathrm{Vol}(T^6)/3$. For the fermions, instead, one simply sets $\theta_a \to \theta_a + 2\pi w_a$. Under this relative rotation one has, modulo irrelevant integer multiples of 2π

$$\begin{aligned}
\phi_1 &\to \phi_1 + 2\pi(w_4 + w_6 + w_8) = \phi_1 \ , \\
\phi_2 &\to \phi_2 + 2\pi(-w_4 - w_6 + w_8) = \phi_2 + 4\pi w_8 \ , \\
\phi_3 &\to \phi_3 + 2\pi(w_4 - w_6 - w_8) = \phi_3 - 4\pi w_4 \ , \\
\phi_4 &\to \phi_4 + 2\pi(-w_4 + w_6 - w_8) = \phi_4 + 4\pi w_6 \ .
\end{aligned} \tag{5.61}$$

The averaging procedure has the important consequence of projecting out the contribution depending on the non-invariant ϕ_2, ϕ_3, ϕ_4, with respect to the T^6 case. Indeed,

$$\frac{1}{3}\sum_{\{w_a\}}\prod_a \cos(\theta_a + 2\pi w_a) = \frac{1}{4}\cos\phi_1 \ , \quad \frac{1}{3}\sum_{\{w_a\}}\prod_a \sin(\theta_a + 2\pi w_a) = -\frac{1}{4}\sin\phi_1 \ . \tag{5.62}$$

One is therefore left with the contribution of the sole e_1, g_1 charges

$$\alpha_{even} = \left(\hat{e}_1^{(1)}\hat{e}_1^{(2)} + \hat{g}_1^{(1)}\hat{g}_1^{(2)}\right) \ , \quad \alpha_{odd} = \left(\hat{e}_1^{(1)}\hat{g}_1^{(2)} - \hat{g}_1^{(1)}\hat{e}_1^{(2)}\right) \ . \tag{5.63}$$

Thus, after the $\mathbb{Z}_3$ gauging, only one pair of electric and magnetic charges survives, consistently with the fact that, as already pointed out at the beginning of this section, only one vector field survives to the projection in the low energy effective theory, namely the graviphoton. The Dirac quantization still holds, like in the T^6 case. Indeed, due to the cancellation of the $1/3$ in the projection with the 3 coming from the volume, the averaging procedure is equivalent to superpose three replica of the D3-brane forming $2\pi/3$ angles between them. Since the Dirac quantization condition holds for each of pair of these, it holds also for the sum of the interactions.

Summarizing, wrapping a D3-brane on $T^6/\mathbb{Z}_3$ one obtains a one-parameter family of four-dimensional dyons whose effective couplings depend on the orientation of the D3-brane in the compact part of spacetime. Recall finally that, as discussed in Chapter 4, the $\mathbb{Z}_3$ projection, which reduces the four independent gauge fields to one, is also responsible for the decoupling of the scalars fields from the D3-brane. Thus, the D3-brane wrapped on $T^6/\mathbb{Z}_3$ looks like an extremal R-N configuration, being a source of gravity and Maxwell field only.

5.3 R-N black hole as D3-branes wrapped on CY threefolds

In this section, we will confirm the evidence found by computing interactions that the D3-brane wrapped on $T^6/\mathbb{Z}_3$ is a R-N black hole of the low-energy N=2 SUGRA. More in general, we will show how an extremal R-N black hole solution can be obtained by wrapping a dyonic 3-brane solution of Type IIB SUGRA on a CY manifold. In the orbifold limit $T^6/\mathbb{Z}_3$, we explicitly show the correspondence between the solution of the SUGRA equations of motion and the D3-brane boundary state description of such a black hole already discussed several times.

5.3.1 Black hole and CY compactification

In the last couple of years there has been much effort in finding a microscopic description of both extremal and non-extremal black holes arising as compactifications of different p-brane solutions of ten-dimensional SUGRA theories. This has been done by considering various solitonic configurations in string theory, such as bound states of D-branes and solitons of different kinds [71] or as intersecting (both orthogonally and at angles) D-branes alone [157, 158]. As far as the microscopic description is concerned, these studies have been mainly devoted to toroidal compactifications and less has been said about CY ones. On the contrary, from a macroscopic SUGRA point of view, these black hole solutions have been known for a long time in both cases and many progresses have been made in the last few years (see [159, 160, 161] and many subsequent works). Different problems arise when trying to find an appropriate D-brane description of these solutions in a non-flat asymptotic space. Moreover, some general results that are valid in the toroidal case no longer hold for CY compactifications. In particular, it is not straightforward to generalize the so called *harmonic function rule* and it is also no longer true that the minimum number of "different" charges (that is, carried by different microscopic objects) must be four in order to obtain a regular black hole in four dimensions.

We will be interested in discussing R-N black hole in four dimensions within a CY compactification (whose relevance for obtaining non-singular four-dimensional black hole was already pointed out, see for instance ref. [162]). The R-N solution defined as the usual non-singular black hole solution of Maxwell-Einstein gravity, can also be seen as a particular solution of a wider class of field theories in four dimensions in which the only fields having a non-trivial coordinate dependence are the metric $g_{\mu\nu}$ and a gauge field A_μ, whereas any other field is taken to be constant. In particular, in four-dimensional N=2 SUGRA this solution, known as *double-extreme* black hole [163], arises in the specific case in which one assumes that the moduli fields belonging to vector multiplets (as well as those belonging to hyper-multiplets which are anyhow constant in any N=2 black hole solution) take the same constant values from the horizon to spatial infinity. In order to be consistent with the field equations such constant values are not arbitrary but must coincide with the so called *fixed values*: these are determined in terms of the electric and magnetic charges of all the existing gauge fields by a variational principle that extremizes the central charge and leads to classical formulæ expressing the horizon area as a quartic invariant of the U-duality group (see for instance [156, 164, 165, 166] and references therein).

When ten-dimensional SUGRA is compactified on a CY threefold $\mathcal{M}_3^{CY}$, one obtains D=4 N=2 SUGRA coupled to matter. As well known, the field content of the four-dimensional theory and its interaction structure is completely determined by the *topological and analytical type* of $\mathcal{M}_3^{CY}$ but depends in no way on its metric structure. Indeed the standard counting of hyper and vector multiplets tells us that $n_V = h^{(1,2)}$ and $n_H = h^{(1,1)} + 1$, the numbers $h^{(p,q)}$ being the dimensions of the Dolbeault cohomology groups. Furthermore, the geometrical datum that completely specifies the vector multiplet coupling, namely the choice of the special Kähler manifold and its special Kähler metric, is provided by the moduli space geometry of complex structure deformations. To determine this latter no reference has ever to be made to the Kähler metric $g_{ij^\star}$ installed on $\mathcal{M}_3^{CY}$ (for a review of this well established results see for instance [167]). Because of this crucial property careful thought is therefore needed when one tries to *oxidize* the solutions of D=4 N=2 SUGRA obtained through compactification on $\mathcal{M}_3^{CY}$ to *bona fide* solutions of the original D=10 Type IIB SUGRA. To see the four-dimensional configuration as a configuration in ten dimension one has to choose a metric on the internal manifold in such a way to satisfy the full set of ten-dimensional equations.

5.3.2 The 3-brane wrapped on $T^6/\mathbb{Z}_3$ as a SUGRA solution

In this subsection we will explicitly show how an four-dimensional extreme R-N black hole solution can be obtained by compactifying the self-dual 3-brane on $\mathcal{M}_3^{CY} = T^6/\mathbb{Z}_3$, which is the orbifold limit of a CY manifold with Hodge numbers $h^{(1,1)} = 9$ and $h^{(1,2)} = 0$. In this case, the effective four-dimensional theory is D=4 N=2 SUGRA coupled to 10 hyper-multiplets and 0 vector multiplets, the only vector field in the game being the graviphoton. Since there are no vector multiplet scalars, the only regular black hole solution is the double-extreme one. From a SUGRA point of view, this is somewhat obvious and the same conclusion holds for every Type IIB compactification on CY manifolds with $h^{(1,2)} = 0$. The interest of the $T^6/\mathbb{Z}_3$ case lies in the fact that an explicit and simple D-brane boundary state description is available. It would be obviously very interesting to find more complicated configurations which correspond to regular N=2 black hole solutions for which an analogous D-brane description can be found.

We will start by showing that the *oxidization* of a *double-extreme* black-hole solution of N=2 SUGRA to a *bona fide* solution of Type IIB SUGRA is possible and quite straightforward. It just suffices to choose for the CY metric the Ricci-flat one whose existence in every Kähler class is guaranteed by Yau's theorem [168]. Our exact solution of Type IIB SUGRA in ten dimensions corresponds to a 3-brane wrapped on a 3-cycle of the generic threefold

$\mathcal{M}_3^{CY}$ and dimensionally reduced to four dimensions is a double-extreme black hole. Let us then argue how this simple result is obtained.

As well known, prior to the recent work by Bandos, Sorokin and Tonin [169] Type IIB SUGRA had no supersymmetric spacetime action. Only the field equations could be written as closure conditions of the supersymmetry algebra [170]. The same result could be obtained from the rheonomy superspace formalism as shown in [171, 172]. Indeed, the condition of self-duality for the RR 5-form $F_{(5)}$ that is necessary for the equality of Bose and Fermi degrees of freedom cannot be easily obtained as a variational equation and has to be stated as a constraint. In the new approach of [169] such problems are circumvented by introducing more fields and more symmetries that remove spurious degrees of freedom. However, for our purposes these subtleties are not relevant since our goal is that of showing the existence of a classical solution. Hence, we just need the field equations which are unambiguous and reduce, with our ansatz, to the following ones:

$$R_{MN} = T_{MN} \ , \ \ \nabla_M F_{(5)}^{MABCD} = 0 \ , \tag{5.64}$$

where $T_{MN} = 1/(2\cdot 4!)\, F^2_{(5)MN}$ is the energy-momentum tensor of the RR 4-form $A_{(4)}$ to which the 3-brane couples and $F_{(5)}$ the corresponding self-dual field strength, satisfying the constraint ${}^*F_{(5)} = F_{(5)}$. It is noteworthy that if we just disregarded the self-duality constraint and we considered the ordinary action of the system composed by the graviton and an unrestricted 4-form

$$\mathcal{S} = \frac{1}{2\kappa^2_{(10)}} \int d^{10}x \sqrt{g_{(10)}} \left(R_{(10)} - \frac{1}{2\cdot 5!} F^2_{(5)} \right) \ , \tag{5.65}$$

then, by ordinary variation with respect to the metric, we would anyhow obtain, as source of the Einstein equation, a traceless stress-energy tensor:

$$T_{MN} = \frac{1}{2\cdot 4!} \left(F^2_{(5)MN} - \frac{1}{2\cdot 5} g_{MN} F^2_{(5)} \right) \ . \tag{5.66}$$

The tracelessness of T_{MN} is peculiar to the 4-form and signals its conformal invariance. This, together with the absence of couplings to the dilaton, allows for zero curvature solutions in ten dimensions.

For the metric, we make a block-diagonal ansatz with a Ricci-flat compact part depending only on the internal coordinates y^a (this corresponds to choosing the unique Ricci-flat Kähler metric on $\mathcal{M}_3^{CY}$), and a non-compact part which depends only on the corresponding non-compact coordinates x^μ

$$ds^2 = g^{(4)}_{\mu\nu}(x) dx^\mu dx^\nu + g^{(6)}_{ab}(y) dy^a dy^b \ . \tag{5.67}$$

For $g^{(4)}_{\mu\nu}$ we take the extremal R-N black hole solution, as will be justified below. This ansatz is consistent with the physical situation under consideration. In general, the compact components of the metric depend on the non-compact coordinates x^μ, being some of the scalars of the $N = 2$ effective theory. More precisely, using complex notation, the components $g_{ij^\star}$ are related to the $h^{(1,1)}$ moduli parameterizing the deformations of the Kähler class while the g_{ij} $(g_{i^\star j^\star})$ ones are related to the $h^{(1,2)}$ moduli parameterizing the deformations of the complex structure. In Type IIB compactifications, as already stressed, such moduli belong to hyper and vector multiplets respectively. In our case, however, there are no vector multiplet scalars, that would couple non-minimally to the gauge fields (it is usually said that they "dress" the field strengths), and the hypermultiplet scalars can be set to zero since they do not couple to the unique gauge field in the problem, namely the graviphoton (therefore $g_{ab}(x, y) = g_{ab}(y)$).

The 5-form field strength can be generically decomposed in the basis of all the harmonic 3-forms of the CY manifold $\Omega^{(i,j)}$

$$F_{(5)}(x, y) = F^0_{(2)}(x) \wedge \Omega^{(3,0)}(y) + \sum_{k=1}^{h^{(2,1)}} F^k_{(2)}(x) \wedge \Omega^{(2,1)}_k(y) + \text{c.c.} \; . \tag{5.68}$$

In the case at hand, however, only the graviphoton $F^0_{(2)}$ appear in the general ansatz (5.68), without any additional vector multiplet field strength $F^k_{(2)}$, and conveniently normalizing one can take (from now on $F^0_{(2)} = F_{(2)}$)

$$F_{(5)}(x, y) = \frac{1}{\sqrt{2}} F_{(2)}(x) \wedge \left(\Omega^{(3,0)} + \bar{\Omega}^{(0,3)}\right) \; . \tag{5.69}$$

Notice that this same ansatz is the consistent one for any double-extreme solution even for a more generic CY (i.e. with $h^{(1,2)} \neq 0$).

With these ansätze, Eq. (5.64) reduces to the usual four-dimensional Einstein equation with a graviphoton source, the compact part being identically satisfied. The latter leads to a non-trivial consistency condition that our ansatz has to fulfill. Indeed, Eq. (5.64) taken with compact indices gives rise (after integration on the compact manifold) in general to various equations for the scalar fields. Indeed, the compact part of the ten-dimensional Ricci tensor R_{ab} is made of the CY Ricci tensor (that with our choice of the metric is zero by definition) plus mixed components (i.e. $R^{\mu}_{a\mu b}$) containing, in particular, kinetic terms of the scalars. The corresponding compact components of the energy-momentum tensor on the right hand side of the equation would represent coupling terms of the scalars with the gauge fields. In our case, however, these mixed components of R_{ab} are absent. Therefore, the complete ten-dimensional Ricci tensor vanishes ($R_{ab} = 0$) and self-consistency of the solution requires that also the complete energy-momentum tensor T_{ab} should vanish. This follows from our ansatz (5.69) as it is evident by doing an explicit computation. This conclusion can also be reached by observing that the kinetic term of the 4-form does not depend on g_{ab} when $g_{ij} = 0$, see Eq. (5.70) below.

The four-dimensional Lagrangian is obtained by carrying out explicitly the integration over the CY. Indeed, choosing the normalization of $\Omega^{(3,0)}$ and $\bar{\Omega}^{(0,3)}$ such that $\|\Omega^{(3,0)}\|^2 = V_3^2/V_{CY}$ (since the volume of the corresponding 3-cycle is precisely the volume V_3 of the wrapped 3-brane) one has ($z^a = (y^a + iy^{a+1})/\sqrt{2}$ and $d^6y = id^3z d^3\bar{z}$)

$$\int_{CY} d^6y \sqrt{g_{(6)}} = V_{CY} \; , \quad i \int_{CY} \Omega^{(3,0)} \wedge \bar{\Omega}^{(0,3)} = V_3^2 = \int_{CY} d^6y \sqrt{g_{(6)}} \left\|\Omega^{(3,0)}\right\|^2 \; . \tag{5.70}$$

In terms of $\kappa^2_{(4)} = \kappa^2_{(10)}/V_{CY}$ one then finds

$$\mathcal{S} = \frac{1}{2\kappa^2_{(4)}} \int d^4x \sqrt{g_{(4)}} \left(R_{(4)} - \frac{1}{2 \cdot 2!} \frac{V_3^2}{V_{CY}} F_{\mu\nu} F^{\mu\nu}\right) \; . \tag{5.71}$$

In the more general case corresponding to Eq. (5.68), the integration over the CY gives rise to a gauge field kinetic term of the standard form $\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}F^{\Sigma} + \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda *}F^{\Sigma}$, where $\Lambda, \Sigma = 0, 1, ..., h^{(1,2)}$. In our simpler case, there is only $F^0_{(2)} = F$ with $\mathrm{Im}\mathcal{N}_{00} = V_3^2/V_{CY}$. As well known, the four-dimensional Maxwell-Einstein equations of motion following from this Lagrangian admit the extremal R-N black hole solution (in coordinates in which the horizon is located at $r = 0$)

$$\begin{cases} g_{00} = -H(r)^{-2} \ , \quad g_{mn} = \delta_{mn}\, H(r)^2 \\ F_{m0} = \dfrac{\sqrt{V_{CY}}}{V_3} \cos\alpha\, \partial_m H(r)\, H(r)^{-2} \ , \quad F_{mn} = \dfrac{\sqrt{V_{CY}}}{V_3} \sin\alpha\, \epsilon_{mnp}\, \partial^p H(r) \end{cases} . \tag{5.72}$$

where $m, n, p = 1, 2, 3$ and $H(r) = 1 + 2\kappa^2_{(4)} M \Delta_{(3)}(r)$. Notice that the kinetic term of the gauge field A^μ is not canonically normalized, and therefore the effective charges appearing in scattering amplitude are rescaled by a factor $V_3/\sqrt{V_{CY}}$. Taking into account this fact, the couplings are

$$\hat{m} = \hat{M} \ , \quad \hat{e} = \frac{\hat{M}}{2}\cos\alpha \ , \quad \hat{g} = \frac{\hat{M}}{2}\sin\alpha \ . \tag{5.73}$$

and satisfy the extremality condition $\hat{m}^2 = (\hat{e}^2 + \hat{g}^2)/4$. As usual, hatted charges are expressed in inverse units of the effective coupling $\sqrt{2}\kappa_{(4)}$. The parameter $\hat{M}$ depends directly on the 3-brane tension $\hat{\mu}_3$ through the relation $\hat{M} = V_3/\sqrt{V_{CY}}\,\hat{\mu}_3$, and the arbitrary angle α depends on the way the 3-brane is wrapped on the CY. At the quantum level, the electric and magnetic charges $\hat{e}$ and $\hat{g}$ are quantized as a consequence of Dirac's condition $\hat{e}\hat{g} = 2\pi n$. Correspondingly, the angle α can take only discrete values and this turns out to be automatically implemented in the compactification, as seen in previous section.

5.3.3 The D3-brane wrapped on $T^6/\mathbb{Z}_3$ in string theory

The problem of describing curved D-branes, such as D-branes wrapped on a cycle of the internal manifold in a generic compactification of string theory, is in general too difficult to be solved. In fact, Polchinski's description of D-branes as hypersurfaces on which open strings can end relies on the possibility of implementing the corresponding boundary conditions in the CFT describing open string dynamics. Very little has been done for a generic target space compactification (for a recent discussion of this and related issues, see [173, 174]) but there exist special cases, such as orbifold compactifications, which capture all the essential features of more general situations, in which ordinary techniques can be applied.

The phase-shift computations of Chapter 4 lead to evidence that the D3-brane wrapped on $T^6/\mathbb{Z}_3$ represents a R-N black hole. Moreover, the results of the previous section clearly show that this black hole is actually dyonic. An equivalent but more direct way to see that this configuration indeed correctly fits the general solution R-N $\times$ CY discussed above, is to compute one-point functions $\langle\Psi\rangle = \langle\Psi|B\rangle$ of the massless fields of SUGRA and compare them to the linearized long range fields of the SUGRA R-N black hole solution (5.72). This second method presents the advantage of yielding direct informations on the couplings to the massless fields of the low energy theory.

Recall that the original ten-dimensional coordinates are organized as follows: the four non-compact directions x^0, x^1, x^2 and x^3 span $\mathbb{R}_4$, whereas the six compact directions x^a, x^{a+1}, $a = 4, 6, 8$, span $T^6/\mathbb{Z}_3$. The three T^2's composing T^6 are parameterized by the

3 pairs x^a, x^{a+1}, and the $\mathbb{Z}_3$ action is generated by $2\pi/3$ rotations in these planes. The boundary state $|B\rangle$ of the D3-brane wrapped on a generic $\mathbb{Z}_3$-invariant 3-cycle is obtained from the boundary state $|B_3(\theta_0)\rangle$ of a D3-brane in ten dimensions with N directions x^0 and $x'^a(\theta_0)$, where the $x'^a(\theta_0)$ directions form an arbitrary common angle θ_0 with the x^a directions in each of the 3 planes (x^a, x^{a+1}) (actually, we could have chosen 3 different angles in the 3 planes, but only their sum will be relevant, as could be inferred from Eq. (5.80) below). First, one projects onto the $\mathbb{Z}_3$-invariant part and then compactifies the directions x^a, x^{a+1}. The $\mathbb{Z}_3$ projection is implemented by applying the projector $P = 1/3(1 + g + g^2)$ on $|B_3(\theta_0)\rangle$, where $g = \exp\{i2\pi/3(J^{45} + J^{67} + J^{89})\}$ is the generator of the $\mathbb{Z}_3$ action and J^{aa+1} is the x^a, x^{a+1} component of the angular momentum operator. This yields

$$|B\rangle = \frac{1}{3} \sum_{\{\Delta\theta\}} |B_3(\theta = \Delta\theta + \theta_0)\rangle \ , \tag{5.74}$$

where the sum is over $\Delta\theta = 0, 2\pi/3, 4\pi/3$. It is obvious form this formula that $|B\rangle$ is a periodic function of the parameter θ_0 with period $2\pi/3$. Therefore, the physically distinct values of θ_0 are in $[0, 2\pi/3]$ and define a one parameter family of $\mathbb{Z}_3$-invariant boundary states, corresponding to all the possible harmonic 3-forms on $T^6/\mathbb{Z}_3$, as we will see. Recall from previous section that requiring a fixed finite volume V_3 for the 3-cycle on which the D3-brane is wrapped implies discrete values for θ_0. The compactification process restricts the momenta entering the Fourier decomposition of $|B\rangle$ to belong the momentum lattice of $T^6/\mathbb{Z}_3$. Since the massless supergraviton states $|\Psi\rangle$ carry only space time momentum, the compact part of the boundary state will contribute a volume factor which turns the ten-dimensional D3-brane tension $\hat{\mu}_3 = \sqrt{2\pi}$ into the four-dimensional black hole mass $\hat{M} = V_3/\sqrt{V_{CY}}\,\hat{\mu}_3$, and some trigonometric functions of θ_0 to be discussed below.

Using the technique described in Chapter 3 (see [130]), the relevant one-point functions on $|B_3(\theta)\rangle$ for the graviton and 4-form states $|h\rangle$ and $|A\rangle$ with polarization h^{MN} and A^{MNPQ}, are

$$\langle B_3(\theta)|h\rangle = -\hat{M}\, T\, h_{MN}\, M^{MN}(\theta) \ , \tag{5.75}$$

$$\langle B_3(\theta)|A\rangle = -\frac{\hat{M}}{8}\, T\, A_{MNPQ}\, M_{ab}(\theta)\, \Gamma^{MNPQ}_{ba} \ . \tag{5.76}$$

Here T is the total time. The numerical coefficients appearing in (5.75) have been chosen at our convenience by relying on the phase-shift computations of Chapter 4, where the relative normalization is easily fixed, as already discussed. The matrices $M(\theta) = \Sigma(\theta) M \Sigma^T(\theta)$ are obtained from the usual ones corresponding to Neumann boundary conditions along x^0, x^4, x^6 and x^8

$$M_{MN} = \mathrm{diag}(-1,-1,-1,-1,1,-1,1,-1,1,-1) \ , \ \ M_{ab} = \Gamma^{0468}_{ab} \ , \tag{5.77}$$

through a rotation of angle θ in the three planes (x^a, x^{a+1}), generated in the vector and spinor representations of each SO(2) subgroup of the rotation group SO(8) by

$$\Sigma_V(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \ , \ \ \Sigma_S(\theta) = \cos\frac{\theta}{2}\mathbb{1} - \sin\frac{\theta}{2}\Gamma^{aa+1} \ . \tag{5.78}$$

After some simple algebra, one finds

$$\langle B_3(\theta)|h\rangle = \hat{M}\,T\left\{h^{00}+h^{11}+h^{22}+h^{33}-\sum_a\left[\cos 2\theta\,\left(h^{aa}-h^{a+1a+1}\right)-2\sin 2\theta\,h^{aa+1}\right]\right\}\ ,$$

$$\begin{aligned}\langle B_3(\theta)|A\rangle = 2\hat{M}\,T\Big\{&\cos^3\theta\left(A^{0468}-A^{0479}-A^{0569}-A^{0578}\right)\\ &+\sin^3\theta\left(A^{0579}-A^{0568}-A^{0478}-A^{0469}\right)\\ &+\cos\theta\left(A^{0479}+A^{0569}+A^{0578}\right)+\sin\theta\left(A^{0568}+A^{0478}+A^{0469}\right)\Big\}\ .\end{aligned} \tag{5.79}$$

The one-point functions for the D3-brane wrapped on $T^6/\mathbb{Z}_3$ are then obtained by averaging over the allowed $\Delta\theta$'s: $\langle\Psi\rangle = 1/3\sum_{\{\Delta\theta\}}\langle B_3(\theta)|\Psi\rangle$. One easily finds the only non-vanishing averages of the trigonometric functions appearing in Eqs. (5.79) to be

$$\frac{1}{3}\sum_{\{\Delta\theta\}}\cos^3\theta = \frac{1}{4}\cos 3\theta_0\ ,\quad \frac{1}{3}\sum_{\{\Delta\theta\}}\sin^3\theta = -\frac{1}{4}\sin 3\theta_0\ , \tag{5.80}$$

so that finally, meaning now with h and A all the four-dimensional fields arising from the graviton and the 4-form respectively upon compactification,

$$\langle h\rangle = \hat{M}\,T\left(h^{00}+h^{11}+h^{22}+h^{33}\right)\ , \tag{5.81}$$

$$\langle A\rangle = \frac{\hat{M}}{2}T\left(\cos 3\theta_0 A^0 - \sin 3\theta_0 B^0\right)\ , \tag{5.82}$$

where we have defined the graviphoton fields

$$A^\mu = A^{\mu 468}-A^{\mu 479}-A^{\mu 569}-A^{\mu 578}\ , \tag{5.83}$$

$$B^\mu = A^{\mu 579}-A^{\mu 568}-A^{\mu 478}-A^{\mu 469}\ . \tag{5.84}$$

Using the self-duality of the 5-form field strength in ten dimension, one can easily derive that $F_B^{\mu\nu} = {}^*F_A^{\mu\nu}$, so that A^μ and B^μ are not independent fields, but rather magnetically dual. Using the A^μ field, we get the electric and magnetic charges

$$\hat{e} = \frac{\hat{M}}{2}\cos 3\theta_0\ ,\quad \hat{g} = \frac{\hat{M}}{2}\sin 3\theta_0\ , \tag{5.85}$$

or vice versa using the B^μ field. Comparing with Eqs. (5.73) one finds that $\alpha = 3\theta_0$ and therefore the ratio between e and g depends on the choice of the 3-cycle, as anticipated. Also, as explained, only discrete values of θ_0 naturally emerge requiring a finite volume. The identifications (5.85) are in agreement with the diagonal and off-diagonal phase-shifts found in the previous section between two of these configurations with different θ_0's, call them $\theta_0^{(1,2)}$. Indeed,

$$\mathcal{A}_{even} \sim \frac{\hat{M}^2}{4}\cos 3\left(\theta^{(1)}-\theta^{(2)}\right) = \hat{e}^{(1)}\hat{e}^{(2)}+\hat{g}^{(1)}\hat{g}^{(2)}\ , \tag{5.86}$$

$$\mathcal{A}_{odd} \sim \frac{\hat{M}^2}{4}\sin 3\left(\theta^{(1)}-\theta^{(2)}\right) = \hat{e}^{(1)}\hat{g}^{(2)}-\hat{g}^{(1)}\hat{e}^{(2)}\ . \tag{5.87}$$

Notice that all the compact components h^{ab} of the graviton have canceled in (5.81), reflecting the fact the black hole has no scalar hairs. Moreover, the one-point function (5.82) of the RR 4-form is precisely of the form of our ansatz (5.69), with the unique holomorphic and antiholomorphic 3-forms $\Omega^{(3,0)}$ and $\bar{\Omega}^{(0,3)}$ showing up. Indeed

$$\Omega^{(3,0)} = \Omega\, dz^4 \wedge dz^6 \wedge dz^8 \ , \quad \bar{\Omega}^{(0,3)} = \Omega^*\, d\bar{z}^4 \wedge d\bar{z}^6 \wedge d\bar{z}^8 \ , \tag{5.88}$$

so that the real 3-form appearing in (5.69) is given by

$$\Omega^{(3,0)} + \bar{\Omega}^{(0,3)} = \mathrm{Re}\Omega \left(\omega^{468} - \omega^{479} - \omega^{569} - \omega^{578}\right) + \mathrm{Im}\Omega \left(\omega^{579} - \omega^{568} - \omega^{478} - \omega^{469}\right) \tag{5.89}$$

where $\omega^{abc} = 1/\sqrt{2}\, dy^a \wedge dy^b \wedge dy^c$. The precise correspondence between the boundary state result (5.82) and the purely geometric identity (5.89) is then evident. The combination of components of the 4-form appearing in (5.82) is proportional to the integral over the D3-brane world-volume W_{1+3}

$$\langle A \rangle = \frac{\hat{\mu}_3}{2} \mathrm{Re} \int_{W_{1+3}} (A + iB) \wedge \Omega^{(3,0)} = \int_{W_1} (\hat{e}A + \hat{g}B) \ . \tag{5.90}$$

This formula yields an interesting relation between the parameters $\hat{\mu}_3, \hat{M}, \theta_0$ and the complex component Ω in (5.88) defining the 3-cycle. One gets $\Omega = (\hat{M}/\hat{\mu}_3)\exp\{-i3\theta_0\}$. Notice that one correctly recovers $|\Omega| = V_3/\sqrt{V_{CY}}$, the arbitrary phase being the sum of the arbitrary overall angles θ_0 appearing in the boundary state construction. Finally, dropping the overall time T, inserting a propagator $\Delta = 1/\vec{q}^2$ and Fourier transforming Eqs. (5.81) with the identification (5.90), one recovers the asymptotic gravitational and electromagnetic fields of the R-N black hole, Eqs. (5.72).

This definitively confirms that our boundary state describes a D3-brane wrapped on $T^6/\mathbb{Z}_3$, falling in the class of regular four-dimensional R-N extremal black holes obtained by wrapping the self-dual D3-brane on a generic CY threefold. This boundary state encodes the leading order couplings to the massless fields of the theory, and allows the direct determination of their long range components, falling off like $1/r$ in four dimensions. The sub-leading post-Newtonian corrections to these fields arise instead as open string higher loop corrections, corresponding to string world-sheets with more boundaries. From a classical field theory point of view, this is the standard replica of the source in the tree-level perturbative evaluation of a non-linear theory. In a series expansion for $r \to \infty$, a generic term going like $1/r^l$ comes from a diagram with l open string loops, that is l branches of a tree-level closed string graph (each branch brings an integration over the transverse 3-momentum, two propagators and a SUGRA vertex involving two powers of momentum, yielding an overall contribution of dimension $1/r$).

As pointed out by the authors of [162], heuristically speaking the reason why single D-brane black holes are non-singular in CY compactifications, as opposed to the toroidal case, is that the brane is wrapped on a topologically non-trivial manifold and therefore can intersect with itself. This intersection mimics the actual intersection of different D-branes necesary in toroidal compactifications to get a non-singular solution. In our case, such analogy is particularly manifest since the boundary state $\mathbb{Z}_3$-invariant projection (5.74) could be seen as a three D3-branes superposition at $(2\pi/3)$ angles in a T^6 compactification. As illustrated in [175, 176] such intersection would preserve precisely 1/8 of the supersymmetry, as a single D3-brane does on $T^6/\mathbb{Z}_3$. For toroidal compactification this is not enough, because at least four intersecting D3-branes are needed in order to get a regular solution [157, 158]. Finally, since this extremal R-N configuration is constructed with a single D3-brane, it naturally arises the question of understanding the microscopic origin of its entropy.

Chapter 6

Spin effects in D-brane dynamics

In this chapter, we study the spin-dependence in Dp-Dp and Dp-D(p+4) dynamics using the the boundary state formalism in the Green-Schwarz (G-S) formulation of superstring theory, by applying broken supersymmetry transformations to the usual scalar boundary state. We focus on the leading terms for small velocities v, which are found to behave as v^{4-n}/r^{7-p+n} and v^{2-n}/r^{3-p+n} for Dp-Dp and the Dp-D(p+4). These interactions receive contributions only from BPS intermediate states, massive states contributions canceling as a consequence of the residual supersymmetry. This implies the scale-invariance of these leading spin-effects, supporting the equivalence between their matrix model and supergravity descriptions. We give also a field theory interpretation of our results, that allows in particular to deduce the gyromagnetic ratio $g = 1$ and its quadrupole analog $\tilde{g} = 1$. We follow [103] and especially [104].

6.1 Boundary states in the G-S formalism

In this section we shall review the boundary state formalism in the G-S formulation of superstring theory [177, 178, 179] (see also [180]) and construct the boundary state for a generic spinning Dp-brane.

Consider the Type II theory in the light-cone gauge, and concentrate for instance on the IIB chiral version for which the notation is somewhat friendlier. One has $X^+ = x^+ + p^+\tau$ whereas X^- is completely fixed in terms of the transverse fields and after fixing the κ-symmetry, one is left with two left and right spinors S^a and $\tilde{S}^a$, in the $\mathbf{8_s}$ representation of SO(8). The Fock space is constructed by applying negative frequency creation operators to a vacuum representing the Clifford algebra of the fermionic z.m. S_0^a and $\tilde{S}_0^a$. The representation is $\mathbf{8_v} \oplus \mathbf{8_c}$ both for the left and right parts, and the fermionic z.m. acts as SO(8) γ-matrices

$$S_0^a|i\rangle = \frac{1}{\sqrt{2}}\gamma^i_{a\dot{a}}|\dot{a}\rangle \ , \ \ S_0^a|\dot{a}\rangle = \frac{1}{\sqrt{2}}\gamma^i_{a\dot{a}}|i\rangle \ , \tag{6.1}$$

$$\tilde{S}_0^a|\tilde{i}\rangle = \frac{1}{\sqrt{2}}\gamma^i_{a\dot{a}}|\tilde{\dot{a}}\rangle \ , \ \ \tilde{S}_0^a|\tilde{\dot{a}}\rangle = \frac{1}{\sqrt{2}}\gamma^i_{a\dot{a}}|\tilde{i}\rangle \ . \tag{6.2}$$

Recall finally that the 32 supersymmetry charges of the theory in absence of D-branes are directly related to the fermion fields S^a and $\tilde{S}^a$ playing the role of the spin-fields of the covariant RNS formulation, and are given by

$$Q^a = \sqrt{2p^+}\oint d\sigma S^a \ , \ Q^{\dot{a}} = \frac{1}{\sqrt{p^+}}\gamma^i_{\dot{a}a}\oint d\sigma \partial X^i S^a \ , \tag{6.3}$$

$$\tilde{Q}^a = \sqrt{2p^+}\oint d\sigma \tilde{S}^a \ , \ \tilde{Q}^{\dot{a}} = \frac{1}{\sqrt{p^+}}\gamma^i_{\dot{a}a}\oint d\sigma \bar{\partial} X^i \tilde{S}^a \ . \tag{6.4}$$

and satisfy the N=2 supersymmetry algebra.

The fields in the $\pm$ light-cone directions automatically satisfy D b.c. due to the light-cone gauge [178, 179], whereas the b.c. for the fields in the $i = 1, 2, ..., 8$ transverse directions can be chosen freely. It is therefore possible to define a configuration similar but not quite identical to a Dp-brane by choosing N b.c. for the directions $\mu = 1, 2, ..., p+1$ and D b.c. for the directions $I = p+2, ..., 8-p$. In this way one obtains the right number of N and D directions, but the 0 directions is D, so that "time" is temporarily identified with one of the transverse N directions, say the 1 direction. In order to recover the usual covariant description with the 0 direction as time, it will be sufficient to perform the double analytic continuation $0 \leftrightarrow i\,1$ in the final results.

The boundary state describing a Dp-brane configuration in the sense described above is defined as the eigenstate of appropriate b.c. for the bosonic and fermionic fields. The bosonic b.c. are the conventional N,D ones, and are chosen as discussed above. The fermionic b.c. are then unambiguously determined by the requirement that the boundary states must preserve a combination of left and right supersymmetries, that is 1/2 of the original 32. Let us therefore introduce the following generic combinations of left and right supercharges

$$Q^a_\pm = \frac{1}{\sqrt{2}}\left(Q^a \pm iM_{ab}\tilde{Q}^b\right) \ , \tag{6.5}$$

$$Q^{\dot{a}}_\pm = \frac{1}{\sqrt{2}}\left(Q^{\dot{a}} \pm iM_{\dot{a}\dot{b}}\tilde{Q}^{\dot{b}}\right) \ , \tag{6.6}$$

acting as creation-annihilation operators with the algebra

$$\left\{Q^a_+, Q^b_-\right\} = 2p^+\delta^{ab} \ , \ \left\{Q^{\dot{a}}_+, Q^{\dot{b}}_-\right\} = P^-\delta^{\dot{a}\dot{b}} \ , \tag{6.7}$$

$$\left\{Q^a_+, Q^{\dot{a}}_-\right\} = \frac{1}{\sqrt{2}}\left[\gamma^i_{a\dot{a}}p^i + (M\gamma^i M^T)_{a\dot{a}}\tilde{p}^i\right] \ . \tag{6.8}$$

We then impose the following BPS conditions on the boundary state

$$Q^a_+|B\rangle = 0 \ , \ Q^{\dot{a}}_+|B\rangle = 0 \ \Rightarrow \ Q^a_+, Q^{\dot{a}}_+ \text{ unbroken} \ , \tag{6.9}$$

$$Q^a_-|B\rangle \neq 0 \ , \ Q^{\dot{a}}_-|B\rangle \neq 0 \ \Rightarrow \ Q^a_-, Q^{\dot{a}}_- \text{ broken} \ . \tag{6.10}$$

The bosonic b.c. imply

$$(\alpha^i_n + M_{ij}\tilde{\alpha}^j_{-n})|B\rangle = 0 \ , \tag{6.11}$$

where

$$M_{ij} = \begin{pmatrix} -1\!\!1_{p+1} & 0 \\ 0 & 1\!\!1_{7-p} \end{pmatrix} \ . \tag{6.12}$$

For the fermionic b.c., we make the reasonable ansatz

$$(S^a_n + iM_{ab}\tilde{S}^b_{-n})|B\rangle = 0 \ . \tag{6.13}$$

involving the same matrix appearing also in the broken and unbroken supercharge combinations. Consistency with the BPS conditions (6.9) and (6.10) then implies the orthogonality and triality conditions

$$(MM^T)_{ab} = \delta_{ab} \ , \ \ (M\gamma^i M^T)_{a\dot{a}} = M_{ij}\gamma^j_{a\dot{a}} \ , \tag{6.14}$$

which yield finally the solutions

$$M_{ab} = (\gamma^1\gamma^2...\gamma^{p+1})_{ab} \ , \ \ M_{\dot{a}\dot{b}} = (\gamma^1\gamma^2...\gamma^{p+1})_{\dot{a}\dot{b}} \ . \tag{6.15}$$

The solution for the boundary state $|B\rangle$ is obtained through the Bogoliubov transformation

$$|B\rangle = \exp\sum_{n>0}\left(\frac{1}{n}M_{ij}\alpha^i_{-n}\tilde{\alpha}^j_{-n} - iM_{ab}S^a_{-n}\tilde{S}^b_{-n}\right)|B_0\rangle \tag{6.16}$$

from the z.m. part $|B_0\rangle$ implementing the b.c. on the fermionic z.m., which is given by

$$|B_0\rangle = M_{ij}|i\rangle|\tilde{j}\rangle - iM_{\dot{a}\dot{b}}|\dot{a}\rangle|\tilde{\dot{b}}\rangle \ . \tag{6.17}$$

Finally, the localized configuration space boundary state is as usual a superposition of D momentum eigenstates

$$\begin{aligned} |B,\vec{x}\rangle &= (2\pi\sqrt{\alpha'})^{4-p}\delta^{(9-p)}(\vec{x}-\vec{Y})|B\rangle\otimes|\vec{0}\rangle \\ &= (2\pi\sqrt{\alpha'})^{4-p}\int\frac{d^{9-p}q}{(2\pi)^{9-p}}\,e^{i\vec{q}\cdot\vec{Y}}\,|B\rangle\otimes|\vec{q}\rangle \ . \end{aligned} \tag{6.18}$$

Being BPS states, Dp-branes fill supermultiplets realizing the broken half of the supersymmetries. By performing an arbitrary broken supersymmetry transformation to the above scalar boundary state, one can obtain informations one the couplings of every component of this Dp-brane supermultiplet. In particular, D0-branes fill a short-multiplet containing $2^8 = 256$ components grouped in the **44**⊕**84**⊕**128** representations of the little group SO(9) for massive states, with "spin" 1, 3/2 and 2, which is precisely the Kaluza-Klein reduction of the mass gravitational multiplet of D=11 SUGRA, with $2^8 = 256$ components grouped into the same representations of the little group SO(9), now for massless states.

In the formalism of previous section, the boundary state represents the semiclassical source formed by an "in" and an "out" Dp-brane. Its overlap $\langle B|\Psi\rangle$ with a string state $|\Psi\rangle$ represents therefore semiclassical 3-point functions as shown in Fig. 6.1. The generic state obtained by applying a generic supersymmetry transformation to the scalar one is

$$|B,\eta\rangle = e^{\eta Q^-}|B\rangle = \sum_{m=0}^{16}\frac{1}{m!}(\eta Q^-)^m|B\rangle \ . \tag{6.19}$$

where we have used the SO(9) notation $\eta = (\eta_a, \tilde{\eta}_{\dot{a}})$ and $Q^- = (Q^-_a, Q^-_{\dot{a}})$. The free fermionic number η parameterizes all the possible semiclassical currents obtained by choosing arbitrarily the incoming and outcomming Dp-branes among the 2^8 components of the Dp-brane multiplet. Notice that in principle there could be 2^{16} possibilities, but restricting to linearly realized supersymmetries corresponding to the z.m. part of the supercharges, there actually only 2^8, as we shall see. The sum in Eq. (6.19) is a generalized multipole expansion in powers of the fermionic number η carrying the dimensions $M^{-1/2}$. Terms with a even or odd number of Q^- are relevant for globally bosonic or fermionic currents, coupling to bosonic or fermionic closed string states Ψ_B and Ψ_F respectively. The situation is similar also for p-brane solutions of Type II SUGRAs. Indeed, the p-brane solution has a Killing spinor corresponding to the unbroken supersymmetries, and zero modes, corresponding to the broken supersymmetries. These are related to orthogonal projections of the supersymmetry parameter η.

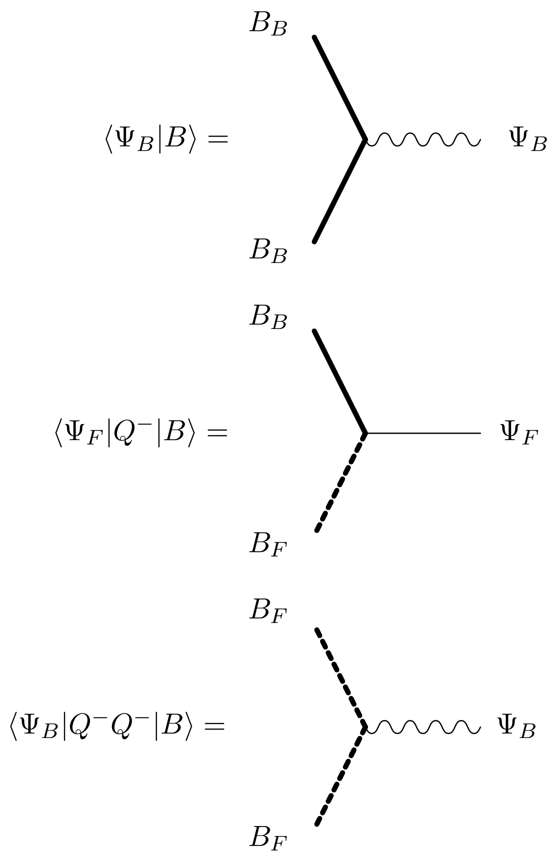


Figure 6.1:

The generic boundary state (6.19) encodes all two-brane one-particle couplings, relevant in a general inelastic scattering of Dp-branes which can change their spin. For elastic scatterings, to which we will limit our attention here, only those terms with an even power of Q^- are relevant, and the current will be automatically bosonic. Moreover, each pair of supercharges gives, in light-cone notation,

$$(\eta Q^-)^2 = (\eta_a Q^-_a + \tilde{\eta}_{\dot{a}} Q^-_{\dot{a}})^2 = \eta_a \eta_b Q^-_a Q^-_b + \tilde{\eta}_{\dot{a}} \tilde{\eta}_{\dot{b}} Q^-_{\dot{a}} Q^-_{\dot{b}} + 2 \eta_a \tilde{\eta}_{\dot{b}} Q^-_a Q^-_{\dot{b}} \ . \tag{6.20}$$

Each of the three distinct factors has the dimension of a momentum and is proportional respectively to p^+, $p^- = \vec{p}^2/p^+$ and $\vec{p}$. Clearly, this corresponds precisely to the decomposition of scalar products into light-cone components, and the role of the first two contributions is simply to supply appropriate terms that, when added to the third contribution corresponding to the $SO(8)$ part of the scalar products, will reconstruct the complete $SO(9,1)$ scalar products. We have checked that this indeed always happens to work perfectly. For simplicity, in the following we will simply omit to write the first two kind of terms and focus on the third kind, since this will be sufficient to clearly fix the complete result. Before going on, let us further comment on the analogy of the expansion in Eq. (6.19) and a multipole expansion, which is particularly clear in the special case we are considering here. Indeed, for a bosonic current associated to an "elastic" current, the term with 2n supercharges will produce, when acting on the boundary state, components with n powers of the D momentum q, which are the momentum space representation of an n-derivative n-pole coupling. In field theory, this corresponds to the expansion of the source in powers of the transfered momentum q.

Consider therefore the operator

$$V_\eta = \eta_a \tilde{\eta}_{\dot{a}} Q^-_a Q^-_{\dot{a}} \ . \tag{6.21}$$

When applied n times to the scalar boundary state $|B\rangle$, corresponding to 2n supersymmetry transformations, it produce the SO(8) part of n-pole term in the bosonic Dp-brane current

$$|B\rangle_{(n)} = V_\eta^n |B\rangle \ . \tag{6.22}$$

In total, the SO(8) part of the boundary state $|B,\eta\rangle$ describing the whole current is

$$|B,\eta\rangle = \sum_{n=0}^{8} \binom{2n}{n} \frac{V_\eta^n}{(2n)!} |B\rangle = \sum_{n=0}^{8} \frac{1}{(n!)^2} |B\rangle_{(n)} \ . \tag{6.23}$$

Consider now the action of the z.m. part $V_{\eta 0}$ of V_η, which will be relevant in the following. After simple algebra one finds

$$|B_0\rangle_{(n)} = V_{\eta 0}^n |B_0\rangle = q_{i_1}...q_{i_n} \left[\eta_{[a_1}(\tilde{\eta}\gamma^{i_1})_{a_2}...\eta_{a_{2n-1}}(\tilde{\eta}\gamma^{i_n})_{a_{2n}]}\right] S_0^{-a_1}...S_0^{-a_{2n}}|B_0\rangle \ , \tag{6.24}$$

where $S_0^{\pm a} = (S_0^a \pm i M_{ab} \tilde{S}_0^b)/\sqrt{2}$ and satisfy $\{S_0^{\pm a}, S_0^{\mp b}\} = \delta^{ab}$. Since $\{S_0^{-a}, S_0^{-b}\} = 0$, it follows that $V_{\eta 0}^n \neq 0$ only for $n \leq 4$, corresponding to the fact that there are 2^8 and not 2^{16} different currents at the linearized level. Using the b.c. implemented by the boundary state $|B_0\rangle$ and the antisymmetry of the factor in [...], each S_0^- can be converted into $\sqrt{2} S_0$, in terms of the sole right-moving fermionic z.m.. Furthermore, the z.m. S_0 satisfy the following Fiertz identity (which is essentially a decomposition into commutator and anticommutator)

$$S_0^a S_0^b = \frac{1}{2}\delta^{ab} + \frac{1}{4}\gamma^{ij}_{ab} R_0^{ij} \ , \tag{6.25}$$

in terms of the SO(8) generators

$$R_0^{ij} = \frac{1}{4} S_0^a \gamma^{ij}_{ab} S_0^b \ . \tag{6.26}$$

Using this property, the effective form of $V_{\eta 0}^n$ acting on $|B_0\rangle$ is found to be

$$V_{\eta 0}^n = q_{i_1}...q_{i_n}\, \omega^{i_1...i_n}_{j_1...j_{2n}}(\eta) R_0^{j_1 j_2}...R_0^{j_{2n-1}j_{2n}} \ , \tag{6.27}$$

with

$$\omega^{i_1...i_n}_{j_1...j_{2n}}(\eta) = \frac{1}{2^n}\left[\eta_{[a_1}(\tilde{\eta}\gamma^{i_1})_{a_2}...\eta_{a_{2n-1}}(\tilde{\eta}\gamma^{i_n})_{a_{2n}]}\right]\gamma^{j_1j_2}_{a_1a_2}...\gamma^{j_{2n-1}j_{2n}}_{a_{2n-1}a_{2n}} \qquad (6.28)$$

encoding the dependence on the supersymmetry parameter.

Finally, using the explicit form of the action of the generators R_0^{ij} in the $\mathbf{8_v}$ and $\mathbf{8_c}$ representations,

$$R_0^{mn}|i\rangle = (\delta^{ni}\delta^{mj} - \delta^{mi}\delta^{nj})|j\rangle \; , \qquad (6.29)$$

$$R_0^{mn}|\dot{a}\rangle = -\frac{1}{2}\gamma^{mn}_{\dot{a}\dot{b}}|\dot{b}\rangle \; , \qquad (6.30)$$

the boundary state can be written in the standard form

$$|B_0\rangle_{(n)} = M^{(n)}_{ij}|i\rangle|\tilde{j}\rangle - iM^{(n)}_{\dot{a}\dot{b}}|\dot{a}\rangle|\tilde{\dot{b}}\rangle \; , \qquad (6.31)$$

in terms of the matrices

$$M^{(n)}_{ij} = 2^n\, q_{i_1}...q_{i_n}\,\omega^{i_1...i_n}_{ik_1k_1...k_{n-1}k_{n-1}k_n}(\eta)M_{k_nj} \; , \qquad (6.32)$$

$$M^{(n)}_{\dot{a}\dot{b}} = \frac{1}{2^n}\, q_{i_1}...q_{i_n}\,\omega^{i_1...i_n}_{j_1...j_{2n}}(\eta)(\gamma^{j_1j_2}...\gamma^{j_{2n-1}j_{2n}}M)_{\dot{a}\dot{b}} \; . \qquad (6.33)$$

For the oscillator part, one can proceed in a similar way. However, the algebra becomes more heavy and since we we will use only the z.m. part, we do not discuss it.

It is straightforward to generalize the above construction two a bosonic current corresponding to Dp-branes moving with a constant velocity $v = \tanh\pi\epsilon$. The corresponding boundary state is obtained by applying a simple Lorentz transformation to the static one. Assuming that before the analytic continuation the “time” is identified with the 1 direction and the velocity is along the 8 direction, the boundary state for the “moving” Dp-brane is actually obtained through the rotation $|B,\eta,\epsilon\rangle = \exp\{-i\pi\epsilon J^{18}\}|B,\eta\rangle$ The z.m. part of the angular momentum operator is $J_0^{ij} = x^ip^j - x^jp^i - 2iR_0^{ij}$. The bosonic part changes the momentum spectrum of the boundary state, whereas the fermionic part acts directly on $|B_0\rangle_{(n)}$, with the net effect of rotating the matrices (6.32) and (6.33) appearing in the static boundary state (6.31) as $M^{(n)} \to M^{(n)}(\epsilon) = \Sigma(\epsilon)M^{(n)}\Sigma^T(\epsilon)$, where $\Sigma(\epsilon)$ is the appropriate representation of the SO(8) rotation, that is

$$\Sigma_V(\epsilon) = \begin{pmatrix} \cos\pi\epsilon & 0 & -\sin\pi\epsilon \\ 0 & 1\!\!1_6 & 0 \\ \sin\pi\epsilon & 0 & \cos\pi\epsilon \end{pmatrix} \; , \quad \Sigma_S(\epsilon) = \cos\frac{\pi\epsilon}{2}\,1\!\!1 - \sin\frac{\pi\epsilon}{2}\,\gamma^{18} \; . \qquad (6.34)$$

Again, for the oscillators one could proceed in the same way.

In principle, since the angular momentum operator is quadratic in the string modes, one could find explicitly the moving boundary state and work exactly in the rapidity ϵ. However, this leads to heavy algebra and masks the extremely important role of supersymmetry in the cancellations which occur in the computation of interactions. Rather, in the following we shall proceed perturbatively in the rapidity, expanding the boost operator in power series for $\epsilon \to 0$. The corresponding vertex operator is simply

$$V_\epsilon = -i\pi\epsilon J^{18} \; , \qquad (6.35)$$

whose fermionic z.m. part is

$$V_{\epsilon 0} = -2\pi\epsilon R_0^{18} \ . \tag{6.36}$$

In this way, the exact result for the boundary state $|B,\eta,\epsilon\rangle$ describing a current of Dp-branes with supersymmetry parameter η and moving with rapidity ϵ is given by the infinite series

$$|B,\eta,\epsilon\rangle = \sum_{m=0}^{\infty} \frac{V_\epsilon^m}{m!}|B,\eta\rangle = \sum_{n=0}^{8}\sum_{m=0}^{\infty} \frac{V_\eta^n V_\epsilon^m}{(n!)^2 m!}|B\rangle \ , \tag{6.37}$$

in terms of the static scalar boundary state $|B\rangle$.

6.2 One-point functions

The first important information that one can extract form the boundary state constructed in previous section is on the spin-dependent non-minimal couplings of the Dp-brane current to closed string states. In particular, we shall compute the one-point functions $\Psi_{(n)} = \langle\Psi|B_0\rangle_{(n)}$ of all the massless closed string fields $|\Psi\rangle$, in order to extract the new couplings occurring at each multipole order. Inserting a propagator, one can also compute the asymptotic fields of the corresponding p-brane solution of SUGRA, obtaining the complete dependence on the spin.

Recall that in the G-S formulation of the Type II superstring, the massless bosonic states (in the covariant language) are written as

$$|\Psi_{NSNS}\rangle = \xi_{mn}|m\rangle|\tilde{n}\rangle \ , \ \ \xi_{mn} \sim \delta_{mn}\,\phi + g_{mn} + b_{mn} \ , \tag{6.38}$$

$$|\Psi_{RR}\rangle = C_{\dot{a}\dot{b}}|\dot{a}\rangle|\tilde{\dot{b}}\rangle \ , \ \ C_{\dot{a}\dot{b}} \sim \sum_k \frac{1}{k!} C^{(k)}_{m_1\ldots m_k}\gamma^{m_1\ldots m_k}_{\dot{a}\dot{b}} \ . \tag{6.39}$$

Apart form normalizations, that we shall disregard in this section, one finds

$$\Psi_{(n)}^{NSNS} = q_{i_1}\ldots q_{i_n}\,\xi^{ij}\omega^{i_1\ldots i_n}_{ik_1k_1\ldots k_{n-1}k_{n-1}k_n}(\eta)M_{k_nj} \ , \tag{6.40}$$

$$\Psi_{(n)}^{RR} = q_{i_1}\ldots q_{i_n}\sum_k \frac{1}{k!}C^{(k)}_{m_1\ldots m_k}\omega^{i_1\ldots i_n}_{j_1\ldots j_{2n}}(\eta)\mathrm{Tr}_S[\gamma^{m_1\ldots m_k}\gamma^{j_1j_2}\ldots\gamma^{j_{2n-1}j_{2n}}M] \ . \tag{6.41}$$

These expressions encode all the coupling to RR and NSNS states (again in the covariant language), organized in a multipole expansion, $n = 0,1,..,4$. At the n-th multipole order, there are n power of the transfered momentum q^i which, upon Fourier transforming, will become n derivatives, reflecting a non-minimal coupling. Denoting N indices with $\mu,\nu,\ldots$ and D indices with $I,J,\ldots$, and using the symmetry properties of the tensor $\omega^{i_1\ldots i_n}_{j_1\ldots j_{2n}}(\eta)$ entering the boundary state, one finds the following couplings

$$\Psi_{(n)}^{NSNS} \ \Rightarrow \ \begin{cases} \phi, g_{\mu\nu}, g_{IJ}, b_{\mu I} \ , & n \text{ even} \\ g_{\mu I}, b_{\mu\nu}, b_{IJ} \ , & n \text{ odd} \end{cases} \ , \tag{6.42}$$

$$\Psi_{(n)}^{RR} \ \Rightarrow \ C_{(k)} \ , \ \ k = p+1-2n,\ldots,p+1+2n \ . \tag{6.43}$$

Let us consider more in detail each case separately.

n=0 : universal coupling

The n=0 boundary state encodes the usual universal couplings which are independent of the spin of the Dp-brane and are therefore the same for each component of the Dp-brane supermultiplet. One finds

$$\Psi_{(0)}^{NSNS} = \xi_{ij} M^{ij} \ , \tag{6.44}$$

$$\Psi_{(0)}^{RR} = \sum_k \frac{1}{k!} C^{(k)}_{m_1 \ldots m_k} \mathrm{Tr}_S[\gamma^{m_1 \ldots m_k} M] \ . \tag{6.45}$$

These SO(8) expressions can be covariantized by performing the double analytic continuation described in previous section and generalizing the SO(8) matrices M^{ij} and M_{ab} to the SO(9,1) ones $M^{\mu\nu}$, with -1 entry in each N direction and $+1$ entry in each D direction, and $\mathcal{M} = \Gamma^0 \ldots \Gamma^p$. One finds simply

$$\Psi_{(0)}^{NSNS} = \xi_{\mu\nu} M^{\mu\nu} \ , \tag{6.46}$$

$$\Psi_{(0)}^{RR} = \sum_k \frac{1}{k!} C^{(k)}_{\mu_1 \ldots \mu_k} \mathrm{Tr}_S[\gamma^{\mu_1 \ldots \mu_k} \mathcal{M}] \ . \tag{6.47}$$

These expressions lead to the usual couplings to the dilaton and the graviton in the NSNS sector and to the (p+1)-form in the RR.

n=1 : dipole coupling

The n=1 boundary state encodes the dipole couplings which depend directly on the spin of the Dp-brane and are therefore different for each component of the Dp-brane supermultiplet. One finds

$$\Psi_{(1)}^{NSNS} = \xi_{ik} M^k{}_j \, (\eta \gamma^{ijl} \tilde{\eta}) \, q_l \ , \tag{6.48}$$

$$\Psi_{(1)}^{RR} = \sum_k \frac{1}{k!} C^{(k)}_{m_1 \ldots m_k} \mathrm{Tr}_S[\gamma^{m_1 \ldots m_k} \gamma_{ij} M] \, (\eta \gamma^{ijl} \tilde{\eta}) \, q_l \ . \tag{6.49}$$

In order to covariantize these expression, we need to introduce an SO(9,1) Majorana-Weyl supersymmetry parameter ψ, which in a chiral representation is given by $\psi = \binom{\eta}{0}$ with $\eta = \binom{\eta_a}{\tilde{\eta}_{\dot{a}}}$. Defining

$$J^{\mu\nu\rho} = \bar{\psi} \Gamma^{\mu\nu\rho} \psi \ , \tag{6.50}$$

the covariant expression is found to be

$$\Psi_{(1)}^{NSNS} = \xi_{\mu\sigma} M^\sigma{}_\nu \, J^{\mu\nu\rho} \, q_\rho \ , \tag{6.51}$$

$$\Psi_{(1)}^{RR} = \sum_k \frac{1}{k!} C^{(k)}_{\mu_1 \ldots \mu_k} \mathrm{Tr}_S[\Gamma^{\mu_1 \ldots \mu_k} \Gamma_{\mu\nu} \mathcal{M}] \, J^{\mu\nu\rho} \, q_\rho \ . \tag{6.52}$$

These expressions lead to non-minimal couplings to various NSNS and RR fields depending on the D-brane.

n=2 : quadrupole coupling

The n=2 boundary state encodes the quadrupole couplings. Proceeding as before, one finds expressions which can again be covariantized quite easily, obtaining

$$\Psi^{NSNS}_{(2)} = \xi_{\mu\sigma} M^{\sigma}{}_{\nu} \, J^{\mu\rho\alpha} J^{\nu}{}_{\rho}{}^{\beta} \, q_\alpha q_\beta \,, \tag{6.53}$$

$$\Psi^{RR}_{(2)} = \sum_k \frac{1}{k!} C^{(k)}_{\mu_1 \ldots \mu_k} \mathrm{Tr}_S[\Gamma^{\mu_1 \ldots \mu_k} \Gamma_{\nu_1\nu_2} \Gamma_{\nu_3\nu_4} \mathcal{M}] \, J^{\nu_1\nu_2\alpha} J^{\nu_3\nu_4\beta} \, q_\alpha q_\beta \,. \tag{6.54}$$

n=n : n-pole coupling

Looking at the previous expressions for n=0, 1, 2, it is easy to guess the result for generic n. Indeed, defining the fermionic bilinears

$$K^{\mu\nu}(q) = J^{\mu\nu\rho} \, q_\rho \;, \;\; \not{K}(q) = J^{\mu\nu\rho} \Gamma_{\mu\nu} \, q_\rho \;, \tag{6.55}$$

one finds

$$\begin{aligned} \Psi^{NSNS}_{(n)} &= \mathrm{Tr}_V[\xi M K^n(q)] \\ &= \xi_{\mu\sigma} M^{\sigma}{}_{\nu} K^{\mu}{}_{\alpha_1}(q) K^{\alpha_1}{}_{\alpha_2}(q) \ldots K^{\alpha_n}{}_{\nu}(q) \;, \end{aligned} \tag{6.56}$$

$$\begin{aligned} \Psi^{RR}_{(n)} &= \mathrm{Tr}_S[C \mathcal{M} \not{K}^n(q)] \\ &= \sum_k \frac{1}{k!} C^{(k)}_{\mu_1 \ldots \mu_k} \mathrm{Tr}_S[\Gamma^{\mu_1 \ldots \mu_k} \Gamma_{\nu_1\nu_2} \ldots \Gamma_{\nu_{2n-1}\nu_{2n}} \mathcal{M}] K^{\nu_1\nu_2}(q) \ldots K^{\nu_{2n-1}\nu_{2n}}(q) \,. \end{aligned} \tag{6.57}$$

Notice that the tensor structure is unique, due the Fiertz identity $K_{\mu\nu}(q) K^{\mu\nu}(q) = 0$.

This concludes our analysis of the spin-dependent one-point functions. The asymptotic fields for the complete p-brane solution can be obtained simply by inserting a propagator. The correct normalizations can be extracted very efficiently from the interactions amplitude that we shall discuss below.

6.3 Leading interactions and spin effects

A second important application of the boundary state that we have constructed is the computation of the phase-shift for a generic D-brane scattering, yielding the complete spin-dependent interaction potential between D-branes. We will work perturbatively in the rapidity, showing that the leading non-relativistic terms of each n-pole interaction are determined by the sole fermionic z.m. and are therefore *scale-invariant*. We focus on the Dp-Dp and Dp-D(p+4) systems preserving 16 and 8 supercharges respectively.

6.3.1 Dp-Dp system

Consider the usual system of two parallel Dp-branes moving with rapidities ϵ_i and supersymmetry parameters η_i. The phase-shift is

$$\mathcal{A}_{p,p} = \frac{1}{16} \int_0^\infty dt \, \langle B_p, \eta_1, \epsilon_1, \vec{Y}_1 | e^{-2\pi\alpha' t p^+ (P^- - p^-)} | B_p, \eta_2, \epsilon_2, \vec{Y}_2 \rangle \;, \tag{6.58}$$

where

$$P^- = \frac{1}{2p^+}\left[(p^i)^2 + \frac{1}{\alpha'}\sum_{n=1}^{\infty}(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + n\,S^a_{-n}S^a_n + n\,\tilde{S}^a_{-n}\tilde{S}^a_{-n})\right] \tag{6.59}$$

is the light-cone Hamiltonian. Defining $\epsilon = \epsilon_1 - \epsilon_2$ and $\vec{b} = \vec{Y}_1 - \vec{Y}_2$, and computing explicitly the bosonic z.m. part, this can be written as

$$\mathcal{A}_{p,p} = \frac{V_p\,(4\pi^2\alpha')^{4-p}}{16\sinh\pi\epsilon}\int_0^\infty dt\int\frac{d^{8-p}q}{(2\pi)^{8-p}}\,e^{i\vec{q}\cdot\vec{b}}\,e^{-\pi\alpha' t\vec{q}^2}\,Z_0(\eta_i,\epsilon_i)Z_{osc}(t,\eta_i,\epsilon_i)\ , \tag{6.60}$$

where we have defined the partition functions of the fermionic z.m. and of all the bosonic and fermionic oscillator as

$$Z_0(\eta_i,\epsilon_i) = \langle B_{p0},\eta_1,\epsilon_1|B_{p0},\eta_2,\epsilon_2\rangle\ , \tag{6.61}$$

$$Z_{osc}(t,\eta_i,\epsilon_i) = \langle B_{posc},\eta_1,\epsilon_1|e^{-2\pi\alpha' tp^+P^-}|B_{posc},\eta_2,\epsilon_2\rangle\ . \tag{6.62}$$

In order to understand the role of supersymmetry, let us reconsider first the simple case $\eta_i = 0$ and $\epsilon_i = 0$, and then analyze the effect of switching on $\eta_i \neq 0$ and/or $\epsilon_i \neq 0$.

$\eta_i = 0$ and $\epsilon_i = 0$

In the simple case in which both Dp-branes are at rest and one neglects their spin, the amplitude vanishes as a consequence of the 16 unbroken supersymmetries. In the G-S formalism, this is very well encoded in the fermionic z.m., which are indeed associated to the surviving supercharges. In fact, the contribution of the fermionic z.m. gives a vanishing result, whereas the oscillator contribution is simply 1, since the bosonic and fermionic contributions cancel

$$Z_0 = \mathrm{Tr}_V[\mathbb{1}] - \mathrm{Tr}_S[\mathbb{1}] = 8 - 8 = 0\ , \tag{6.63}$$

$$Z_{osc}(t) = \prod_{n=1}^{\infty}\frac{(1-e^{-2\pi tn})^8}{(1-e^{-2\pi tn})^8} = 1\ . \tag{6.64}$$

Recalling that for $\epsilon_i = 0$, translational invariance in the time direction is recovered

$$\frac{V_p}{\sinh\pi\epsilon}\Delta_{(8-p)}(b) \underset{\epsilon\to 0}{\longrightarrow} V_{p+1}\Delta_{(9-p)}(r)\ , \tag{6.65}$$

one finds finally

$$\mathcal{A}_{p,p} = V_{p+1}\,\hat{T}_p^2\,(1-1)\,\Delta_{(9-p)}(r)\ , \tag{6.66}$$

where $\hat{T}_p = \sqrt{2\pi}(4\pi^2\alpha')^{(3-p)/2}$. Notice that this expression is vanishing, but in some sense *exact* in α', since all the oscillator contributions have simplified.

$\eta_i = 0$ and $\epsilon_i \neq 0$

The effect of a relative velocity, as we have seen already several times, is to break definitively the residual supersymmetry which was left over in the static case. One finds in this case the following results for the partition functions

$$Z_0(\epsilon) = \mathrm{Tr}_V[M^T(\epsilon_2)M(\epsilon_1)] - \mathrm{Tr}_S[M^T(\epsilon_2)M(\epsilon_1)] = (6 + 2\cos 2\pi\epsilon) - 8\cos\pi\epsilon$$
$$= 16\sin^4\frac{\pi\epsilon}{2} \sim v^4 \ , \tag{6.67}$$
$$Z_{osc}(t,\epsilon) = \prod_{n=1}^{\infty} \frac{|1 - e^{i\pi\epsilon/2}e^{-2\pi tn}|^8}{|1 - e^{i\pi\epsilon}e^{-2\pi tn}|^2(1 - e^{-2\pi tn})^6} \sim 1 \ , \tag{6.68}$$

Performing the analytic continuation $\epsilon \to i\epsilon$, one correctly recovers the amplitude already discussed in Chapter 2

$$\mathcal{A}_{p,p} = \frac{V_p}{8}(4\pi^2\alpha')^{4-p}\int_0^\infty \frac{dt}{(4\pi\alpha' t)^{\frac{8-p}{2}}} e^{-\frac{b^2}{4\pi\alpha' t}} \frac{\vartheta_1^4(i\frac{\epsilon}{2}|2it)}{\vartheta_1(i\epsilon|2it)\eta^9(2it)} \ . \tag{6.69}$$

The important point to notice here is that the behavior for $\epsilon \to 0$ is completely determined by supersymmetry. This statement can be understood as follows. Notice first that the fermionic z.m. partition function $Z_0(\epsilon)$ can be thought as receiving a non-trivial contribution only from the left-movers, the right movers being related to the right-movers by the b.c. implemented by the boundary state. It can therefore be rewritten as a z.m. trace of a Type I theory associated to the right movers

$$Z_0(\epsilon) = \mathrm{Tr}_{S_0}[e^{V_{\epsilon 0}}] \ . \tag{6.70}$$

This is precisely the analogous of the integral over fermionic z.m. in the open string path-integral giving the amplitude as a one-loop effective action. Here $V_{\epsilon 0} = -2\pi\epsilon_i R_0^{1i}$ is the z.m. part of the vertex operator associated to the rapidity, whose exponential reconstructs the boost operator. As well known from Type I one-loop amplitudes, the z.m. trace is vanishing unless at least 8 fermionic z.m. are inserted. The first non vanishing trace is conveniently summarized by considering the insertion of the fermionic z.m. part R_0^{ij} of 4 SO(8) generators, each of them being bilinear in the fermionic z.m.. One finds

$$\begin{aligned} t^{i_1\ldots i_8} &= \mathrm{Tr}_{S_0}[R_0^{i_1 i_2}R_0^{i_3 i_4}R_0^{i_5 i_6}R_0^{i_7 i_8}] \\ &= -\frac{1}{2}\epsilon^{i_1\ldots i_8} - \frac{1}{2}\left[\delta^{i_1 i_4}\delta^{i_2 i_3}\delta^{i_5 i_8}\delta^{i_6 i_7} + \mathrm{perm.}\right] \\ &\quad + \frac{1}{2}\left[\delta^{i_2 i_3}\delta^{i_4 i_5}\delta^{i_6 i_7}\delta^{i_8 i_1} + \mathrm{perm.}\right] \ , \end{aligned} \tag{6.71}$$

where "perm." means permutations over the pairs $i_{2n-1}i_{2n}$ and antisymmetrization within each pair. Now each vertex operator $V_{\epsilon 0}$ provides two fermionic z.m. and therefore, expanding the boost operator e^{V_ϵ} in powers of ϵ, we see that the leading contribution to $Z_0(\epsilon)$ for $\epsilon \to 0$ comes form the ϵ^4 term which has just enough fermionic z.m., that is 8, to give a non-vanishing contribution in the trace. Thus, one immediately finds the behavior $Z_0(\epsilon) \sim |v|^4$, without needing to first compute the exact result. Adopting the same strategy both for the z.m. and the oscillator, we expand the whole boost operator e^{V_ϵ} in powers of ϵ and write the partition functions as infinite series of vertex operator correlation functions

$$Z_0(\epsilon) = \sum_{m=0}^{\infty} \frac{1}{m!}\langle B_{p0}|V_{\epsilon 0}^m|B_{p0}\rangle \ , \tag{6.72}$$
$$Z_{osc}(t,\epsilon) = \sum_{q=0}^{\infty} \frac{1}{q!}\langle B_{posc}|V_\epsilon^q\, e^{-2\pi\alpha' tp^+P^-}|B_{posc}\rangle \ . \tag{6.73}$$

Since we are interested in computing the leading order behavior, the effect of the boost on the bosonic z.m. can be omitted. By doing so, one obtains directly the non-relativistic potential times the total time, instead of the non-relativistic integrated phase-shift. It is now clear that the leading term of the total $Z(t,\epsilon) = Z_0(\epsilon) Z_{osc}(t,\epsilon)$ for $\epsilon \to 0$ receives a unique contribution corresponding to $m = 4$ e $q = 0$ in Eqs. (6.72) and (6.73), and one finds

$$Z(t,\epsilon) \underset{\epsilon\to 0}{\longrightarrow} \frac{1}{4!}\mathrm{Tr}_{S_0}[V_{\epsilon 0}^4] \;, \tag{6.74}$$

which is independent of the modulus t. Importantly enough, the oscillator part is the same as in the static case, and continues to give 1, all the dependence on the rapidity coming from the fermionic z.m. part. This means that only the exchange of BPS states (in this case the massless states associated to the fermionic z.m.) contribute, since all the massive states cancel a priori. The non-relativistic amplitude is therefore

$$\mathcal{A}_{p,p} = \frac{|v|^4}{8}\, V_{p+1}\, T_p^2\, \Delta_{(9-p)}(r) \;, \tag{6.75}$$

and is exact in α', that is scale-invariant.

Obviously, this result could have been inferred by simply taking the non-relativistic limit of the exact amplitude, as done in Chapter 2. The interest of the present discussion is that it can now be applied to the complete spin-dependent amplitude, whose exact form in the rapidity is very complicated and unknown.

$\eta_i \neq 0$ and $\epsilon_i \neq 0$

The dependence on the supersymmetry parameter η_i can be treated by using the same strategy as for the rapidity dependence, expanding the supersymmetry transformations in powers of η_i. In this case, there is only a finite number of terms, due to the anticommuting properties of η_i, and the partition functions can be written as

$$Z_0(\eta_{1,2},\epsilon) = \sum_{n_1,n_2}^{n_1+n_2\leq 4}\sum_{m=0}^{\infty}\frac{1}{(n_1!)^2(n_2!)^2 m!}\langle B_{p0}|V_{\eta_1 0}^{n_1}V_{\eta_2 0}^{n_2}V_{\epsilon 0}^{m}|B_{p0}\rangle \;, \tag{6.76}$$

$$Z_{osc}(t,\eta_{1,2},\epsilon) = \sum_{p_1,p_2}^{p_1+p_2\leq 8}\sum_{q=0}^{\infty}\frac{1}{(p_1!)^2(p_2!)^2 q!}\langle B_{posc}|V_{\eta_1}^{p_1}V_{\eta_2}^{p_2}V_{\epsilon}^{q}\, e^{-2\pi\alpha' t p^+ P^-}|B_{posc}\rangle \;. \tag{6.77}$$

Consider now some fixed values for the numbers n_1+p_1 and n_2+p_2 of broken supersymmetries applied to the two boundary states. In order to get the maximum of fermionic z.m., we take also $p_1 = p_2 = 0$ to get the maximum possible n_1 and n_2. Then, since $n_1 + n_2 \leq 4$, in order to have $Z_0(\eta_{1,2},\epsilon) \neq 0$ one has to consider terms with $m \geq 4 - n_1 - n_2$. The leading behavior of the total partition function $Z(t,\epsilon) = Z_0(\epsilon)Z_{osc}(t,\epsilon)$ for $\epsilon \to 0$ is therefore obtained by taking the minimum number of $V_{\epsilon 0}$ insertions for the z.m. part, $m = 4 - n_1 - n_2$, and no V_ϵ insertion for the oscillator part, $q = 0$, which will therefore give 1 as in the static case. Therefore, one finds

$$Z(t,\epsilon) \underset{\epsilon\to 0}{\longrightarrow} \frac{1}{(n_1!)^2(n_2!)^2(4-n_1-n_2)!}\mathrm{Tr}_{S_0}[V_{\eta_1 0}^{n_1}V_{\eta_2 0}^{n_2}V_{\epsilon 0}^{4-n_1-n_2}] \;, \tag{6.78}$$

which is independent of the modulus t. Each $V_{\eta 0}$ brings a power of the transfered momentum q which, when integrating over the momentum in the amplitude, will produce a derivative on the propagator $\Delta_{(9-p)}$. The leading behavior of the amplitude is therefore

$$\mathcal{A}_{p,p}^{(n_1,n_2)} \sim \eta_1^{2n_1}\eta_2^{2n_2}|v|^{4-n_1-n_2}\partial^{n_1+n_2}\Delta_{(9-p)}(r) \sim \eta_1^{2n_1}\eta_2^{2n_2}\frac{v^{4-n_1-n_2}}{r^{7-p+n_1+n_2}} \,. \tag{6.79}$$

Notice in particular that there is a static spin-spin interaction. All these interactions have a simple power-law behavior and are exact in α', that is scale-invariant. The explicit expressions for the amplitudes can be easily worked out in terms of the tensor $t^{i_1\ldots i_8}$ arising from the trace over the fermionic z.m. and the tensor $\omega^{i_1\ldots i_n}_{j_1\ldots j_{2n}}(\eta)$ entering the effective form of $V^n_{\eta 0}$. One finds the following results

$$\mathcal{A}_{p,p}^{(0,0)} = \frac{V_{p+1}}{8\cdot 4!}\hat{T}_p^2\, v_{m_1}v_{m_2}v_{m_3}v_{m_4}\, t^{1m_1 1m_2 1m_3 1m_4}\,\Delta_{(9-p)}(r)\,, \tag{6.80}$$

$$\mathcal{A}_{p,p}^{(1,0)} = \frac{V_{p+1}}{8\cdot 3!}\hat{T}_p^2\, v_{m_1}v_{m_2}v_{m_3}\, t^{i_1 i_2 1m_1 1m_2 1m_3}\,\omega^{j_1}_{i_1 i_2}(\eta_1)\,\partial_{j_1}\Delta_{(9-p)}(r)\,, \tag{6.81}$$

$$\mathcal{A}_{p,p}^{(2,0)} = \frac{V_{p+1}}{8\cdot 2!^3}\hat{T}_p^2\, v_{m_1}v_{m_2}\, t^{i_1\ldots i_4 1m_1 1m_2}\,\omega^{j_1 j_2}_{i_1\ldots i_4}(\eta_1)\,\partial_{j_1}\partial_{j_2}\Delta_{(9-p)}(r)\,, \tag{6.82}$$

$$\mathcal{A}_{p,p}^{(1,1)} = \frac{V_{p+1}}{8\cdot 2!}\hat{T}_p^2\, v_{m_1}v_{m_2}\, t^{i_1\ldots i_4 1m_1 1m_2}\,\omega^{j_1}_{i_1 i_2}(\eta_1)\,\omega^{j_2}_{i_3 i_4}(\eta_2)\,\partial_{j_1}\partial_{j_2}\Delta_{(9-p)}(r)\,, \tag{6.83}$$

$$\mathcal{A}_{p,p}^{(3,0)} = \frac{V_{p+1}}{8\cdot 3!^2}\hat{T}_p^2\, v_{m_1}\, t^{i_1\ldots i_6 1m_1}\,\omega^{j_1 j_2 j_3}_{i_1\ldots i_6}(\eta_1)\,\partial_{j_1}\partial_{j_2}\partial_{j_3}\Delta_{(9-p)}(r)\,, \tag{6.84}$$

$$\mathcal{A}_{p,p}^{(2,1)} = \frac{V_{p+1}}{8\cdot 2!^2}\hat{T}_p^2\, v_{m_1}\, t^{i_1\ldots i_6 1m_1}\,\omega^{j_1 j_2}_{i_1\ldots i_4}(\eta_1)\,\omega^{j_3}_{i_5 i_6}(\eta_2)\,\partial_{j_1}\partial_{j_2}\partial_{j_3}\Delta_{(9-p)}(r)\,, \tag{6.85}$$

$$\mathcal{A}_{p,p}^{(4,0)} = \frac{V_{p+1}}{8\cdot 4!^2}\hat{T}_p^2\, t^{i_1\ldots i_8}\,\omega^{j_1\ldots j_4}_{i_1\ldots i_8}(\eta_1)\,\partial_{j_1}\partial_{j_2}\partial_{j_3}\partial_{j_4}\Delta_{(9-p)}(r)\,, \tag{6.86}$$

$$\mathcal{A}_{p,p}^{(3,1)} = \frac{V_{p+1}}{8\cdot 3!^2}\hat{T}_p^2\, t^{i_1\ldots i_8}\,\omega^{j_1 j_2 j_3}_{i_1\ldots i_6}(\eta_1)\,\omega^{j_4}_{i_7 i_8}(\eta_2)\,\partial_{j_1}\partial_{j_2}\partial_{j_3}\partial_{j_4}\Delta_{(9-p)}(r)\,, \tag{6.87}$$

$$\mathcal{A}_{p,p}^{(2,2)} = \frac{V_{p+1}}{8\cdot 2!^4}\hat{T}_p^2\, t^{i_1\ldots i_8}\,\omega^{j_1 j_2}_{i_1\ldots i_4}(\eta_1)\,\omega^{j_3 j_4}_{i_5\ldots i_8}(\eta_2)\,\partial_{j_1}\partial_{j_2}\partial_{j_3}\partial_{j_4}\Delta_{(9-p)}(r)\,. \tag{6.88}$$

6.3.2 Dp-D(p+4) system

Consider now the system of two parallel Dp and D(p+4)-branes with rapidities ϵ_i and supersymmetry parameters η_i. The phase-shift is given by

$$\mathcal{A}_{p,p+4} = \frac{1}{16}\int_0^\infty dt\,\langle B_p,\eta_1,\epsilon_1,\vec{Y}_1|e^{-2\pi\alpha' t p^+(P^- - p^-)}|B_{p+4},\eta_2,\epsilon_2,\vec{Y}_2\rangle\,. \tag{6.89}$$

As before, it can be rewritten as

$$\mathcal{A}_{p,p+4} = \frac{V_p\,(4\pi^2\alpha')^{-\frac{p(4-p)}{2}}}{16\sinh\pi\epsilon}\int_0^\infty dt\int\frac{d^{4-p}q}{(2\pi)^{4-p}}\,e^{i\vec{q}\cdot\vec{b}}\,e^{-\pi\alpha' t\vec{q}^2}\,Z_0(\eta_i,\epsilon_i)Z_{osc}(t,\eta_i,\epsilon_i)\,, \tag{6.90}$$

where now

$$Z_0(\eta_i,\epsilon_i) = \langle B_{p0},\eta_1,\epsilon_1|B_{p+40},\eta_2,\epsilon_2\rangle\,, \tag{6.91}$$

$$Z_{osc}(t,\eta_i,\epsilon_i) = \langle B_{posc},\eta_1,\epsilon_1|e^{-2\pi\alpha' t p^+ P^-}|B_{p+4osc},\eta_2,\epsilon_2\rangle\,. \tag{6.92}$$

Again, we reconsider first the simple case $\eta_i = 0$ and $\epsilon_i = 0$, and then analyze the effect of switching on $\eta_i \neq 0$ and/or $\epsilon_i \neq 0$.

$\eta_i = 0$ **and** $\epsilon_i = 0$

When both the Dp and the D(p+4)-branes are at rest and one neglects their spin, the amplitude vanishes as a consequence of the 8 unbroken supersymmetries. In the G-S formalism, this is again encoded in the fermionic z.m., associated to the surviving supercharges. In fact, the contribution of the fermionic z.m. gives a vanishing result, whereas in the oscillator contribution bosons and fermions cancel to give simply 1. One finds

$$Z_0 = \mathrm{Tr}_V[N] - \mathrm{Tr}_S[N] = (2-2) - 0 = 0 \,, \tag{6.93}$$

$$Z_{osc}(t) = \prod_{n=1}^{\infty} \frac{(1-e^{-2\pi tn})^4(1+e^{-2\pi tn})^4}{(1-e^{-2\pi tn})^4(1+e^{-2\pi tn})^4} = 1 \,, \tag{6.94}$$

where we have defined the matrices

$$N^{ij} = (M_p^T M_{p+4})^{ij} = \begin{pmatrix} 1\!\!1_{p+1} & 0 & 0 \\ 0 & -1\!\!1_4 & 0 \\ 0 & 0 & 1\!\!1_{3-p} \end{pmatrix} \,, \tag{6.95}$$

$$N_{\dot a \dot b} = (M_p^T M_{p+4})_{\dot a \dot b} = (\gamma^{p+2}...\gamma^{p+5})_{\dot a \dot b} \,. \tag{6.96}$$

Using these results, the static amplitude is found to be

$$\mathcal{A}_{p,p+4} = V_{p+1}\, \hat T_p \hat T_{p+4}\, (1-1)\, \Delta_{(5-p)}(r) \,. \tag{6.97}$$

$\eta_i = 0$ **and** $\epsilon_i \neq 0$

The effect of a relative velocity is again to break the residual supersymmetry which was left over in the static case. One finds

$$\begin{aligned} Z_0(\epsilon) &= \mathrm{Tr}_V[M_p^T(\epsilon_2)M_{p+4}(\epsilon_1)] - \mathrm{Tr}_S[M_p^T(\epsilon_2)M_{p+4}(\epsilon_1)] = (2-2\cos 2\pi\epsilon) - 0 \\ &= 16\cos^2\frac{\pi\epsilon}{2}\sin^2\frac{\pi\epsilon}{2} \sim 4v^2 \,, \end{aligned} \tag{6.98}$$

$$Z_{osc}(t,\epsilon) = \prod_{n=1}^{\infty} \frac{|1-e^{i\pi\epsilon/2}e^{-2\pi tn}|^4|1+e^{i\pi\epsilon/2}e^{-2\pi tn}|^4}{|1-e^{i\pi\epsilon}e^{-2\pi tn}|^2(1-e^{-2\pi tn})^2(1+e^{-2\pi tn})^4} \sim 1 \,. \tag{6.99}$$

Performing the analytic continuation $\epsilon \to i\epsilon$ one finally recovers the correct amplitude already discussed in Chapter 2,

$$\mathcal{A}_{p,p+4} = \frac{V_p}{8}\,(4\pi^2\alpha')^{-\frac{p(4-p)}{2}} \int_0^\infty \frac{dt}{(4\pi\alpha' t)^{\frac{4-p}{2}}}\, e^{-\frac{b^2}{4\pi\alpha' t}}\, \frac{\vartheta_1^2(i\frac{\epsilon}{2}|2it)\vartheta_2^2(i\frac{\epsilon}{2}|2it)}{\vartheta_1(i\epsilon|2it)\vartheta_2^2(0|2it)\eta^3(2it)} \,. \tag{6.100}$$

Again, the behavior for $\epsilon \to 0$ is completely determined by supersymmetry. The fermionic z.m. partition function $Z_0(\epsilon)$ can again be written as a z.m. trace of a Type I theory associated to the right mover, but now with only four z.m.. More precisely,

$$Z_0(\epsilon) = \mathrm{Tr}'_{S_0}[e^{V_{\epsilon 0}}] = \mathrm{Tr}_{S_0}[e^{V_{\epsilon 0}} N] \,, \tag{6.101}$$

where N is the operator corresponding to Eqs. (6.95) and (6.96). The trace is vanishing unless at least 4 fermionic z.m. are inserted, and the first non-vanishing is

$$\begin{aligned} t^{i_1 \dots i_4} &= \mathrm{Tr}'_{S_0}\, R_0^{i_1 i_2} R_0^{i_3 i_4} \\ &= 2\, \epsilon^{i_1 \dots i_4 p+2 \dots p+5} + 2\left[\delta^{i_1 p+2}\delta^{i_2 p+3}\delta^{i_3 p+4}\delta^{i_4 p+5} + \mathrm{perm.}\right] \\ &\quad +2\left[\delta^{i_1 i_3} N^{i_2 i_4} + \mathrm{perm.}\right] . \end{aligned} \tag{6.102}$$

Each vertex operator $V_{\epsilon 0}$ provides two fermionic z.m. and therefore, expanding the boost operator e^{V_ϵ} in powers of ϵ, the leading contribution to $Z_0(\epsilon)$ for $\epsilon \to 0$ comes from the ϵ^2 term which has just enough fermionic z.m., that is in this case 4, to give a non-vanishing contribution. In this way we recover $Z_0(\epsilon) \sim 4|v|^2$, as we already know from the exact amplitude. Adopting the same strategy the oscillators, we expand the whole boost operator e^{V_ϵ} in powers of ϵ and write the partition functions as infinite series of vertex operator correlation functions

$$Z_0(\epsilon) = \sum_{m=0}^{\infty} \frac{1}{m!} \langle B_{p0}| V_{\epsilon 0}^m |B_{p+40}\rangle \;, \tag{6.103}$$

$$Z_{osc}(t,\epsilon) = \sum_{q=0}^{\infty} \frac{1}{q!} \langle B_{posc}| V_{\epsilon}^q \, e^{-2\pi\alpha' t p^+ P^-} |B_{p+4osc}\rangle \;. \tag{6.104}$$

We see that the leading contributions to the total partition function $Z(t,\epsilon) = Z_0(\epsilon) Z_{osc}(t,\epsilon)$ for $\epsilon \to 0$ comes from the term with $m = 2$ and $q = 0$. The oscillators cancel as in the static case, and one finds

$$Z(t,\epsilon) \underset{\epsilon \to 0}{\longrightarrow} \frac{1}{2!} \mathrm{Tr}'_{S_0} [V_{\epsilon 0}^2] \;, \tag{6.105}$$

which is independent of the modulus t. This means that again only the exchange of BPS states contribute, since all the massive modes have canceled. The non-relativistic amplitude

$$\mathcal{A}_{p,p+4} = \frac{|v|^2}{2}\, V_{p+1}\, \hat{T}_p \hat{T}_{p+4}\, \Delta_{(5-p)}(r) \tag{6.106}$$

is therefore exact in α', that is scale-invariant.

$\eta_i \neq 0$ and $\epsilon_i \neq 0$

In the general case, we use the same strategy and write the partition functions as

$$Z_0(\eta_{1,2},\epsilon) = \sum_{n_1,n_2}^{n_1+n_2\le 6} \sum_{m=0}^{\infty} \frac{1}{(n_1!)^2 (n_2!)^2 m!} \langle B_{p0}| V_{\eta_1 0}^{n_1} V_{\eta_2 0}^{n_2} V_{\epsilon 0}^m |B_{p+40}\rangle \;, \tag{6.107}$$

$$Z_{osc}(t,\eta_{1,2},\epsilon) = \sum_{p_1,p_2}^{p_1+p_2\le 12} \sum_{q=0}^{\infty} \frac{1}{(p_1!)^2 (p_2!)^2 q!} \langle B_{posc}| V_{\eta_1}^{p_1} V_{\eta_2}^{p_2} V_{\epsilon}^q \, e^{-2\pi\alpha' t p^+ P^-} |B_{p+4osc}\rangle \,. \tag{6.108}$$

Consider as before some fixed values for the number $n_1 + p_1$ and $n_2 + p_2$ of broken supersymmetries applied to the two boundary states. Again we take $p_1 = p_2 = 0$ in order to maximize n_1 and n_2 and get the maximum of fermionic z.m. Moreover, let us concentrate on the cases with $n_1 + n_2 \leq 2$. In order in order to have $Z_0(\eta_{1,2}, \epsilon) \neq 0$ one has to consider terms with $m \geq 2 - n_1 - n_2$. The leading behavior of the total partition function $Z(t, \epsilon) = Z_0(\epsilon) Z_{osc}(t, \epsilon)$ for $\epsilon \to 0$ is obtained by taking the minimum number of $V_{\epsilon 0}$ insertions for the z.m. part, $m = 2 - n_1 - n_2$, and no V_ϵ insertion for the oscillator part, $q = 0$, which will therefore give 1 as in the static case. One finds

$$Z(t, \epsilon) \underset{\epsilon \to 0}{\longrightarrow} \frac{1}{(n_1!)^2 (n_2!)^2 (2 - n_1 - n_2)!} \mathrm{Tr}'_{S_0} [V_{\eta_1 0}^{n_1} V_{\eta_2 0}^{n_2} V_{\epsilon 0}^{2-n_1-n_2}] \;, \tag{6.109}$$

which is independent of the modulus t. As before, each $V_{\eta 0}$ brings a power of the transfered momentum q which will eventually produce a derivative on the propagator $\Delta_{(5-p)}$. The behavior is therefore

$$\mathcal{A}_{p,p+4}^{(n_1,n_2)} \sim \eta_1^{2n_1} \eta_2^{2n_2} |v|^{2-n_1-n_2} \partial^{n_1+n_2} \Delta_{(5-p)}(r) \sim \eta_1^{2n_1} \eta_2^{2n_2} \frac{v^{2-n_1-n_2}}{r^{3-p+n_1+n_2}} \;. \tag{6.110}$$

All these interactions are exact in α', that is scale invariant. They can be expressed in terms of the tensor $t^{i_1 \ldots i_4}$ emerging from the z.m. trace and the tensor $\omega_{j_1 \ldots j_{2n}}^{i_1 \ldots i_n}(\eta)$ coming from $V_{\eta 0}^n$. One finds the following expressions

$$\mathcal{A}_{p,p+4}^{(0,0)} = \frac{V_{p+1}}{8 \cdot 2!} \hat{T}_p \hat{T}_{p+4}\, v_{m_1} v_{m_2}\, t^{1 m_1 1 m_2}\, \Delta_{(5-p)}(r) \;, \tag{6.111}$$

$$\mathcal{A}_{p,p+4}^{(1,0)} = \frac{V_{p+1}}{8} \hat{T}_p \hat{T}_{p+4}\, v_{m_1}\, t^{i_1 i_2 1 m_1}\, \omega_{i_1 i_2}^{j_1}(\eta_1)\, \partial_{j_1} \Delta_{(5-p)}(r) \;, \tag{6.112}$$

$$\mathcal{A}_{p,p+4}^{(2,0)} = \frac{V_{p+1}}{8 \cdot 2!^2} \hat{T}_p \hat{T}_{p+4}\, t^{i_1 \ldots i_4}\, \omega_{i_1 \ldots i_4}^{j_1 j_2}(\eta_1)\, \partial_{j_1} \partial_{j_2} \Delta_{(5-p)}(r) \;, \tag{6.113}$$

$$\mathcal{A}_{p,p+4}^{(1,1)} = \frac{V_{p+1}}{8} \hat{T}_p \hat{T}_{p+4}\, t^{i_1 \ldots i_4}\, \omega_{i_1 i_2}^{j_1}(\eta_1)\, \omega_{i_3 i_4}^{j_2}(\eta_2)\, \partial_{j_1} \partial_{j_2} \Delta_{(5-p)}(r) \;. \tag{6.114}$$

6.4 Field theory interpretation

In the present section we discuss the field theory interpretation of our results. We will show in particular that the knowledge of all the one-point functions of the massless fields of Type IIA/B supergravity allows to infer the complete and generic asymptotic form of the corresponding p-brane solution. Moreover, the spin-effects in scattering amplitudes that we have computed and the supersymmetric cancellation of some of their leading orders proves to constitute an extremely efficient way to fix unambiguously the various coefficients entering the solution, and in particular the relative strength of the NSNS attraction and the RR repulsion (the fact that normalizations are better encoded in scattering amplitude than in one-point functions, especially through the vanishing of leading order, was already appreciated in Polchinski's computation of the Dp-brane charge [56]). As we will see, this approach yields a powerful technique to extract informations about a generic component of the p-brane multiplet. The analogous computation in supergravity would consist in performing supersymmetry transformations to the usual p-brane solution, to determine all the spinning superpartners; this requires looking up to eight variations, a program that, as can be appreciated from previous works [181, 182, 183], is out of reach within the component fields formalism.

From the results obtained for the one-point functions, one can extract the NSNS and RR asymptotic fields for a generic component of the Dp-brane multiplet. These can be written as a multipole expansion in momentum space:

$$\xi^{\mu\nu} = \kappa_{(10)}^2 \left[A_0 M^{\mu\nu} + A_1 J^{\mu\sigma\alpha} M^\nu_\sigma q_\alpha + A_2 J^{\mu\alpha\rho} J^{\sigma\beta}{}_\rho M^\nu_\sigma q_\alpha q_\beta + ...\right] \ , \tag{6.115}$$

$$\begin{aligned} C^{\mu_1\ldots\mu_k}_{(k)} = \frac{\kappa_{(10)}^2}{k!} & [B_0 \mathrm{Tr}_S[\Gamma^{\mu_1\ldots\mu_k}\mathcal{M}] + B_1 \mathrm{Tr}_S[\Gamma^{\mu_1\ldots\mu_k}\Gamma_{\nu_1\nu_2}\mathcal{M}] J^{\nu_1\nu_2\alpha} q_\alpha \\ & + B_2 \mathrm{Tr}_S[\Gamma^{\mu_1\ldots\mu_k}\Gamma_{\nu_1\nu_2}\Gamma_{\nu_3\nu_4}\mathcal{M}] J^{\nu_1\nu_2\alpha} J^{\nu_3\nu_4\beta} q_\alpha q_\beta + ...\Big] \ . \end{aligned} \tag{6.116}$$

We have restored the ten-dimensional Plank constant $\kappa_{(10)}^2$ for convenience. Dots stand for six and eight supercharge insertions, corresponding to three and four powers of momentum, that we shall not consider. The constants A_i, B_i could in principle be fixed by correctly normalizing the one-point functions; however, this is highly awkward, and since any final conclusion will eventually depend in a crucial way on these constants, we will take advantage of our results for the scattering amplitude to fix them unambiguously.

From now on we specialize to the D0-brane, for which $M^0_0 = -1$, $M^i_j = \delta^i_j$ and $\mathcal{M} = \Gamma^0$; the other cases can be treated in the same way. Recall that in the NSNS sector, a generic field $\xi_{\mu\nu}$ is decomposed into trace, symmetric and antisymmetric parts ϕ, $h_{\mu\nu}$ and $b_{\mu\nu}$ as

$$\epsilon^{(\phi)}_{\mu\nu} = \frac{1}{4}(\eta_{\mu\nu} - q_\mu l_\nu - q_\nu l_\mu) \ , \ \ \epsilon^{(h)}_{\mu\nu} = \xi_{(\mu\nu)} \ , \ \ \epsilon^{(b)}_{\mu\nu} = \xi_{[\mu\nu]} \ , \tag{6.117}$$

where l^μ is a vector satisfying $q \cdot l = 1$, $l^2 = 0$. The asymptotic fields in the NSNS sector are then found to be

$$\begin{aligned} & \phi = \frac{3}{2}\kappa_{(10)}^2 M G_9(r) + \frac{1}{4}\kappa_{(10)}^2 C J^{mpq} J^n{}_{pq} \partial_m \partial_n \Delta_{(9)}(r) + ... \ , \\ & \left\{ \begin{array}{l} h_{00} = \kappa_{(10)}^2 M \Delta_{(9)}(r) + \kappa_{(10)}^2 C J^{m0q} J^n{}_{0q} \partial_m \partial_n \Delta_{(9)}(r) + ... \\ h_{ij} = \delta_{ij}\kappa_{(10)}^2 M \Delta_{(9)}(r) + \kappa_{(10)}^2 C J^m{}_i{}^\rho J^n{}_{j\rho} \partial_m \partial_n \Delta_{(9)}(r) + ... \\ h_{0i} = 2\kappa_{(10)}^2 A J_{0i}{}^m \partial_m \Delta_{(9)}(r) + ... \end{array} \right. \ , \\ & \left\{ \begin{array}{l} b_{ij} = \kappa_{(10)}^2 A J_{ij}{}^m \partial_m \Delta_{(9)}(r) + ... \\ b_{0i} = 2\kappa_{(10)}^2 C J^m_{0q} J^n{}_i{}^q \partial_m \partial_n \Delta_{(9)}(r) + ... \end{array} \right. \ , \end{aligned} \tag{6.118}$$

whereas Eq. (6.116) in the RR sector become

$$\begin{aligned} & \left\{ \begin{array}{l} C_0 = 2\kappa_{(10)}^2 Q \Delta_{(9)}(r) + \kappa_{(10)}^2 D J^{m\rho\tau} J^n{}_{\rho\tau} \partial_m \partial_n \Delta_{(9)}(r) + ... \\ C_i = 2\kappa_{(10)}^2 B J_{0i}{}^m \partial_m \Delta_{(9)}(r) + ... \end{array} \right. \ , \\ & \left\{ \begin{array}{l} C_{0ij} = \kappa_{(10)}^2 B J_{ij}{}^m \partial_m \Delta_{(9)}(r) + ... \\ C_{ijk} = 2\kappa_{(10)}^2 D J^m_{0[i} J^n{}_{jk]} \partial_m \partial_n \Delta_{(9)}(r) + ... \end{array} \right. \ . \end{aligned} \tag{6.119}$$

The constants A_i, B_i have been redefined and called M, A, B, Q, C, D for later convenience, and again, dots stand for higher derivative terms associated to further supercharge insertions.

Comparing Eqs. (6.118) and (6.119) with the usual 0-brane solution [184] and the general result valid in D dimensions derived in [185], we conclude that M is the mass and Q the electric charge, so the charge-mass ratio is $\alpha = Q/M$, whereas $2AJ_{0ij} = J_{ij}$ is the angular momentum and $BJ_{0ij} = \mu_{ij}$ the magnetic moment, so that the gyromagnetic ratio, defined by the relation $\mu^{ij} = (gQ)/(2M)J^{ij}$, is given by $g = (MB)/(QA)$. Also, the electric and gravitational dipole moments vanish, since they would correspond to one-derivative terms in C_0 and h_{00}, h_{ij} respectively. The presence of two-derivative terms in the gravitational and gauge fields signals potential quadrupole moments for D-particles. Notice however that the quadrupole term in C^0 vanishes due to the Fiertz identity $J^{m\mu\nu}J^{n}{}_{\mu\nu} = 0$, and therefore the corresponding quadrupole moment is zero. Keeping in mind this fact, we nevertheless define the quadrupole analog $\tilde{g}$ of gyromagnetic ratio g, constructed as the ratio of the electric and gravitational quadrupole moments as $\tilde{g} = 4(MD)/(QC)$.

It is now straightforward to show how the semiclassical analysis of the phase-shift between two of these configurations can be used to determine in a simple way the value of the gyromagnetic ratio g and its quadrupole analogue $\tilde{g}$ associated to D0-branes. According to [186, 183], massive Kaluza-Klein states present a common value $g = 1$, contrarily to the usual and *natural* [187, 188, 189] value $g = 2$ shared by all the known elementary particles (neglecting radiative corrections, of course). This particular signature of Kaluza-Klein states can be useful to establish the eleven-dimensional nature of D0-branes, implying $g = 1$. This consistency check has been recently performed in [183] considering D0-branes as extended extremal 0-brane solution of IIA supergravity. We present an alternative and independent argument that relies on the *stringy* nature of D0-branes as points on which open strings can end. In particular, much in the same way as the cancellation of the static and quadratic velocity parts in the universal amplitude $\mathcal{A}^{(0,0)}$ implies a charge-mass ratio $\alpha = 1$, we will show that $g = 1$ is the only possible value compatible with the cancellation of the linear term in velocity in the first spin effect $\mathcal{A}^{(1,0)}$. Similarly, we will show that our stringy analysis predicts for the quadrupole analog the value $\tilde{g} = 1$ from the cancellation of the static contribution to the second spin effect $\mathcal{A}^{(2,0)}$.

The string theory results for the non-relativistic amplitude is

$$\mathcal{A}_{0,0} = \mathcal{A}_{0,0}^{(0,0)} + \mathcal{A}_{0,0}^{(1,0)} + \mathcal{A}_{0,0}^{(2,0)} + \dots \,. \tag{6.120}$$

The results obtained in previous section for the these spin-effects can be explicated, finding the following covariant results for two D0-branes

$$\mathcal{A}_{0,0}^{(0,0)} = \frac{|v|^4}{8} V_{p+1} \hat{T}_p^2 \, \Delta_{(9)}(r) \,, \tag{6.121}$$

$$\mathcal{A}_{0,0}^{(1,0)} = \frac{V_{p+1}}{4} \hat{T}_p^2 \, |v|^2 v_i \, J_0{}^{ij} \, \partial_j \Delta_{(9)}(r) \,, \tag{6.122}$$

$$\mathcal{A}_{0,0}^{(2,0)} = \frac{V_{p+1}}{96} \hat{T}_p^2 \, |v|^2 (2J^{m0q} J^{n}{}_{0q} - J^{mpq} J^{n}{}_{pq} + 4J^{m\rho}{}_{i} J^{n}{}_{\rho j} \hat{v}^i \hat{v}^j) \partial_m \partial_n \Delta_{(9)}(r) \,. \tag{6.123}$$

To compute the phase-shift in field-theory, we use the world-line effective action of a scalar 0-brane probe

$$\mathcal{S} = -M \int d\tau \, e^{-\phi} \sqrt{-g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} - Q \int d\tau C_\mu \dot{X}^\mu \,, \tag{6.124}$$

which in the weak-field limit $\kappa_{(10)} \to 0$ reduces to

$$\mathcal{S}_0 = \int d\tau \left(M\phi + \frac{1}{2} M h_{\mu\nu} \dot{X}^\mu \dot{X}^\nu - Q C_\mu \dot{X}^\mu \right) \,. \tag{6.125}$$

This can also be obtained as the dimensional reduction to D=10 of the D=11 superparticle action. It is straightforward to check that this reproduces the universal part of the asymptotic fields of the 0-brane for a static source $X^0 = \tau$, $X^i = 0$. To compute the interaction between a scalar and a spinning 0-branes, one simply evaluates the above effective action for the moving scalar probe with $X^0 = \cosh \pi\epsilon\,\tau$, $X^i = \hat{v}^i \sinh \pi\epsilon\,\tau$, in the background of the spinning one. Expanding for $v \to 0$ one finds

$$\mathcal{S} = \int d\tau \sum_{n\geq 0} v^n V_n \ , \qquad (6.126)$$

with

$$\begin{aligned}
&V_0 = M\phi + \frac{1}{2}Mh_{00} - QC_0 \ , \\
&V_1 = Mh_{0i}\hat{v}^i - QC_i\hat{v}^i \ , \ \ V_2 = \frac{1}{2}M(h_{00} + h_{ij}\hat{v}^i\hat{v}^j) - \frac{1}{2}QC_0 \ , \\
&V_3 = Mh_{0i}\hat{v}^i - \frac{1}{2}QC_i\hat{v}^i \ , \ \ V_4 = \frac{1}{2}M(h_{00} + h_{ij}\hat{v}^i\hat{v}^j) - \frac{3}{8}QC_0 \ , \\
&\ldots
\end{aligned} \qquad (6.127)$$

Comparing with the string theory result, one finds in particular the following implications

$$\begin{aligned}
V_{0,2}|_{\Delta} = 0 \ &\Rightarrow \quad M = Q \quad &\Rightarrow \ \alpha = 1 \\
V_1|_{\partial\Delta} = 0 \ &\Rightarrow \ MA = QB \ &\Rightarrow \ g = 1 \\
V_0|_{\partial^2\Delta} = 0 \ &\Rightarrow \ MC = 4QD \ &\Rightarrow \ \tilde{g} = 1
\end{aligned} \quad . \qquad (6.128)$$

As an important consistency check, we have explicitly verified that with these values the correct tensor structures come out.

A comment is in order on how our boundary state formalism for describing higher spin Dp-branes is related to the supergravity description, where p-branes appear as solitonic solutions. As already said, the asymptotic fields found by applying the procedure of this section correspond to supergravity solutions obtained by taking supersymmetric variations of the usual scalar ones. This has been partially done in [183] for the D0-brane solution, where the second supersymmetry variation was used to compute the angular momentum dependence of $h_{\mu\nu}$ and C_μ. Using the same strategy, we have similarly checked that the angular momentum contributions to the higher forms $b_{\mu\nu}$ and $C_{\mu\nu\rho}$ (which have not been considered in [183]) correctly reproduce those in Eqs. (6.118) and (6.119). We have also checked that the fourth supersymmetry variation reproduces all the two-derivative terms we find, but it is unrealistic to compute and trust the coefficient because of the increasing complexity of the involved expressions.

Finally, another interesting outcome of the knowledge of the asymptotic fields (6.118), (6.119) is the possibility to derive the supersymmetric completion of the linearized 0-brane world-line effective action (6.125) *in an arbitrary Type IIA background*, at least for weak fields. The complete action will be of the form

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{\eta^2} + \mathcal{S}_{\eta^4} + \ldots \ . \qquad (6.129)$$

For example, it is not difficult to verify that, in much the same way as the part of the asymptotic fields going like Δ can be derived from the action (6.125), the part of the fields going like $\partial\Delta$ can be derived from the following non-minimal couplings

$$\mathcal{S}_{\eta^2} = \int d\tau \left(-\partial_i h_{0j} J^{0ij} + \frac{1}{4}\partial_i b_{jk} J^{ijk} + \partial_i C_j J^{0ij} - \frac{1}{4}\partial_i C_{0jk} J^{ijk} \right) . \quad (6.130)$$

The covariant form this action should be obtained by replacing each 0 index by a contraction with the momentum $\dot{X}^\mu$; in such a way, the fields generated by a moving 0-brane are given by the boost of those produced by a static one. One obtains

$$\mathcal{S}_{\eta^2} = \int d\tau \left(\Gamma^\rho{}_{\sigma\mu} \dot{X}^\mu \dot{X}^\nu J_\rho{}^\sigma{}_\nu + \frac{g}{12} F_{\mu\nu\rho\sigma} \dot{X}^\mu J^{\nu\rho\sigma} + \frac{1}{12} H_{\mu\nu\rho} J^{\mu\nu\rho} + \frac{g}{2} F_{\mu\nu} \dot{X}^\rho J^{\mu\nu}{}_\rho \right) \quad (6.131)$$

where $F_{\mu\nu} = 2\partial_{[\mu} C_{\nu]}$, $F_{\mu\nu\rho\sigma} = 4\partial_{[\mu} C_{\nu\rho\sigma]}$ and $H_{\mu\nu\rho} = 3\partial_{[\mu} b_{\nu\rho]}$. The coefficients have been further checked by computing the static force contribution of order $\partial^2\Delta$ between two spinning 0-branes, that vanishes as expected. Finally, notice that *if and only if* $g = 1$, the action (6.131) is the Kaluza-Klein dimensional reduction to D=10 of a D=11 action containing only the first two terms.

6.5 Scale-invariance and the SYM-SUGRA correspondence

An extremely important issue in the study of D0-brane dynamics is about the conjecture of [79] that the dynamics of M-theory in the infinite momentum frame (IMF) is governed by the degrees of freedom of a large number of D0-branes. The essence of the conjecture is a relation between effective loop interactions in the SYM gauge theory describing nearby D0-branes and tree interaction between D0-branes in SUGRA, seen as Kaluza-Klein states coming from compactification of D=11 SUGRA on a circle. The IMF automatically provides a kinematics which automatically selects the non-relativistic limit of the theory, keeping only the leading order interactions. Form a field theory point of view, the matching between SYM and SUGRA computations seems at first sight miraculous. Actually, the matching of the $v \to 0$ limit of the leading tree-level SUGRA interactions and the SYM one-loop effective action is dictated by supersymmetry.

Form the results of this chapter, the nature of the SYM-SUGRA correspondence emerges in a very clear way. Indeed, we have shown that all the leading non-relativistic interactions, including all the effects related to spin, are completely determined by a trace over fermionic z.m.. This leads to the extremely important conclusion that these interactions are exact in α' and are valid at any distance. This means in particular that the two truncations to the exact cylinder amplitude to open and closed string massless states, corresponding to the SYM and SUGRA low-energy effective theories relevant at short and large distances, agree in the non-relativistic limit, implying the exact matching between the complete spin-dependent SUGRA interactions and the complete one-loop SYM effective action. Significantly enough, the simple power-low non-relativistic interactions discussed here depend in no way on the string scale l_s, and the apparently miraculous matching of the limits $r \ll l_S$ (SYM) and $r \gg l_s$ (SUGRA) is actually a trivial consequence of the fact that these leading interactions are *completely determined by supersymmetry*. In [190], it was indeed demonstrated that the leading non-relativistic terms of the D0-brane effective action is completely fixed by requiring sixteen unbroken supersymmetries. This implies that whatever microscopic theory one uses to describe D0-brane dynamics, it has to reproduce this leading terms in the LEEA.

Moreover, these leading terms enjoy a non-renormalization theorem protecting them form additional corrections.

It is nevertheless interesting to check that the detailed tensor structure of our results is reproduced both in SUGRA and in SYM. To this aim, it is convenient to explicit the amplitudes $\mathcal{A}_{0,0}^{(n_1,n_2)}$ in terms of the SO(9) spinor $\binom{\eta^a}{\tilde{\eta}^{\dot{a}}}$ that we shall call here θ (instead of η). One finds, after heavy algebra and using several SO(9) Fiertz identities (for details see [191, 192]), the following *complete interaction potential*

$$
\begin{aligned}
V = \frac{1}{8} \Big[& v^4 + 2i\, v^2 v_m (\theta\gamma^{mn}\theta)\, \partial_n - 2 v_p\, v_q (\theta\gamma^{pm}\theta)(\theta\gamma^{qn}\theta)\, \partial_m \partial_n \\
& - \frac{4i}{9} v_i (\theta\gamma^{im}\theta)(\theta\gamma^{nl}\theta)(\theta\gamma^{pl}\theta)\, \partial_m \partial_n \partial_p \\
& + \frac{2}{63} (\theta\gamma^{ml}\theta)(\theta\gamma^{nl}\theta)(\theta\gamma^{pk}\theta)(\theta\gamma^{qk}\theta)\, \partial_m \partial_n \partial_p \partial_q \Big]\, \Delta_{(9)}(r)\,.
\end{aligned}
\tag{6.132}
$$

The first, second, third and last terms of this potential have been calculated in the SYM context in [89], [193], [194] and [195] respectively. All the terms and coefficients have been shown also to agree with the eikonal approximation of the complete four-supergraviton scattering amplitude in SUGRA [191, 192]. Finally, let us notice that the scattering of D0-branes can be mapped to the scattering of fundamental strings by a chain of duality. More precisely, compactifying on a circle and performing a T-duality, the four D-particles are turned into four D-strings winding around the circle, which are finally turned to F-strings by an S-duality transformation. The spin dependent terms in the D0-brane scattering amplitude are then related to the the corresponding spin-dependent terms in the wound F-strings scattering amplitude, and the correct dependence on the distance, spin and velocity comes out [196].

Since the leading part of the D0-brane effective action is completely determined by supersymmetry, it cannot be used to probe significantly the matrix model conjecture. Rather, one has to study sub-leading interactions corresponding to two or more loops in the SYM effective action, and compare them to tree-level post-Newtonian SUGRA corrections. In the two-body case, perfect agreement of the leading non-relativistic behavior has been found up to two loops [197, 198], but again, it seems [199] that supersymmetry constrains the corrections in such a way to determine them almost completely, as happening at one-loop. The first really non-trivial check invoques three-body interactions, which starts receiving contributions at two-loops on the SYM side. Due to the extreme complexity of the computation there has been a variety of partial results, but finally perfect agreement has been found between SYM and SUGRA [200].

The strategy that we have used to study spin-effects in D-brane dynamics can be summarized as follows. Instead of considering the full configuration of moving branes, where supersymmetry is broken, we have perturbed through appropriate vertex operators the supersymmetric vacuum associated to the static Dp-Dp and Dp-D(p+4) systems, deriving in this way important results on the structure of the exact (in powers of α') leading spin interactions in a velocity expansion. The cylinder amplitudes corresponding to these interactions collapses to its zero mode contribution, supporting an equivalent description in terms of either the open (matrix model) or closed (supergravity) massless degrees of freedom. This strategy is actually quite general and can be easily extended to several other D-brane configurations preserving some supersymmetry, like those studied for example in

[201, 202, 203, 204, 205, 206, 207], where a similar short-long distance matching of the leading interactions was observed, in the leading cylinder amplitude. Again, the reason of the matching lies in the fact that they are actually scale-invariant and completely determined by supersymmetry. Indeed, starting form a supersymmetric D-brane configuration with a vanishing interaction energy and a given number of fermionic z.m., and perturbing it through a supersymmetry breaking deformation associated to some small parameter ϵ, it is clear that the leading order interactions in a expansion in powers of ϵ will be determined by the sole fermionic z.m. and will therefore be scale-invariant.

Conclusion

In this thesis, we have studied various aspects of D-brane dynamics in string theory, using prevalently the boundary state formalism. For instance, we have studied four-dimensional point-like objects arising as wrapped or dimensionally reduced D-branes in toroidal and orbifold compactifications. From the velocity dependence of the phase-shift, it clearly emerges that these four-dimensional configurations couple in general to the scalar fields of the corresponding LEEA, and correspond therefore to singular extremal black hole solutions. The D3-brane wrapped on $T^6/\mathbb{Z}_3$ seems to be an interesting exception, since the orbifold projection kills any coupling to scalars, yielding a regular R-N extremal black hole solution. The same conclusions are obtained by studying the emission of a massless closed string state form two of these D-brane configurations in interaction. In the large distance limit, the emission amplitude is in perfect agreement with SUGRA. A careful analysis shows that actually the point-like object obtained by wrapping a D3-brane is a dyon. A detailed analysis of the electric and magnetic phase-shifts allows to compute the electric and magnetic charges as functions of the orientation of the D3-brane in the internal compact space. Comparison with SUGRA is achieved by constructing an explicit ten-dimensional solution, with a metric factorizing in a four-dimensional dyonic R-N extremal black hole and a six-dimensional CY internal part. Final evidence for the identification of this configuration with a wrapped D3-brane is obtained by computing the one-point functions of all the massless fields, leading to couplings which are in exact agreement with those extracted from the phase-shifts. Finally, we have studied leading spin-effects in D-brane dynamics for small velocities v, finding contributions of the form v^{4-n}/r^{7-p+n} and v^{2-n}/r^{3-p+n} for the Dp-Dp and the Dp-D(p+4) systems preserving 16 and 8 supersymmetries in the static limit. These interactions receive contributions only from massless BPS intermediate states, massive states canceling as a consequence of supersymmetry. This implies the scale-invariance of these leading spin-effects and in particular the equivalence between their SYM and SUGRA descriptions. The supersymmetry cancellations occurring in the interaction potential imply a particular value for the gyromagnetic ratio $g = 1$ and its quadrupole analog $\tilde{g} = 1$ for D0-branes, in agreement with their eleven-dimensional Kaluza-Klein nature.

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Appendix A

ϑ-functions

In this appendix we quote some important definitions and properties about Jacobi elliptic ϑ-functions.

A.1 $\vartheta\left[{a \atop b}\right]$-functions.

A.1.1 Definition

The function $\vartheta\left[{a \atop b}\right](v|\tau)$ is defined as the infinite series

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(v|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-a)^2} e^{2\pi i(n-a)(v-b)} \ , \tag{A.1}$$

where $a, b \in [0,1]$ and $q = e^{2\pi i\tau}$. Equivalently, there is also an infinite product representation

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(v|\tau) = q^{a^2} e^{2\pi i a(b+v)} \prod_{n=1}^{\infty}\left(1-q^{2n}\right)$$
$$\prod_{n=1}^{\infty}\left[\left(1+q^{2(n+a)-1}e^{2\pi i(b+v)}\right)\left(1+q^{2(n-a)-1}e^{-2\pi i(b+v)}\right)\right] \ . \tag{A.2}$$

It has the obvious properties

$$\vartheta\begin{bmatrix}a+1\\b\end{bmatrix}(v|\tau) = \vartheta\begin{bmatrix}a\\b\end{bmatrix}(v|\tau) \ , \quad \vartheta\begin{bmatrix}a\\b+1\end{bmatrix}(v|\tau) = e^{2\pi i a}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(v|\tau) \ , \tag{A.3}$$

so that the periods a and b are actually defined essentially modulo 1.

A.1.2 Transformation properties

The functions $\vartheta\left[{a \atop b}\right](v|\tau)$ are essentially the most general functions with definite monodromy around the cycles on a torus of modulus τ. In fact, under the shift $v \to v + \alpha\tau + \beta$ which circles α and β times the two cycles of the torus, the functions $\vartheta\left[{a \atop b}\right](v|\tau)$ transform as follows

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(v+\alpha\tau+\beta|\tau) = e^{-2\pi i\alpha(v-b+\frac{\alpha}{2}+\beta)}\vartheta\begin{bmatrix}a-\alpha\\b-\beta\end{bmatrix}(v|\tau) \ . \tag{A.4}$$

In particular, if α and β are integers, the function simply picks up a phase around the two cycles of the torus. Under modular transformations of the torus, one has

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(v|\tau+1) = e^{-\pi i a(a-1)}\vartheta\begin{bmatrix}a\\b+a-\frac{1}{2}\end{bmatrix}(v|\tau)\ , \tag{A.5}$$

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\frac{v}{\tau}|-\frac{1}{\tau}) = \sqrt{-i\tau}e^{2\pi i(ab+\frac{v^2}{\tau})}\vartheta\begin{bmatrix}b\\-a\end{bmatrix}(v|\tau)\ . \tag{A.6}$$

A.1.3 Riemann identity

The functions $\vartheta\left[{a \atop b}\right](v|\tau)$ satisfy the extremely important identity

$$\frac{1}{2}\sum_{a,b=0}^{\frac{1}{2}}(-1)^{2(a+b)}\prod_{i=1}^{4}\vartheta\begin{bmatrix}a+h_i\\b+g_i\end{bmatrix}(v_i|\tau) = -\prod_{i=1}^{4}\vartheta\begin{bmatrix}\frac{1}{2}-h_i\\\frac{1}{2}-g_i\end{bmatrix}(v'_i|\tau)\ , \tag{A.7}$$

where h_i and g_i are subject to the condition $\sum_i h_i = n$ and $\sum_i g_i = 0$, and the arguments v'_i are given by

$$\begin{aligned} v'_1 &= \frac{1}{2}(v_1+v_2+v_3+v_4)\ ,\quad v'_2 = \frac{1}{2}(v_1+v_2-v_3-v_4)\ ,\\ v'_3 &= \frac{1}{2}(v_1-v_2+v_3-v_4)\ ,\quad v'_4 = \frac{1}{2}(v_1+v_2-v_3-v_4)\ . \end{aligned} \tag{A.8}$$

A.2 ϑ_α-functions.

A.2.1 Definition

The function $\vartheta\left[{a \atop b}\right](v|\tau)$ for the special values $a, b = 0, 1/2$ are particularly important, so that they have a name:

$$\begin{aligned} \vartheta_1(v|\tau) &= \vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(v|\tau)\ ,\quad \vartheta_2(v|\tau) = \vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(v|\tau)\ ,\\ \vartheta_3(v|\tau) &= \vartheta\begin{bmatrix}0\\0\end{bmatrix}(v|\tau)\ ,\quad \vartheta_4(v|\tau) = \vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(v|\tau)\ . \end{aligned} \tag{A.9}$$

It is worth to report their important infinite product representations which are particular cases of Eq. (A.2)

$$\vartheta_1(v|\tau) = 2\sin\pi v\, q^{\frac{1}{4}}\prod_{n=1}^{\infty}\left(1-q^{2n}\right)\prod_{n=1}^{\infty}\left[\left(1-q^{2n}e^{2\pi iv}\right)\left(1-q^{2n}e^{-2\pi iv}\right)\right]\ , \tag{A.10}$$

$$\vartheta_2(v|\tau) = 2\cos\pi v\, q^{\frac{1}{4}}\prod_{n=1}^{\infty}\left(1-q^{2n}\right)\prod_{n=1}^{\infty}\left[\left(1+q^{2n}e^{2\pi iv}\right)\left(1+q^{2n}e^{-2\pi iv}\right)\right]\ , \tag{A.11}$$

$$\vartheta_3(v|\tau) = \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\prod_{n=1}^{\infty}\left[\left(1+q^{2n-1}e^{2\pi iv}\right)\left(1+q^{2n-1}e^{-2\pi iv}\right)\right]\ , \tag{A.12}$$

$$\vartheta_4(v|\tau) = \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\prod_{n=1}^{\infty}\left[\left(1-q^{2n-1}e^{2\pi iv}\right)\left(1-q^{2n-1}e^{-2\pi iv}\right)\right]\ . \tag{A.13}$$

Another very important modular function related to bosonic partition functions is the Dedekind η-function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \ . \tag{A.14}$$

It is related to the θ_α-functions by the useful relation

$$\vartheta_1'(0|\tau) = 2\pi\eta^3(\tau) \ . \tag{A.15}$$

A.2.2 Transformation properties

The monodromy properties around the cycles of the torus follow from Eq. (A.4). The important modular transformation Eq. (A.6) becomes

$$\vartheta_1(\frac{v}{\tau}| - \frac{1}{\tau}) = \sqrt{i\tau} e^{2\pi\frac{v^2}{\tau}} \vartheta_1(v|\tau) \ , \tag{A.16}$$

$$\vartheta_2(\frac{v}{\tau}| - \frac{1}{\tau}) = \sqrt{i\tau} e^{2\pi\frac{v^2}{\tau}} \vartheta_4(v|\tau) \ , \tag{A.17}$$

$$\vartheta_3(\frac{v}{\tau}| - \frac{1}{\tau}) = \sqrt{-i\tau} e^{2\pi\frac{v^2}{\tau}} \vartheta_3(v|\tau) \ , \tag{A.18}$$

$$\vartheta_4(\frac{v}{\tau}| - \frac{1}{\tau}) = \sqrt{-i\tau} e^{2\pi\frac{v^2}{\tau}} \vartheta_2(v|\tau) \ , \tag{A.19}$$

whereas the η-function transforms as

$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau) \ . \tag{A.20}$$

A.2.3 Identities

The Riemann identity Eq. (A.7) specialized to $h_i, g_i = 0, \pm\frac{1}{2}$ yields important relations between ϑ_α-functions. For instance, taking $h_i = 0$, $g_i = 0$, or $h_{1,2} = 0$, $h_{3,4} = \pm 1/2$, $g_i = 0$, or $h_i = 0$, $g_{1,2} = 0$, $g_{3,4} = \pm 1/2$, one finds

$$\prod_{i=1}^{4} \vartheta_1(v_i|\tau) - \prod_{i=1}^{4} \vartheta_2(v_i|\tau) + \prod_{i=1}^{4} \vartheta_3(v_i|\tau) - \prod_{i=1}^{4} \vartheta_4(v_i|\tau) = -2 \prod_{i=1}^{4} \vartheta_1(v_i'|\tau) \ , \tag{A.21}$$

$$\begin{aligned} &\prod_{i=1}^{2} \vartheta_1(v_i|\tau) \prod_{i=3}^{4} \vartheta_4(v_i|\tau) - \prod_{i=1}^{2} \vartheta_2(v_i|\tau) \prod_{i=3}^{4} \vartheta_3(v_i|\tau) \\ &+ \prod_{i=1}^{2} \vartheta_3(v_i|\tau) \prod_{i=3}^{4} \vartheta_2(v_i|\tau) - \prod_{i=1}^{2} \vartheta_4(v_i|\tau) \prod_{i=3}^{4} \vartheta_1(v_i|\tau) = -2 \prod_{i=1}^{2} \vartheta_1(v_i'|\tau) \prod_{i=3}^{4} \vartheta_4(v_i'|\tau) \ , \end{aligned} \tag{A.22}$$

$$\begin{aligned} &\prod_{i=1}^{2} \vartheta_1(v_i|\tau) \prod_{i=3}^{4} \vartheta_2(v_i|\tau) - \prod_{i=1}^{2} \vartheta_2(v_i|\tau) \prod_{i=3}^{4} \vartheta_1(v_i|\tau) \\ &- \prod_{i=1}^{2} \vartheta_3(v_i|\tau) \prod_{i=3}^{4} \vartheta_4(v_i|\tau) + \prod_{i=1}^{2} \vartheta_4(v_i|\tau) \prod_{i=3}^{4} \vartheta_3(v_i|\tau) = -2 \prod_{i=1}^{2} \vartheta_1(v_i'|\tau) \prod_{i=3}^{4} \vartheta_2(v_i'|\tau) \ , \end{aligned} \tag{A.23}$$

with v_i' given by Eq. (A.8). Further important special cases are

$$\vartheta_2(v|\tau)\vartheta_2^3(0|\tau) - \vartheta_3(v|\tau)\vartheta_3^3(0|\tau) + \vartheta_4(v|\tau)\vartheta_4^3(0|\tau) = 2\vartheta_1^4(\frac{v}{2}|\tau) \ , \tag{A.24}$$

$$\vartheta_2(v|\tau)\vartheta_2(0|\tau)\vartheta_3^2(0|\tau) - \vartheta_3(v|\tau)\vartheta_3(0|\tau)\vartheta_2^2(0|\tau) = 2\vartheta_1^2(\frac{v}{2}|\tau)\vartheta_4^2(\frac{v}{2}|\tau) \ , \tag{A.25}$$

$$\vartheta_4(v|\tau)\vartheta_4(0|\tau)\vartheta_3^2(0|\tau) - \vartheta_3(v|\tau)\vartheta_3(0|\tau)\vartheta_4^2(0|\tau) = 2\vartheta_1^2(\frac{v}{2}|\tau)\vartheta_2^2(\frac{v}{2}|\tau) \ . \tag{A.26}$$

Appendix B

Partition functions

In this appendix, we summarize the partition functions on the torus for bosons and fermions with all the possible periodicities around the two cycles and generic twists.

B.1 Untwisted partition functions

Indicating with the symbol

$$ {}_{P_2}\square_{P_1} \qquad \text{(B.1)} $$

the contribution of a boson or a fermion with periodicities P_1 and P_2 around the two cycles of the torus, one finds

Bosons

$$ {}_{P}\square_{P} \;:\; Z^B_{osc}(t) = q^{-\frac{1}{12}} \prod_{n=1}^{\infty} \left(1 - q^{2n}\right)^{-1} = \frac{1}{\eta(\frac{it}{2})} \,, \qquad \text{(B.2)} $$

$$ {}_{P}\square_{A} \;:\; Z^B_{osc}(t) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 - q^{2n-1}\right)^{-1} = \sqrt{\frac{\eta(\frac{it}{2})}{\vartheta_4(0|\frac{it}{2})}} \,, \qquad \text{(B.3)} $$

Fermions

$$ {}_{P}\square_{P} \;:\; Z^{F(P-)}(t) = \frac{1}{\sqrt{2}} q^{\frac{1}{12}} \prod_{n=0}^{\infty} \left(1 - q^{2n}\right) = \sqrt{\frac{\vartheta_1(0|\frac{it}{2})}{i\eta(\frac{it}{2})}} = 0 \,, \qquad \text{(B.4)} $$

$$ {}_{A}\square_{P} \;:\; Z^{F(P+)}(t) = \frac{1}{\sqrt{2}} q^{\frac{1}{12}} \prod_{n=0}^{\infty} \left(1 + q^{2n}\right) = \sqrt{\frac{\vartheta_2(0|\frac{it}{2})}{\eta(\frac{it}{2})}} \,, \qquad \text{(B.5)} $$

$$ {}_{P}\square_{A} \;:\; Z^{F(A-)}(t) = q^{-\frac{1}{24}} \prod_{n=0}^{\infty} \left(1 - q^{2n+1}\right) = \sqrt{\frac{\vartheta_4(0|\frac{it}{2})}{\eta(\frac{it}{2})}} \,, \qquad \text{(B.6)} $$

$$ {}_{A}\square_{A} \;:\; Z^{F(A+)}(t) = q^{-\frac{1}{24}} \prod_{n=0}^{\infty} \left(1 + q^{2n+1}\right) = \sqrt{\frac{\vartheta_3(0|\frac{it}{2})}{\eta(\frac{it}{2})}} \,, \qquad \text{(B.7)} $$

where $q = e^{-\pi t}$.

B.2 Twisted partition functions

More in general, indicating with the symbol

$$ {}^{P_2}\Box_{P_{1\pm\gamma}} \tag{B.8}$$

the contribution of a boson or a fermion with periodicities P_1 and P_2 around the two cycles of the torus, with an additional twist $\pm\gamma$ around the first cycle, one finds

Bosons

$$ {}^{P}\Box_{P_\gamma} \times {}^{P}\Box_{P_{-\gamma}} \quad : \quad \begin{aligned} Z^B(t,\gamma) &= q^{-\frac{1}{6}+\gamma(1-\gamma)} \prod_{n=0}^{\infty}\left(1-q^{2(n+\gamma)}\right)^{-1} \prod_{n=1}^{\infty}\left(1-q^{2(n-\gamma)}\right)^{-1} \\ &= \frac{\eta(\frac{it}{2})}{\vartheta\left[{\frac{1}{2}-\gamma \atop \frac{1}{2}}\right](0|\frac{it}{2})} = q^{-\gamma^2} \frac{i\eta(\frac{it}{2})}{\vartheta_1(\frac{i\gamma t}{2}|\frac{it}{2})} \,, \end{aligned} \tag{B.9}$$

$$ {}^{P}\Box_{A_\gamma} \times {}^{P}\Box_{A_{-\gamma}} \quad : \quad \begin{aligned} Z^B(t,\gamma) &= q^{\frac{1}{12}-\gamma^2} \prod_{n=1}^{\infty}\left(1-q^{2(n+\gamma)-1}\right)^{-1} \prod_{n=1}^{\infty}\left(1-q^{2(n-\gamma)-1}\right)^{-1} \\ &= \frac{\eta(\frac{it}{2})}{\vartheta\left[{-\gamma \atop \frac{1}{2}}\right](0|\frac{it}{2})} = q^{-\gamma^2} \frac{\eta(\frac{it}{2})}{\vartheta_4(\frac{i\gamma t}{2}|\frac{it}{2})} \,, \end{aligned} \tag{B.10}$$

Fermions

$$ {}^{P}\Box_{P_\gamma} \times {}^{P}\Box_{P_{-\gamma}} \quad : \quad \begin{aligned} Z^{F(P-)}(t,\gamma) &= q^{\frac{1}{6}-\gamma(1-\gamma)} \prod_{n=0}^{\infty}\left(1-q^{2(n+\gamma)}\right) \prod_{n=1}^{\infty}\left(1-q^{2(n-\gamma)}\right) \\ &= \frac{\vartheta\left[{\frac{1}{2}-\gamma \atop \frac{1}{2}}\right](0|\frac{it}{2})}{\eta(\frac{it}{2})} = q^{\gamma^2} \frac{\vartheta_1(\frac{i\gamma t}{2}|\frac{it}{2})}{i\eta(\frac{it}{2})} \,, \end{aligned} \tag{B.11}$$

$$ {}^{A}\Box_{P_\gamma} \times {}^{A}\Box_{P_{-\gamma}} \quad : \quad \begin{aligned} Z^{F(P+)}(t,\gamma) &= q^{\frac{1}{6}-\gamma(1-\gamma)} \prod_{n=0}^{\infty}\left(1+q^{2(n+\gamma)}\right) \prod_{n=1}^{\infty}\left(1+q^{2(n-\gamma)}\right) \\ &= \frac{\vartheta\left[{\frac{1}{2}-\gamma \atop 0}\right](0|\frac{it}{2})}{\eta(\frac{it}{2})} = q^{\gamma^2} \frac{\vartheta_2(\frac{i\gamma t}{2}|\frac{it}{2})}{\eta(\frac{it}{2})} \,, \end{aligned} \tag{B.12}$$

$$ {}^{P}\Box_{A_\gamma} \times {}^{P}\Box_{A_{-\gamma}} \quad : \quad \begin{aligned} Z^{F(A+)}(t,\gamma) &= q^{-\frac{1}{12}+\gamma^2} \prod_{n=1}^{\infty}\left(1-q^{2(n+\gamma)-1}\right) \prod_{n=1}^{\infty}\left(1-q^{2(n-\gamma)-1}\right) \\ &= \frac{\vartheta\left[{-\gamma \atop \frac{1}{2}}\right](0|\frac{it}{2})}{\eta(\frac{it}{2})} = q^{\gamma^2} \frac{\vartheta_4(\frac{i\gamma t}{2}|\frac{it}{2})}{\eta(\frac{it}{2})} \,, \end{aligned} \tag{B.13}$$

$$ {}^{A}\Box_{A_\gamma} \times {}^{A}\Box_{A_{-\gamma}} \quad : \quad \begin{aligned} Z^{F(A-)}(t,\gamma) &= q^{-\frac{1}{12}+\gamma^2} \prod_{n=1}^{\infty}\left(1+q^{2(n+\gamma)-1}\right) \prod_{n=1}^{\infty}\left(1+q^{2(n-\gamma)-1}\right) \\ &= \frac{\vartheta\left[{-\gamma \atop 0}\right](0|\frac{it}{2})}{\eta(\frac{it}{2})} = q^{\gamma^2} \frac{\vartheta_3(\frac{i\gamma t}{2}|\frac{it}{2})}{\eta(\frac{it}{2})} \,, \end{aligned} \tag{B.14}$$

where $q = e^{-\pi t}$.

Appendix C

Field-theory computations

In this appendix we report some field theory results which are relevant for the interpretation of string theory results for D-brane dynamics.

C.1 Asymptotic fields of a Dp-brane

A Dp-brane localized at position $\vec{Y}$ in transverse space represents a source a dilaton, p-form and gravitational sources

$$J_{(\phi)}(x) = \hat{a}_p \delta^{(9-p)}(\vec{x} - \vec{Y}) \; , \tag{C.1}$$

$$J_{(C)}^{\mu_1 \ldots \mu_{p+1}}(x) = \hat{\mu}_p \bar{\epsilon}^{\mu_1 \ldots \mu_{p+1}} \delta^{(9-p)}(\vec{x} - \vec{Y}) \; , \tag{C.2}$$

$$J_{(h)}^{\mu\nu}(x) = \hat{T}_p \bar{\eta}^{\mu\nu} \delta^{(9-p)}(\vec{x} - \vec{Y}) \; , \tag{C.3}$$

where $\bar{\eta}^{\mu\nu}$ and $\bar{\epsilon}^{\mu_1 \ldots \mu_p+1}$ are the Minkowski and Levi-Civita tensors on the Dp-brane world-volume, with indices running from 0 to p. The asymptotic fields in units of $\sqrt{2}\kappa_{(10)}$ at some point $\vec{Z}$ are then given by

$$\phi = \int d^{10}x \Delta_{(\phi)}(Z - x) J_{(\psi)}(x) \; , \tag{C.4}$$

$$C^{\mu_1 \ldots \mu_{p+1}} = \frac{1}{(p+1)!} \int d^{10}x \Delta_{(C)}^{\mu_1 \ldots \mu_{p+1}, \nu_1 \ldots \nu_{p+1}}(Z - x) J_{(C)\mu_1 \ldots \mu_{p+1}}(x) \; , \tag{C.5}$$

$$h^{\mu\nu} = -\frac{1}{2} \int d^{10}x \Delta_{(h)}^{\mu\nu,\rho\sigma}(Z - x) J_{(h)\mu\nu}(x) \; , \tag{C.6}$$

in terms of the dilaton, p-form and graviton propagators. Taking the Feynman and De Donder gauge for the p-form and graviton, the propagators in d dimensions are

$$\Delta_{(\phi)} = \Delta_{(d)} \; , \tag{C.7}$$

$$\Delta_{(C)}^{\mu_1 \ldots \mu_{p+1}, \nu_1 \ldots \nu_{p+1}} = \left(\eta^{\mu_1 \nu_1} \ldots \eta^{\mu_{p+1} \nu_{p+1}} + \text{ant.} \right) \Delta_{(d)} \; , \tag{C.8}$$

$$\Delta_{(h)}^{\mu\nu,\rho\sigma} = \left(\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho} - \frac{2}{d-2} \eta^{\mu\nu} \eta^{\rho\sigma} \right) \Delta_{(d)} \; , \tag{C.9}$$

where

$$\Delta_{(d)}(x) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip\cdot x}}{p^2} = \frac{1}{4\pi^{d/2}} \Gamma\left(\frac{d-2}{2}\right) \frac{1}{x^{d-2}} \,. \tag{C.10}$$

Taking $d = 10$, multiplying by $\sqrt{2}\kappa_{(10)}$ to get the correct units and explicating the couplings one finds, in the Einstein frame

$$\begin{cases} \phi = \dfrac{3-p}{2} \kappa_{(10)}^2 T_p \Delta_{(9-p)}(r) \\ C^{\mu_1 \ldots \mu_{p+1}} = 2\kappa_{(10)}^2 T_p \epsilon_{(p+1)}^{\mu_1 \ldots \mu_{p+1}} \Delta_{(9-p)}(r) \\ h^{\alpha\beta} = \dfrac{p-7}{4} \kappa_{(10)}^2 T_p \eta^{\alpha\beta} \Delta_{(9-p)}(r) \,, \;\; h^{ij} = \dfrac{p+1}{4} \kappa_{(10)}^2 T_p \delta^{ij} \Delta_{(9-p)}(r) \end{cases} \,, \tag{C.11}$$

with $\vec{r} = \vec{Z} - \vec{Y}$. The result in the string frame is easily obtained through the shift $h_S^{\mu\nu} = h_E^{\mu\nu} + 1/2\eta^{\mu\nu}\phi$. One finds

$$\begin{cases} \phi = \dfrac{3-p}{2} \kappa_{(10)}^2 T_p \Delta_{(9-p)}(r) \\ C^{\mu_1 \ldots \mu_{p+1}} = 2\kappa_{(10)}^2 T_p \epsilon_{(p+1)}^{\mu_1 \ldots \mu_{p+1}} \Delta_{(9-p)}(r) \\ h^{\alpha\beta} = -\kappa_{(10)}^2 T_p \eta^{\alpha\beta} \Delta_{(9-p)}(r) \,, \;\; h^{ij} = \kappa_{(10)}^2 T_p \delta^{ij} \Delta_{(9-p)}(r) \end{cases} \,. \tag{C.12}$$

Here T_p is the true tension of the Dp-brane.

C.2 Interaction between static D-branes

With the knowledge of the sources Eqs. (C.1)-(C.3) associated to a Dp-brane and the propagators Eqs. (C.7)-(C.9), it is easy to compute the interaction amplitude between two D-branes.

C.2.1 Dp-Dp static interaction

Two static Dp-branes can interact exchanging the dilaton and graviton in the NSNS sector and the (p+1)-form in the RR sector. The interaction amplitude is

$$\begin{aligned} \mathcal{A} = \int d^{10}x \int d^{10}\tilde{x} \, \Big[& J_{(\phi)}(x) \Delta_{(\phi)}(x-\tilde{x}) \tilde{J}_{(\phi)}(\tilde{x}) + \frac{1}{4} J_{(h)}(x) \cdot \Delta_{(h)}(x-\tilde{x}) \cdot \tilde{J}_{(h)}(\tilde{x}) \\ & - J_{(C)}(x) \cdot \Delta_{(C)}(x-\tilde{x}) \cdot \tilde{J}_{(C)}(\tilde{x}) \Big] \,, \end{aligned} \tag{C.13}$$

where J and $\tilde{J}$ refer to the first and the second Dp-branes at positions $\vec{Y}_1$ and $\vec{Y}_2$. One finds

$$\begin{aligned} \mathcal{A} &= V_{p+1} \hat{T}_p^2 \left[\left(\frac{3-p}{4}\right)^2 + \frac{(p+1)(7-p)}{16} - 1 \right] \Delta_{(9-p)}(r) \\ &= V_{p+1} \hat{T}_p^2 (1-1) \Delta_{(9-p)}(r) \,, \end{aligned} \tag{C.14}$$

where $\vec{r} = \vec{Y}_1 - \vec{Y}_2$.

C.2.2 Dp-D(p+4) static interaction

A Dp-brane and a D(p+4)-brane can interact exchanging only the dilaton and graviton in the NSNS sector, since there is no form in the RR sector coupling to both of them. The interaction amplitude is therefore

$$\mathcal{A} = \int d^{10}x \int d^{10}\tilde{x} \left[J_{(\phi)}(x)\Delta_{(\phi)}(x-\tilde{x})\tilde{J}_{(\phi)}(\tilde{x}) + \frac{1}{4} J_{(h)}(x) \cdot \Delta_{(h)}(x-\tilde{x}) \cdot \tilde{J}_{(h)}(\tilde{x}) \right] , \quad \text{(C.15)}$$

where J and $\tilde{J}$ refer to the Dp-brane and D(p+4)-brane at transverse positions $\vec{Y}_1$ and $\vec{Y}_2$. One finds

$$\begin{aligned} \mathcal{A} &= V_{p+1}\hat{T}_p\hat{T}_{p+4} \left[-\frac{(3-p)(p+1)}{16} + \frac{(p+1)(3-p)}{16} \right] \Delta_{(5-p)}(r) \\ &= \frac{(3-p)(p+1)}{16} V_{p+1}\hat{T}_p^2 (1-1)\Delta_{(5-p)}(r) \ . \end{aligned} \quad \text{(C.16)}$$

C.3 Interaction between moving D-branes

The sources Eqs. (C.1)-(C.3) can be generalized to a Dp-brane moving with constant velocity $v = \tanh \pi\epsilon$ through a Lorentz transformation. Taking for simplicity the velocity in the x^9 direction, the longitudinal δ-function becomes $\delta(\cosh \pi\epsilon\, x^9 - \sinh \pi\epsilon\, x^0)$, whereas the polarizations transform by multiplying for each index with the matrix

$$(\Sigma_V)^{\mu}{}_{\nu}(\epsilon) = \begin{pmatrix} \cosh \pi\epsilon & -\sinh \pi\epsilon \\ -\sinh \pi\epsilon & \cosh \pi\epsilon \end{pmatrix} . \quad \text{(C.17)}$$

One can then repeat the same computations as in the static case for the interaction between two D-brane moving with constant velocities $v_{1,2} = \tanh \pi\epsilon_{1,2}$. Being $\epsilon = \epsilon_1 - \epsilon_2$ the relative rapidity, the Lorentz transformation gives a $\cosh \pi\epsilon$ dependence to the RR gauge form exchange, since its antisymmetric polarization transform as a vector in the (x^0, x^9) boost plane, and a $\cosh 2\pi\epsilon$ to the exchange of the 0,9 components of the graviton, which transform as a rank two tensor. The dilaton exchange, as well as the exchange of the transverse components of the graviton, do not produce any dependence on the rapidity since they transform as scalars in the boost plane. The rotated argument of the longitudinal δ-function produces an overall $\sinh \pi\epsilon$ in the denominator instead of the total time of interaction.

C.3.1 Dp-Dp interaction

One finds

$$\begin{aligned} \mathcal{A} &= \frac{V_p}{\sinh \pi\epsilon}\hat{T}_p^2 \left[\left(\frac{3-p}{4}\right)^2 + \frac{(p+1)(7-p) - 4 + 4\cosh 2\pi\epsilon}{16} - \cosh \pi\epsilon \right] \Delta_{(8-p)}(b) \\ &= \frac{V_p}{\sinh \pi\epsilon}\hat{T}_p^2 \left(\frac{3}{4} + \frac{1}{4}\cosh 2\pi\epsilon - \cosh \pi\epsilon \right) \Delta_{(8-p)}(b) \\ &= 2V_p\hat{T}_p^2 \frac{\sinh^4 \frac{\pi\epsilon}{2}}{\sinh \pi\epsilon} \Delta_{(8-p)}(b) \ , \end{aligned} \quad \text{(C.18)}$$

where $\vec{b} = \vec{Y}_1 - \vec{Y}_2$.

C.3.2 Dp-D(p+4) interaction

One finds

$$\begin{aligned}\mathcal{A} &= \frac{V_p}{\sinh \pi\epsilon}\hat{T}_p\hat{T}_{p+4}\left[-\frac{(3-p)(p+1)}{16} + \frac{(p+1)(3-p) - 4 + 4\cosh 2\pi\epsilon}{16}\right]\Delta_{(4-p)}(b) \\ &= \frac{V_p}{\sinh \pi\epsilon}\hat{T}_p\hat{T}_{p+4}\left(-\frac{1}{4} + \frac{1}{4}\cosh 2\pi\epsilon\right)\Delta_{(4-p)}(b) \\ &= \frac{1}{2}V_p\hat{T}_p\hat{T}_{p+4}\frac{\sinh\frac{\pi\epsilon}{2}}{\sinh \pi\epsilon}\Delta_{(4-p)}(b) \ ,\end{aligned} \tag{C.19}$$

C.4 U(1) Effective actions

Consider the one-loop effective action of a U(1) gauge theory in D dimensions coupled to a particle of mass m, charge e and gyromagnetic ratio g. Generically, this particle will correspond to some irreducible representation of the ten-dimensional Poicarré group which can be constructed with Wigner's method from a corresponding representation of the massive little group SO(D−1). The Euclidean effective action in the constant field approximation is

$$\Gamma(A) = \mathrm{STr}\ln\left(\mathcal{D}^2 + m^2 + \frac{eg}{2}\Sigma_{\mu\nu}F^{\mu\nu}\right) \ . \tag{C.20}$$

Here $\mathcal{D}_\mu = \partial_\mu - ieA_\mu$ is the usual covariant derivative and $\Sigma_{\mu\nu}$ are the generators of the Lorentz group SO(D−1,1) in the representation corresponding to the particle running in the loop. The supertrace STr counts bosons and fermions with opposite signs, and involves both a functional and a representation trace. Using the usual formula

$$\ln X = \int_0^\infty \frac{dt}{t}e^{-tX} \ , \tag{C.21}$$

the effective action (C.20) can be rewritten as

$$\Gamma(A) = \int_0^\infty \frac{dt}{t}Z(t,A) \ , \tag{C.22}$$

where

$$Z(t,A) = \mathrm{STr}e^{-tH} \tag{C.23}$$

is formally the partition function at temperature t for a particle in a magnetic field in D space dimensions with Hamiltonian

$$H = (p - eA)^2 + m^2 + \frac{eg}{2}\Sigma_{\mu\nu}F^{\mu\nu} \ . \tag{C.24}$$

The corresponding Euclidean lagrangian is

$$L = \frac{1}{4}\dot{q}^\mu\dot{q}_\mu + m^2 + ieA_\mu\dot{q}^\mu + \frac{eg}{2}\Sigma_{\mu\nu}F^{\mu\nu} \ , \tag{C.25}$$

which is equivalent through a canonical transformation to the more conventional

$$L = m\sqrt{-\dot{q}^\mu\dot{q}_\mu} + ieA_\mu\dot{q}^\mu + \frac{eg}{2}\Sigma_{\mu\nu}F^{\mu\nu} \ . \tag{C.26}$$

As well known, the partition function Z(t) admits a path-integral representation. The dependence on the representation factorizes (since $F_{\mu\nu}$ is supposed constant) and one can write $Z(t,A) = Z_r(t,A)\hat{Z}(t,A)$. The universal part can be written as a path-integral over bosonic world-line coordinates

$$\begin{aligned} \hat{Z}(t,A) &= \int \mathcal{D}q^\mu(\tau) \exp\left\{-m \int_0^t d\tau \sqrt{-\dot{q}^\mu \dot{q}_\mu} + ieA_\mu \dot{q}^\mu\right\} \\ &= e^{-m^2 t} \int \mathcal{D}q^\mu(\tau) \exp\left\{-\int_0^t d\tau \frac{1}{4}\dot{q}^\mu \dot{q}_\mu + ieA_\mu \dot{q}^\mu\right\} . \end{aligned} \tag{C.27}$$

The representation dependent part is simply

$$Z_{rep}(t,A) = \mathrm{STr} \exp\left\{-t\frac{eg}{2}\Sigma_{\mu\nu}F^{\mu\nu}\right\} . \tag{C.28}$$

For spinor representations, one can write this as a path-integral over fermionic world-line coordinates. The string world-sheet bosonic and fermionic fields are nothing but the generalization of the world-line fields appearing here, allowing the description of modes with arbitrary mass and “spin”.

The partition functions Eqs. (C.27) and (C.28) can be easily evaluated in the simple case in which there is a constant flux only in some plane, say $F_{ii+1} = B$. The contributions to the universal part of the partition function Eq. (C.27) from each of D coordinates is as follows. Each of the D-2 transverse coordinates gives simply the contribution of a free particle with mass 1/2 and temperature t in one dimension with volume V_1, $Z_{free} = V_1/\sqrt{4\pi t}$. The two coordinates in the flux plane give instead essentially the contribution of a harmonic oscillator with frequency $w = 2eB$ and temperature t, $Z'_{osc} = V_2/(4\pi t)eBt/\sinh eBt$. The normalization is fixed by the requirement that in the limit $B \to 0$, $Z_{osc} \to Z^2_{free}$, reflecting the degeneracy corresponding to the arbitrariness of the center of the Landau orbit in the flux plane. Finally,

$$\hat{Z}(t,A) = V_D e^{-m^2 t}(4\pi t)^{-\frac{D}{2}} \frac{eBt}{\sinh eBt} . \tag{C.29}$$

In Eq. (C.28), only the Cartan generator $\lambda = \Sigma_{ii+1}$ corresponding to the SU(2) subalgebra of the flux plane appears. For this reason, $Z_{rep}(t,A)$ depends only on the SU(2) content of the representation. Since a generic representation of SO(D$-$1) will decompose into various SU(2) representations depending on D, it is enough to study the generic spin j SU(2) representation. As well known, this representation is (2j+1)-dimensional, and the helicity λ has eigenvalues $m_\lambda = -j, -j+1, ...j-1, j$. The supertrace appearing in Eq. (C.28) is then straightforward to evaluate, and one finds [127, 12]

$$\begin{aligned} Z_j(t,A) &= \mathrm{STr} e^{-egBt\lambda} = (-1)^{2j} \sum_{m_\lambda=-j}^{j} e^{-egBtm_\lambda} \\ &= (-1)^{2j} \frac{\sinh(2j+1)\frac{egBt}{2}}{\sinh\frac{egBt}{2}} . \end{aligned} \tag{C.30}$$

For example

$$\begin{aligned} &Z_0(t,A) = 1 && Z_{\frac{1}{2}}(t,A) = -2\cosh\frac{egBt}{2}\ , \\ &Z_1(t,A) = 2\cosh 2\frac{egBt}{2} + 1\ , && Z_{\frac{3}{2}}(t,A) = -2\cosh 3\frac{egBt}{2} - 2\cosh\frac{egBt}{2}\ , \\ &\quad ... \end{aligned} \tag{C.31}$$

Therefore, the result for a loop of a generic particle is

$$\Gamma(A) = \frac{V_D}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dt}{t^{1+\frac{D}{2}}} e^{-m^2 t}\frac{eBt}{\sinh eBt} Z_{rep}(\frac{egBt}{2})\ , \tag{C.32}$$

where Z_{rep} can be obtained from Z_j by decomposing the representation in SU(2) spin j representations. The case of an electric field $F_{0i} = E$ is obtained by analytic continuation by setting $B = iE$.

Consider now the contribution to the effective action from a loop of some supermultiplet representing an N extended supersymmetry with n supercharges. The numerator Z_N is in this case independent of the spacetime dimensionality D and for the multiplet with lowest possible spins one finds [127, 12]

$$Z_{rep}(\frac{egBt}{2}) = \left(2\sinh\frac{egBt}{4}\right)^{\frac{n}{4}}\ . \tag{C.33}$$

In a light-cone gauge path-integral à la Green-Schwarz, with spacetime fermions whose fermionic z.m. are directly associated to the linearly realized supersymmetry, this factors comes out directly form the integral over fermionic z.m..

Contact with the SYM effective actions relevant to D-brane dynamics is now straightforward. The only subtlety is that there is in this case only a one-dimensional translational invariance, rather than a two-dimensional one, in the "flux" plane (because it corresponds to a NN and a DD directions and not two NN directions as for a true electromagnetic field). Correspondingly, the contribution of the two coordinates in the flux plane is now the product of a free particle contribution Z_{free} and that of a true harmonic oscillator Z_{osc}. Taking D=p+1, $m = b/(2\pi\alpha')$, $e = 1/(2\pi\alpha')$, $g = 2$ as appropriate for string modes and $B = i\pi\epsilon$, one finds indeed

$$\mathcal{A} = \frac{V_p}{2(4\pi)^{\frac{p}{2}}}\int_0^\infty \frac{dt}{t^{\frac{p+2}{2}}} e^{-(\frac{b}{2\pi\alpha'})^2 t}\frac{(2\sin\frac{\pi\epsilon}{4\pi\alpha'}t)^{\frac{n}{4}}}{\sin\frac{\pi\epsilon}{2\pi\alpha'}t}\ , \tag{C.34}$$

with n=16 for the Dp-Dp interaction and n=8 for the Dp-D(p+4) interaction.

Appendix D

Boundary states and propagators

In this appendix, we report some details about the construction of the boundary state for the D=4 point-like D-brane configurations studied in Chapter 4.

D.1 Boundary state and partition functions

As explained in Chapter 4, the boundary state, as well as the partition functions, split into a universal Minkowski part and a compact part depending on the compactification scheme and on which ten-dimensional D-brane configuration one starts with.

D.1.1 Non-compact part

As for the universal four-dimensional part, one has N b.c. in the time direction and D b.c. in all the three space directions. Call $\vec{Y}_{1,2}$ the positions in the (x^2, x^3) transverse plane and $v_{1,2} = \tanh \pi\epsilon_{1,2}$ the constant velocities in the x^1 direction. It will be convenient to group the fields along the time direction and the longitudinal direction, that we shall take to be for simplicity x^1, into the light-cone combinations $X^\pm = (X^0 \pm X^1)/\sqrt{2}$ and $\psi^\pm = (\psi^0 \pm \psi^1)/\sqrt{2}$, whose modes satisfy $[a_n^\pm, a_m^{\dagger\mp}] = -\delta_{m,n}$ and $\{\psi_n^\pm, \psi_m^{\dagger\mp}\} = -\delta_{m,n}$. Similarly, the fields along the two transverse directions are grouped into the complex combinations $Y, Y^* = (X^2 \pm iX^3)/\sqrt{2}$ and $\xi, \xi^* = (\psi^2 \pm i\psi^3)/\sqrt{2}$, whose modes satisfy $[b_n, b_m^{\dagger *}] = \delta_{m,n}$ and $\{\xi_n, \xi_m^{\dagger *}\} = \delta_{m,n}$.

Consider first the bosons. The z.m. part of the bosonic boundary state is

$$|B_0, \epsilon\rangle_B = \delta(\cosh \pi\epsilon\, x^1 - \sinh \pi\epsilon\, x^0)\delta^{(2)}\left(\vec{x} - \vec{Y}\right) = \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot Y} |k(\epsilon)\rangle \ , \tag{D.1}$$

where $k^\mu(\epsilon_i) = (\sinh \pi\epsilon_i\, k^1, \cos \pi\epsilon_i\, k^1, k^2, k^3)$. Notice that a static D-brane can transfer momentum but no energy, whereas a moving D-brane can transfer both of them in a combination orthogonal to its own four-momentum. Correspondingly, the zero mode contribution to the partition function is

$$\langle B_0|e^{-lH}|B_0, \epsilon\rangle_B = \frac{1}{\sinh \pi\epsilon} \int \frac{d^2\vec{k}}{(2\pi)^2} e^{i\vec{k}\cdot\vec{b}} e^{-\frac{\vec{k}^2}{2}} = \frac{1}{\sinh \pi\epsilon} \frac{e^{-\frac{b^2}{2l}}}{(2\pi l)} \ , \tag{D.2}$$

where $\vec{b} = \vec{Y}_1 - \vec{Y}_2$ is now the impact parameter in the (x^2, x^3) transverse plane and $\epsilon = \epsilon_1 - \epsilon_2$ the relative rapidity. Consider next the bosonic oscillators. The static b.c. would be imply

$$\left(a_n^{\pm} + \tilde{a}_n^{\dagger\mp}\right) |B_{osc}\rangle_B = 0 \ , \tag{D.3}$$

$$\left(b_n - \tilde{b}_n^{\dagger}\right) |B_{osc}\rangle_B = \left(b_n^* - \tilde{b}_n^{\dagger *}\right) |B_{osc}\rangle_B = 0 \ , \tag{D.4}$$

which are solved by

$$|B_{osc}\rangle_B = \exp\left\{ - \sum_{n=1}^{\infty} \left(a_n^{\dagger +} \tilde{a}_n^{\dagger +} + a_n^{\dagger -} \tilde{a}_n^{\dagger -} + b_n^{\dagger} \tilde{b}_n^{\dagger *} + b_n^{\dagger *} \tilde{b}_n^{\dagger} \right) \right\} |0\rangle \ . \tag{D.5}$$

The effect of the velocity is to transform the light-cone oscillators, which pick up an imaginary phase, $a_n^{\pm} \to e^{\pm \pi \epsilon_i} a_n^{\pm}$, so that the boosted boundary state reads

$$|B_{osc}, \epsilon_i\rangle_B = \exp\left\{ - \sum_{n=1}^{\infty} \left(e^{2\pi\epsilon_i} a_n^{\dagger +} \tilde{a}_n^{\dagger +} + e^{-2\pi\epsilon_i} a_n^{\dagger -} \tilde{a}_n^{\dagger -} + b_n^{\dagger} \tilde{b}_n^{\dagger *} + b_n^{\dagger *} \tilde{b}_n^{\dagger} \right) \right\} |0\rangle \ . \tag{D.6}$$

It is straightforward to compute the contribution of the bosonic oscillators of the (0,1) and (2,3) pairs. One finds, taking into account the corresponding zero-point energy

$$\langle B_{osc}, \epsilon_1 | e^{-lH} | B_{osc}, \epsilon_2 \rangle^B_{(0,1)} = q^{-\frac{1}{12}} \prod_{n=0}^{\infty} \left[\left(1 - q^{2n} e^{2\pi\epsilon}\right) \left(1 - q^{2n} e^{-2\pi\epsilon}\right) \right]^{-1} \ , \tag{D.7}$$

$$\langle B_{osc}, \epsilon_1 | e^{-lH} | B_{osc}, \epsilon_2 \rangle^B_{(2,3)} = q^{-\frac{1}{12}} \prod_{n=0}^{\infty} \left(1 - q^{2n}\right)^{-2} \ , \tag{D.8}$$

where $q = e^{-2\pi l}$. The contribution of the b,c ghosts exactly cancels that of the (2,3) pair, so that the total bosonic part of the non-compact partition function is

$$Z_B^{(nc)}(l, \epsilon) = 2 \, \frac{e^{-\frac{b^2}{2l}}}{(2\pi l)} \frac{\eta(2il)}{\vartheta_1(i\epsilon|2il)} \ . \tag{D.9}$$

Consider now the fermions. In the NSNS sector there are no z.m., and the corresponding boundary state can be taken to be simply the Fock vacuum. In the RR sector, the static zero mode b.c. are

$$\left(\psi_0^{\pm} + i\eta \tilde{\psi}_0^{\mp}\right) |B_0, \eta\rangle_{RR} = 0 \ , \tag{D.10}$$

$$\left(\xi_0 - i\eta \tilde{\xi}_0\right) |B_0, \eta\rangle_{RR} = \left(\xi_0^* - i\eta \tilde{\xi}_0^*\right) |B_0, \eta\rangle_{RR} = 0 \ , \tag{D.11}$$

and the state $|B_0, \eta\rangle_{RR}$ solving these b.c. can be constructed from the vacua $|\omega\rangle$ and $|\tilde{\omega}\rangle$ satisfying $\psi_0^+ |\omega\rangle = \xi_0 |\omega\rangle = 0$ and $\tilde{\psi}_0^+ |\tilde{\omega}\rangle = \tilde{\xi}_0^* |\tilde{\omega}\rangle = 0$. One finds

$$|B_0, \eta\rangle_F = \begin{cases} \exp\left\{ i\eta \left(\psi_0^- \tilde{\psi}_0^- + \xi_0^* \tilde{\xi}_0 \right) \right\} |\omega\rangle \otimes |\tilde{\omega}\rangle \ , & \text{RR} \\ |0\rangle & , \ \text{NSNS} \end{cases} \ . \tag{D.12}$$

The effect of a boost is two transform the light-cone z.m. as $\psi_0^\pm \to e^{\pm\pi\epsilon_i}\psi_0^\pm$. Also the spinor vacua are not invariant, but transform as $|\omega\rangle \to e^{\pi\epsilon_i/2}|\omega\rangle$ and $|\tilde{\omega}\rangle \to e^{\pi\epsilon_i/2}|\tilde{\omega}\rangle$ so that the boosted version of the z.m. boundary state is

$$|B_0,\epsilon_i,\eta\rangle_F = \left\{ \begin{array}{ll} e^{\pi\epsilon_i}\exp\left\{i\eta\left(e^{-2\pi\epsilon_i}\psi_0^-\tilde{\psi}_0^- + \xi_0^*\tilde{\xi}_0\right)\right\}|\omega\rangle\otimes|\tilde{\omega}\rangle & , \ \mathrm{RR} \\ |0\rangle & , \ \mathrm{NSNS} \end{array} \right. . \tag{D.13}$$

The corresponding contributions to the partition functions are found to be

$$\langle B_0,\epsilon_1,\eta|e^{-lH}|B_0,\epsilon_2,\eta'\rangle^F_{(0,1)} = \left\{ \begin{array}{ll} (e^{\pi\epsilon}+\eta\eta' e^{-\pi\epsilon}) & , \ \mathrm{RR} \\ 1 & , \ \mathrm{NSNS} \end{array} \right. , \tag{D.14}$$

$$\langle B_0,\epsilon_1,\eta|e^{-lH}|B_0,\epsilon_2,\eta'\rangle^F_{(2,3)} = \left\{ \begin{array}{ll} (1+\eta\eta') & , \ \mathrm{RR} \\ 1 & , \ \mathrm{NSNS} \end{array} \right. . \tag{D.15}$$

For the oscillator modes, the static b.c. would be

$$\left(\psi_n^\pm + i\eta\tilde{\psi}_n^{\dagger\mp}\right)|B_{osc},\eta\rangle_F = 0 \ , \tag{D.16}$$

$$\left(\xi_n - i\eta\tilde{\xi}_n^\dagger\right)|B_{osc},\eta\rangle_F = \left(\xi_n^* - i\eta\xi_n^{\dagger *}\right)|B_{osc},\eta\rangle_F = 0 \ , \tag{D.17}$$

with integer or half-integer moding in the RR or NSNS sectors. These are solved by

$$|B_{osc},\eta\rangle_F = \exp\left\{i\eta\sum_{n>0}\left(\psi_n^{\dagger+}\tilde{\psi}_n^{\dagger+} + \psi_n^{\dagger-}\tilde{\psi}_n^{\dagger-} + \xi_n^\dagger\tilde{\xi}_n^{\dagger *} + \xi_n^{\dagger *}\tilde{\xi}_n^\dagger\right)\right\}|0\rangle \ , \tag{D.18}$$

with appropriate moding. As for the bosons, the effect of the velocity is to transform the light-cone oscillators, which pick up the same imaginary phase, $\psi_n^\pm \to e^{\pm\pi\epsilon_i}\psi_n^\pm$, so that the boosted boundary state reads

$$|B_{osc},\epsilon_i\rangle_F = \exp\left\{i\eta\sum_{n>0}\left(e^{2\pi\epsilon_i}\psi_n^{\dagger+}\tilde{\psi}_n^{\dagger+} + e^{-2\pi\epsilon_i}\psi_n^{\dagger-}\tilde{\psi}_n^{\dagger-} + \xi_n^\dagger\tilde{\xi}_n^{\dagger *} + \xi_n^{\dagger *}\tilde{\xi}_n^\dagger\right)\right\}|0\rangle \ . \tag{D.19}$$

The contribution to the partition function from the fermionic oscillators of the (0,1) and (2,3) pairs is

$$\langle B_{osc},\epsilon_1,\eta|e^{-lH}|B_{osc},\epsilon_2,\eta'\rangle^F_{(0,1)} = q^b\prod_{n>0}^{\infty}\left[\left(1+\eta\eta' q^{2n}e^{2\pi\epsilon}\right)\left(1+\eta\eta' q^{2n}e^{-2\pi\epsilon}\right)\right] \ , \tag{D.20}$$

$$\langle B_{osc},\epsilon_1,\eta|e^{-lH}|B_{osc},\epsilon_2,\eta'\rangle^F_{(2,3)} = q^b\prod_{n>0}^{\infty}\left(1+\eta\eta' q^{2n}\right)^2 \ , \tag{D.21}$$

with integer and half-integer moding and $b = 1/12$ or $-1/6$ in the RR and NSNS sectors, and as before $q = e^{-2\pi l}$. The contribution of the β,γ superghosts exactly cancels that of the (2,3) pair, so that the total fermionic part of the non-compact partition function is, in each spin structure s,

$$Z_{Fs}^{(nc)}(l,\epsilon) = \frac{\vartheta_\alpha(i\epsilon|2il)}{\eta(2il)} \ , \tag{D.22}$$

with $\alpha = 1,2$ for s=R$\pm$ and $\alpha = 3,4$ for s=NS$\pm$. Actually, for s=R$-$, the result vanishes because of the (2,3) fermionic z.m..

D.1.2 Compact part

In the compact direction, the bosonic zero modes get drastically modified. As discussed in Chapter 4, the z.m. part of the bosonic boundary state is a discrete superposition of Kaluza-Klein and winding states, weighted by the position in the D directions and the Wilson line in the N directions. One finds, in short notation

$$|B_0, \vec{Y}, \vec{W}\rangle_B = \frac{V_p}{V_{\mathcal{M}_6}} \sum_{\vec{k}\in\Gamma_6^*, \vec{w}\in\Gamma_6} e^{i(\vec{k}\cdot\vec{Y}+\vec{w}\cdot\vec{W})}|\vec{k},\vec{w}\rangle \ , \tag{D.23}$$

where we have normalized the states $|\vec{k},\vec{w}\rangle$ such that $\langle\vec{k},\vec{w}|\vec{k}',\vec{w}'\rangle = V_p\delta_{\vec{k},\vec{k}'},\delta_{\vec{w},\vec{w}'}$. The normalization of the boundary state comes from the discretization of the Fourier transform in the continuum boundary state. For simplicity, we keep only the part with zero momentum and winding, neglecting all the higher modes. By doing so, the above boundary state reduces simply to

$$|B_0\rangle_B = \frac{V_p}{V_{\mathcal{M}_6}}|\vec{0},\vec{0}\rangle \ . \tag{D.24}$$

In this limit, the contribution of the bosonic z.m. in the compact directions is just

$$\langle B_0|e^{-lH}|B_0\rangle_B = \frac{V_p^2}{V_{\mathcal{M}_6}} \ , \tag{D.25}$$

instead of the continuum result $V_p(2\pi l)^{p/2-3}e^{-b^2/(2l)}$. Notice, as a dimensional check, that the latter continuum expression has the same dimensions since l carries the dimensions of a length squared. The remaining of the compact contribution has to be analyzed case by case. The fields associated to the six compact directions are grouped into the complex combinations $Z^a, Z^{a*} = (X^a \pm iX^{a+1})/\sqrt{2}$ and $\chi^a, \chi^{a*} = (\psi^a \pm i\psi^{a+1})/\sqrt{2}$, for a=4,6,8. The corresponding modes satisfy $[c_n^a, c_m^{\dagger b*}] = \delta^{ab}\delta_{m,n}$ and $\{\chi_n^a, \chi_m^{\dagger b*}\} = \delta^{ab}\delta_{m,n}$.

D0-brane: untwisted sector

The bosonic oscillators in the compact directions satisfy the following b.c.

$$\left(c_n^a - \tilde{c}_n^{\dagger a}\right)|B_{osc}\rangle_B = \left(c_n^{a*} - \tilde{c}_n^{\dagger a*}\right)|B_{osc}\rangle_B = 0 \ , \tag{D.26}$$

which are solved by

$$|B_{osc}\rangle_B = \exp\left\{-\sum_{n=1}^{\infty}\sum_a \left(c_n^{\dagger a}\tilde{c}_n^{\dagger a*} + c_n^{\dagger a*}\tilde{c}_n^{\dagger a}\right)\right\}|0\rangle \ . \tag{D.27}$$

This boundary state is already invariant under orbifold rotation. Indeed, under generic rotations in the three compact planes, the modes of the complex combinations of fields we are using pick up definite phases, $c_n^a \to e^{2\pi i z_a}c_n^a$, and the boundary state is invariant. This is so because the rotations occur in NN,NN planes, and amount to an irrelevant redefinition of the coordinates. The contribution of the bosonic oscillators of each compact pair is found to be

$$\langle B_{osc}|e^{-lH}|B_{osc}\rangle_{(a,a+1)}^B = q^{-\frac{1}{12}}\prod_{n=0}^{\infty}\left(1-q^{2n}\right)^{-2} \ . \tag{D.28}$$

Considering also the factor contributed by the zero modes, the total bosonic part of the compact partition function is

$$Z_B^{(c)}(l) = \frac{1}{V_{\mathcal{M}_6}} \frac{1}{\eta^6(2il)} \ . \tag{D.29}$$

Consider now the fermions. In the NSNS sector, there are no fermionic z.m., so that the z.m. part of the boundary state is simply the Fock vacuum. In the RR sector, the zero mode b.c. are

$$(\chi_0^a - i\eta\tilde{\chi}_0^a)\,|B_0,\eta\rangle_{RR} = (\chi_0^{a*} - i\eta\tilde{\chi}_0^{a*})\,|B_0,\eta\rangle_{RR} = 0 \ . \tag{D.30}$$

The state $|B_0,\eta\rangle_{RR}$ can be constructed from the vacua $|\omega\rangle$ and $|\tilde{\omega}\rangle$ satisfying $\chi_0^a|\omega\rangle = 0$ and $\tilde{\chi}_0^{a*}|\tilde{\omega}\rangle = 0$. One finds

$$|B_0,\eta\rangle_F = \begin{cases} \exp\left\{i\eta\sum_a \chi_0^{a*}\tilde{\chi}_0^a\right\}|\omega\rangle\otimes|\tilde{\omega}\rangle \ , & \text{RR} \\ |0\rangle & , \ \text{NS} \end{cases} \ . \tag{D.31}$$

Under orbifold rotation, $\chi_0^a \to e^{2\pi z_a i}\chi_0^a$ and $|\omega\rangle \to |\omega\rangle$ and $|\tilde{\omega}\rangle \to |\tilde{\omega}\rangle$, and the z.m. boundary state is already invariant. The corresponding contribution to the partition function is, for each pair

$$\langle B_0,\eta|e^{-lH}|B_0,\eta'\rangle^F_{(a,a+1)} = \begin{cases} (1+\eta\eta') & , \ \text{RR} \\ 1 & , \ \text{NSNS} \end{cases} \ . \tag{D.32}$$

Similarly, the b.c. for the fermionic oscillators are

$$\left(\chi_n^a - i\eta\tilde{\chi}_n^{\dagger a}\right)|B_{osc},\eta\rangle_F = \left(\chi_n^{a*} - i\eta\tilde{\chi}_n^{\dagger a*}\right)|B_{osc},\eta\rangle_F = 0 \ , \tag{D.33}$$

with integer or half-integer moding in the RR and NSNS sectors, and are solved by

$$|B_{osc},\eta\rangle_F = \exp\left\{i\eta\sum_{n>0}^{\infty}\sum_a\left(\chi_n^{\dagger a}\tilde{\chi}_n^{\dagger a*} + \chi_n^{\dagger a*}\tilde{\chi}_n^{\dagger a}\right)\right\}|0\rangle \ . \tag{D.34}$$

As its bosonic counterpart, this is already invariant under orbifold rotations, under which $\chi_n^a \to e^{2\pi i z_a}\chi_n^a$. The contribution of each compact pair of fermionic oscillators is found to be

$$\langle B_{osc},\eta|e^{-lH}|B_{osc},\eta'\rangle^F_{(a,a+1)} = q^{-b}\prod_{n=0}^{\infty}\left(1+\eta\eta' q^{2n}\right)^2 \ , \tag{D.35}$$

with integer and half-integer moding and $b = 1/12$ or $-1/6$ in the RR and NSNS sectors. Finally, the total fermionic part of the compact partition function is

$$Z_{sF}^{(c)}(l) = \frac{\vartheta_\alpha^3(0|2il)}{\eta^3(2il)} \ , \tag{D.36}$$

with $\alpha = 1,2$ for s=R$\pm$ and $\alpha = 3,4$ for s=NS$\pm$.

D0-brane: twisted sectors

Consider now a generic orbifold twisted sector, concentrating on one compact pair with twist α_a. Due to the non-integer moding, the fields have in general no longer zero modes (in the notation of the Chapter 4, there is a mode with n=0, but it is no longer hermitian). The bosonic oscillators in the compact directions satisfy the following b.c.

$$\left(c_n^a - \tilde{c}_n^{\dagger a}\right)|B_{osc}\rangle_B = \left(c_n^{a*} - \tilde{c}_n^{\dagger a*}\right)|B_{osc}\rangle_B = 0\ , \tag{D.37}$$

which are solved by

$$|B_{osc}\rangle_B = \exp\left\{-\sum_{n=1}^{\infty}\sum_a \left(c_n^{\dagger a}\tilde{c}_n^{\dagger a*} + c_n^{\dagger a*}\tilde{c}_n^{\dagger a}\right)\right\}|0\rangle\ . \tag{D.38}$$

Again, this boundary state is already invariant under orbifold rotations, under which $c_n^a \to e^{2\pi z_a i}c_n^a$. The contribution of the bosonic oscillators of each compact pair is

$$\langle B_{osc}|e^{-lH}|B_{osc}\rangle^B_{(a,a+1)} = q^{\frac{1}{2}\left[-\frac{1}{6}+\alpha_a(1-\alpha_a)\right]}\prod_{n=0}^{\infty}\left(1-q^{2(n+\alpha_a)}\right)^{-1}\prod_{n=1}^{\infty}\left(1-q^{2(n-\alpha_a)}\right)^{-1} \tag{D.39}$$

Considering also the factor coming from the zero modes, the total contribution of the compact bosons to the partition function is finally

$$Z_B^{(c)}(l,\alpha_a) = \frac{1}{V_{\mathcal{M}_6}}\prod_a \frac{\eta(2il)}{\vartheta\left[{\frac{1}{2}-\alpha_a \atop \frac{1}{2}}\right](0|2il)}\ . \tag{D.40}$$

Consider next the fermions. The b.c. are

$$\left(\chi_n^a - i\eta\tilde{\chi}_n^{\dagger a}\right)|B_{osc},\eta\rangle_F = \left(\chi_n^{a*} - i\eta\tilde{\chi}_n^{\dagger a*}\right)|B_{osc},\eta\rangle_F = 0\ , \tag{D.41}$$

with n integer or half-integer in the RR and NSNS sectors, and are solved by

$$|B_{osc},\eta\rangle_F = \exp\left\{i\eta\sum_{n>0}^{\infty}\sum_a \left(\chi_n^{\dagger a}\tilde{\chi}_n^{\dagger a*} + \chi_n^{\dagger a*}\tilde{\chi}_n^{\dagger a}\right)\right\}|0\rangle\ . \tag{D.42}$$

Again, this is already invariant under orbifold rotations, under which $\chi_n^a \to e^{2\pi z_a i}\chi_n^a$. The contribution of each compact pair of fermionic oscillators is found to be

$$\langle B_{osc},\eta|e^{-lH}|B_{osc},\eta'\rangle^F_{(a,a+1)} = q^{-b(\alpha_a)}\prod_{n=0}^{\infty}\left(1+\eta\eta' q^{2(n+\alpha_a)}\right)\prod_{n=1}^{\infty}\left(1+\eta\eta' q^{2(n-\alpha_a)}\right)\ , \tag{D.43}$$

with n integer and half-integer and $b(\alpha_a) = 1/2[-1/6+\alpha_a(1-\alpha_a)]$ or $1/2(1/12-\alpha_a^2)$ in the RR and NSNS sectors. Finally, the total fermionic part of the compact partition function is

$$Z_{sF}^{(c)}(l) = \prod_a \frac{\vartheta\left[{a-\alpha_a \atop b}\right](0|2il)}{\eta(2il)}\ , \tag{D.44}$$

with $a = 1/2$, $b = 0, 1/2$ for s=R±, and $a = 0$, $b = 0, 1/2$ for s=NS±.

D3-brane: untwisted sector

The bosonic oscillators in the compact directions satisfy now the following b.c.

$$\left(c_n^a + \tilde{c}_n^{\dagger a*}\right)|B_{osc}\rangle_B = \left(c_n^a + \tilde{c}_n^{\dagger a*}\right)|B_{osc}\rangle_B = 0\;, \tag{D.45}$$

which are solved by

$$|B_{osc}\rangle_B = \exp\left\{\sum_{n=1}^{\infty}\sum_a \left(c_n^{\dagger a}\tilde{c}_n^{\dagger a} + c_n^{\dagger a*}\tilde{c}_n^{\dagger a*}\right)\right\}|0\rangle\;. \tag{D.46}$$

As expected, this boundary state is not invariant under orbifold rotation, under which $c_n^a \to e^{2\pi i z_a} c_n^a$. Rather, it becomes

$$|B_{osc}, z_a\rangle_B = \exp\left\{\sum_{n=1}^{\infty}\sum_a \left(e^{4\pi i z_a} c_n^{\dagger a}\tilde{c}_n^{\dagger a} + e^{-4\pi i z_a} c_n^{\dagger a*}\tilde{c}_n^{\dagger a*}\right)\right\}|0\rangle\;. \tag{D.47}$$

The contribution of the bosonic oscillator of each of compact pair is

$$\langle B_{osc}, z_a|e^{-lH}|B_{osc}, z_a'\rangle^B_{(a,a+1)} = q^{-\frac{1}{12}}\prod_{n=0}^{\infty}\left|1 - q^{2n}e^{4\pi i w_a}\right|^{-2}\;, \tag{D.48}$$

where $w_a = z_a - z_a'$ is the relative twist. Considering also the factor coming from the zero modes, the total bosonic part of the compact partition function is

$$Z_B^{(c)}(l, w_a) = \frac{V_3^2}{V_{\mathcal{M}_6}}\eta^3(2il)\prod_a \frac{2\sin 2\pi w_a}{\vartheta_1(2w_a|2il)}\;. \tag{D.49}$$

Consider now the fermions. In the NSNS sector there are no z.m., and the corresponding boundary state is simply the Fock vacuum. In the RR sector, the zero mode b.c. are

$$(\chi_0^a + i\eta\tilde{\chi}_0^{a*})|B_0,\eta\rangle_{RR} = (\chi_0^{a*} + i\eta\tilde{\chi}_0^a)|B_0,\eta\rangle_{RR} = 0\;, \tag{D.50}$$

and the state $|B_0,\eta\rangle_{RR}$ can be constructed from the vacua $|\omega\rangle$ and $|\tilde{\omega}\rangle$ satisfying now $\chi_0^a|\omega\rangle = 0$ and $\tilde{\chi}_0^a|\tilde{\omega}\rangle = 0$. On finds,

$$|B_0,\eta\rangle_F = \left\{\begin{array}{ll} \exp\left\{-i\eta\sum_a \chi_0^{a*}\tilde{\chi}_0^{a*}\right\}|\omega\rangle\otimes|\tilde{\omega}\rangle\;, & \mathrm{RR} \\ |0\rangle & ,\ \mathrm{NSNS} \end{array}\right.\;. \tag{D.51}$$

Under orbifold rotation, one has $\chi_0^a \to e^{2\pi z_a i}\chi_0^a$ and $|\omega\rangle \to e^{i\pi z_a}|\omega\rangle$, $|\tilde{\omega}\rangle \to e^{i\pi z_a}|\tilde{\omega}\rangle$. The z.m. boundary state therefore becomes

$$|B_0, z_a, \eta\rangle_F = \left\{\begin{array}{ll} e^{2\pi i z_a}\exp\left\{-i\eta\sum_a e^{-4\pi i z_a}\chi_0^{a*}\tilde{\chi}_0^{a*}\right\}|\omega\rangle\otimes|\tilde{\omega}\rangle\;, & \mathrm{RR} \\ |0\rangle & ,\ \mathrm{NSNS} \end{array}\right.\;. \tag{D.52}$$

The corresponding contribution to the partition function is, for each pair

$$\langle B_0, z_a, \eta|e^{-lH}|B_0, z_a', \eta'\rangle^F_{(a,a+1)} = \left\{\begin{array}{ll} \left(e^{2\pi i w_a} + \eta\eta' e^{-2\pi i w_a}\right) & ,\ \mathrm{RR} \\ 1 & ,\ \mathrm{NSNS} \end{array}\right.\;. \tag{D.53}$$

Similarly, the b.c. for the fermionic oscillators are

$$\left(\chi_n^a + i\eta\tilde{\chi}_n^{\dagger a*}\right)|B_{osc},\eta\rangle_F = \left(\chi_n^{a*} + i\eta\tilde{\chi}_n^{\dagger a}\right)|B_{osc},\eta\rangle_F = 0 \ , \tag{D.54}$$

with integer or half-integer moding in the RR and NSNS sectors, and are solved by

$$|B_{osc},\eta\rangle_F = \exp\left\{-i\eta\sum_{n>0}^{\infty}\sum_a\left(\chi_n^{\dagger a}\tilde{\chi}_n^{\dagger a} + \chi_n^{\dagger a*}\tilde{\chi}_n^{\dagger a*}\right)\right\}|0\rangle \ . \tag{D.55}$$

As its bosonic counterpart, this is not invariant under orbifold rotations, under which $\chi_n^a \to e^{2\pi i z_a}\chi_n^a$. Rather, it becomes

$$|B_{osc},z_a,\eta\rangle_F = \exp\left\{-i\eta\sum_{n>0}^{\infty}\sum_a\left(e^{4\pi i z_a}\chi_n^{\dagger a}\tilde{\chi}_n^{\dagger a} + e^{-4\pi i z_a}\chi_n^{\dagger a*}\tilde{\chi}_n^{\dagger a*}\right)\right\}|0\rangle \ . \tag{D.56}$$

The contribution of each compact pair of fermionic oscillators is found to be

$$\langle B_{osc},z_a,\eta|e^{-lH}|B_{osc},z_a',\eta'\rangle_{(a,a+1)}^F = q^{-b}\prod_{n=0}^{\infty}\left|1+\eta\eta' q^{2n}e^{4\pi i w_a}\right|^2 \ , \tag{D.57}$$

with integer and half-integer moding and $b = 1/12$ or $-1/6$ in the RR and NSNS sectors. Finally, the total fermionic part of the compact partition function is

$$Z_{sF}^{(c)}(l,w_a) = \frac{\vartheta_\alpha^3(2w_a|2il)}{\eta^3(2il)} \ , \tag{D.58}$$

with $\alpha = 1,2$ for s=R± and $\alpha = 3,4$ for s=NS±.

D3-brane: twisted sector

As discussed in Chapter 4, mixed b.c. are incompatible with twisting. There is therefore no coupling to closed string states of orbifold twisted sectors.

D.2 Two-point functions

In this section, we use the universal non-compact part of the boundary state constructed in previous section to compute the connected two-point functions needed in the computations of Chapter 4.

Consider first the bosonic oscillators. We shall abbreviate $X^\mu(z) = X^\mu$ and $\bar{X}^\mu(\bar{z}) = \bar{X}^\mu$. Since correlation functions only depend on the relative distance between the arguments, the two-point function of two left or two right fields at the same point is a constant, whereas the two-point function of a left and a right moving fields at image points only depends on $z - \bar{z} = 2i\tau$. In agreement with the b.c. implemented by the boundary states, we introduce the following notation

$$\begin{aligned}
&\langle X^0\bar{X}^0\rangle_{osc} = \langle X^1\bar{X}^1\rangle_{osc} = A_\epsilon(\tau,l) \ , \\
&\langle X^2\bar{X}^2\rangle_{osc} = \langle X^3\bar{X}^3\rangle_{osc} = A(\tau,l) \ , \\
&\langle X^0\bar{X}^1\rangle_{osc} = \langle X^1\bar{X}^0\rangle_{osc} = B_\epsilon(\tau,l) \ , \\
&\langle X^0X^0\rangle_{osc} = \langle \bar{X}^0\bar{X}^0\rangle_{osc} = -\langle X^1X^1\rangle_{osc} = -\langle \bar{X}^1\bar{X}^1\rangle_{osc} = C_\epsilon(l) \ , \\
&\langle X^2X^2\rangle_{osc} = \langle \bar{X}^2\bar{X}^2\rangle_{osc} = \langle X^3X^3\rangle_{osc} = \langle \bar{X}^3\bar{X}^3\rangle_{osc} = -C(l) \ ,
\end{aligned} \tag{D.59}$$

with $A(\tau,l) = A_\epsilon(\tau,l)|_{\epsilon_1=\epsilon_2=0}$ and $C(\tau,l) = C_\epsilon(\tau,l)|_{\epsilon_1=\epsilon_2=0}$. These two-point functions can be computed using the non-compact part of the boundary state constructed in previous section, and the definition Eq. (4.80). Carrying out some heavy oscillator algebra and resumming the results, one finds, using also $l' = l - \tau$, infinite series of logarithms corresponding to the propagation of the whole tower of bosonic modes

$$A_\epsilon = \frac{1}{4\pi}\sum_{n=0}^{\infty}\Big\{\cosh 2\pi[(\epsilon_1-\epsilon_2)n-\epsilon_2]\ln(1-q^{2n}e^{-4\pi\tau}) + \cosh 2\pi[(\epsilon_2-\epsilon_1)n-\epsilon_1]\ln(1-q^{2n}e^{-4\pi l'})\Big\}\ , \tag{D.60}$$

$$B_\epsilon = -\frac{1}{4\pi}\sum_{n=0}^{\infty}\Big\{\sinh 2\pi[(\epsilon_1-\epsilon_2)n-\epsilon_2]\ln(1-q^{2n}e^{-4\pi\tau}) + \sinh 2\pi[(\epsilon_2-\epsilon_1)n-\epsilon_1]\ln(1-q^{2n}e^{-4\pi l'})\Big\}\ , \tag{D.61}$$

$$C_\epsilon = \frac{1}{2\pi}\sum_{n=1}^{\infty}\cosh 2\pi[(\epsilon_1-\epsilon_2)n]\ln(1-q^{2n})\ . \tag{D.62}$$

In the last expression, we have discarded a normal ordering constant that will never contribute in the amplitude because of $p^2=0$. The equal-point correlator C_ϵ can be deduced from the other correlators by using the b.c. to reflect left and right movers at the boundaries. The bosonic exponential correlation is given by

$$\langle e^{ip\cdot X}\rangle_{osc} = e^{-\frac{1}{2}p_\mu p_\nu\langle(X+\bar X)^\mu(X+\bar X)^\nu\rangle_{osc}} = e^{-[(p_0^2+p_1^2)A_\epsilon+\vec p_T^{\,2}(A+C_\epsilon-C)+2p_0p_1B_\epsilon]} \tag{D.63}$$

and, using $p=p^0$ and $\cos\theta = p^1/p$, can be recast in the following form

$$\begin{aligned}\langle e^{ip\cdot X}\rangle_{osc} = &\prod_{n=1}^{\infty}\Big[1-q^{2n}\Big]^{-\frac{p^2}{\pi}\sinh^2\pi[(\epsilon_1-\epsilon_2)n]\sin^2\theta}\\ &\times\prod_{n=0}^{\infty}\Big[1-q^{2n}e^{-4\pi\tau}\Big]^{-\frac{p^2}{2\pi}\cosh^2\pi[(\epsilon_1-\epsilon_2)n-\epsilon_2]\{1+\tanh\pi[(\epsilon_1-\epsilon_2)n-\epsilon_2]\cos\theta\}^2}\\ &\times\prod_{n=0}^{\infty}\Big[1-q^{2n}e^{-4\pi l'}\Big]^{-\frac{p^2}{2\pi}\cosh^2\pi[(\epsilon_2-\epsilon_1)n-\epsilon_1]\{1+\tanh\pi[(\epsilon_2-\epsilon_1)n-\epsilon_1]\cos\theta\}^2}\ .\end{aligned} \tag{D.64}$$

Consider now correlations involving one derivative, and introduce

$$\begin{aligned}&\langle\partial X^0\bar X^0\rangle_{osc} = \langle\partial X^1\bar X^1\langle_{osc} = -\langle\bar\partial\bar X^0X^0\rangle_{osc} = -\langle\bar\partial\bar X^1X^1\rangle_{osc} = \frac{i}{2}K_\epsilon(\tau,l)\ ,\\ &\langle\partial X^2\bar X^2\rangle_{osc} = \langle\partial X^3\bar X^3\langle_{osc} = -\langle\bar\partial\bar X^2X^2\rangle_{osc} = -\langle\bar\partial\bar X^3X^3\rangle_{osc} = -\frac{i}{2}K(\tau,l)\ ,\\ &\langle\partial X^0\bar X^1\rangle_{osc} = \langle\partial X^1\bar X^0\rangle_{osc} = -\langle\bar\partial\bar X^1X^0\rangle_{osc} = -\langle\bar\partial\bar X^0X^1\rangle_{osc} = \frac{i}{2}L_\epsilon(\tau,l)\ ,\\ &\langle\partial X^0X^1\rangle_{osc} = -\langle\bar\partial\bar X^0\bar X^1\rangle_{osc} = \frac{i}{2}W_\epsilon(l)\ ,\end{aligned} \tag{D.65}$$

with $K(\tau,l) = K_\epsilon(\tau,l)|_{\epsilon_1=\epsilon_2=0}$. These correlators can be computed as before by using the non-compact part of the boundary state constructed in previous section and the definition

Eq. (4.80). Due to the derivative, one obtains in this case infinite series of poles, rather than logarithms

$$K_\epsilon = -\sum_{n=0}^{\infty}\left\{\cosh 2\pi[(\epsilon_1-\epsilon_2)n-\epsilon_2]\frac{q^{2n}e^{-4\pi\tau}}{1-q^{2n}e^{-4\pi\tau}}\right.$$
$$\left. -\cosh 2\pi[(\epsilon_2-\epsilon_1)n-\epsilon_1]\frac{q^{2n}e^{-4\pi l'}}{1-q^{2n}e^{-4\pi l'}}\right\}\ , \qquad \text{(D.66)}$$

$$L_\epsilon = \sum_{n=0}^{\infty}\left\{\sinh 2\pi[(\epsilon_1-\epsilon_2)n-\epsilon_2]\frac{q^{2n}e^{-4\pi\tau}}{1-q^{2n}e^{-4\pi\tau}}\right.$$
$$\left. -\sinh 2\pi[(\epsilon_2-\epsilon_1)n-\epsilon_1]\frac{q^{2n}e^{-4\pi l'}}{1-q^{2n}e^{-4\pi l'}}\right\}\ , \qquad \text{(D.67)}$$

$$W_\epsilon = -\frac{\pi(v_1-v_2)}{2\pi l} - 2\sum_{n=1}^{\infty}\sinh 2\pi[(\epsilon_1-\epsilon_2)n]\frac{q^{2n}}{1-q^{2n}}\ . \qquad \text{(D.68)}$$

Again, the equal-point correlator W_ϵ can be deduced from the other correlators by using the b.c..

Consider now the fermions. Again, we shall abbreviate $\psi^\mu(z)=\psi^\mu$ and $\bar\psi^\mu(\bar z)=\bar\psi^\mu$. As for the bosons, correlation functions only depend on the relative distance between the arguments. Therefore the two-point function of two left or two right fields at the same point is a constant, whereas the two-point function of a left and a right moving fields at image points only depends on $z-\bar z=2i\tau$. Taking into account the b.c. implemented by the boundary states, and setting the sign η appearing in the fermionic b.c. always equal to 1 for the first boundary state (since only the relative sign $\eta\eta'$ is relevant) according to the discussion of Chapter 3, we introduce

$$\begin{aligned}
\langle\psi^0\bar\psi^0\rangle_s &= \langle\psi^1\bar\psi^1\rangle_s = iF^s_\epsilon(\tau,l)\ ,\\
\langle\psi^2\bar\psi^2\rangle_s &= \langle\psi^3\bar\psi^3\rangle_s = iF^s(\tau,l)\ ,\\
\langle\psi^0\bar\psi^1\rangle_s &= \langle\psi^1\bar\psi^0\rangle_s = iG^s_\epsilon(\tau,l)\ ,\\
\langle\psi^0\psi^1\rangle_s &= \langle\bar\psi^0\bar\psi^1\rangle_s = U^s_\epsilon(l)\ ,
\end{aligned} \qquad \text{(D.69)}$$

with $F^s(\tau,l)=F^s_\epsilon(\tau,l)|_{\epsilon_1=\epsilon_2=0}$. Each of the correlators is splitted into z.m. and oscillator parts

$$F^s_\epsilon = F^{0s}_\epsilon + \tilde F^s_\epsilon\ ,\ \ G^s_\epsilon = G^{0s}_\epsilon + \tilde G^s_\epsilon\ ,\ \ U^s_\epsilon = U^{0s}_\epsilon + \tilde U^s_\epsilon\ . \qquad \text{(D.70)}$$

Proceeding as in the bosonic case, these correlation functions can be computed by using the non-compact part of the boundary state constructed in the previous section and the definition (4.81). The z.m. contributions are found to be

$$F^{0R+}_\epsilon = -\frac{1}{2}\frac{\cosh\pi(\epsilon_1+\epsilon_2)}{\cosh\pi(\epsilon_1-\epsilon_2)}\ ,\ \ F^{0R-}_\epsilon = -\frac{1}{2}\frac{\sinh\pi(\epsilon_1+\epsilon_2)}{\sinh\pi(\epsilon_1-\epsilon_2)}\ ,\ \ F^{0NS\pm}_\epsilon = 0\ , \qquad \text{(D.71)}$$

$$G^{0R+}_\epsilon = -\frac{1}{2}\frac{\sinh\pi(\epsilon_1+\epsilon_2)}{\cosh\pi(\epsilon_1-\epsilon_2)}\ ,\ \ G^{0R-}_\epsilon = -\frac{1}{2}\frac{\cosh\pi(\epsilon_1+\epsilon_2)}{\sinh\pi(\epsilon_1-\epsilon_2)}\ ,\ \ G^{0NS\pm}_\epsilon = 0\ , \qquad \text{(D.72)}$$

$$U^{0R+}_\epsilon = +\frac{1}{2}\tanh\pi(\epsilon_1-\epsilon_2)\ ,\ \ U^{0R-}_\epsilon = +\frac{1}{2}\coth\pi(\epsilon_1-\epsilon_2)\ ,\ \ U^{0NS\pm}_\epsilon = 0\ . \qquad \text{(D.73)}$$

The oscillator part yields as in the bosonic case infinite series of simple poles, corresponding to the propagation of the whole tower of fermionic modes. One finds

$$
\begin{aligned}
\tilde{F}_\epsilon^{R\pm} &= -\sum_{n=0}^{\infty}(\mp)^n\left\{\cosh 2\pi[(\epsilon_1-\epsilon_2)n-\epsilon_2]\frac{q^{2n}e^{-4\pi\tau}}{1-q^{2n}e^{-4\pi\tau}}\right. \\
&\qquad\left.\pm\cosh 2\pi[(\epsilon_2-\epsilon_1)n-\epsilon_1]\frac{q^{2n}e^{-4\pi l'}}{1-q^{2n}e^{-4\pi l'}}\right\} , \\
\tilde{G}_\epsilon^{R\pm} &= \sum_{n=0}^{\infty}(\mp)^n\left\{\sinh 2\pi[(\epsilon_1-\epsilon_2)n-\epsilon_2]\frac{q^{2n}e^{-4\pi\tau}}{1-q^{2n}e^{-4\pi\tau}}\right. \\
&\qquad\left.\pm\sinh 2\pi[(\epsilon_2-\epsilon_1)n-\epsilon_1]\frac{q^{2n}e^{-4\pi l'}}{1-q^{2n}e^{-4\pi l'}}\right\} , \qquad (D.74) \\
\tilde{U}_\epsilon^{R\pm} &= -\frac{\pi(\epsilon_1-\epsilon_2)}{2\pi l}-2\sum_{n=1}^{\infty}(\mp)^n\sinh 2\pi[(\epsilon_1-\epsilon_2)n]\frac{q^{2n}}{1-q^{2n}} , \qquad (D.75)
\end{aligned}
$$

in the RR sector and

$$
\begin{aligned}
F_\epsilon^{NS\pm} &= -\sum_{n=0}^{\infty}(\mp)^n\left\{\cosh 2\pi[(\epsilon_1-\epsilon_2)n-\epsilon_2]\frac{q^{n}e^{-2\pi\tau}}{1-q^{2n}e^{-4\pi\tau}}\right. \\
&\qquad\left.\pm\cosh 2\pi[(\epsilon_2-\epsilon_1)n-\epsilon_1]\frac{q^{n}e^{-2\pi l'}}{1-q^{2n}e^{-4\pi l'}}\right\} , \qquad (D.76) \\
G_\epsilon^{NS\pm} &= \sum_{n=0}^{\infty}(\mp)^n\left\{\sinh 2\pi[(\epsilon_1-\epsilon_2)n-\epsilon_2]\frac{q^{n}e^{-2\pi\tau}}{1-q^{2n}e^{-4\pi\tau}}\right. \\
&\qquad\left.\pm\sinh 2\pi[(\epsilon_2-\epsilon_1)n-\epsilon_1]\frac{q^{n}e^{-2\pi l'}}{1-q^{2n}e^{-4\pi l'}}\right\} , \qquad (D.77) \\
U_\epsilon^{NS\pm} &= -\frac{\pi(\epsilon_1-\epsilon_2)}{2\pi l}-2\sum_{n=1}^{\infty}(\mp)^n\sinh 2\pi[(\epsilon_1-\epsilon_2)n]\frac{q^{n}}{1-q^{2n}} , \qquad (D.78)
\end{aligned}
$$

in the NSNS sector. The equal-point correlators U_ϵ^s can be deduced from the other correlators by using the b.c. to reflect left and right movers at the boundaries.

Notice that world-sheet supersymmetry is enforced between the bosons and the odd spin-structure fermions. Since $K_v = \tilde{F}_v^{R-}$, $L_v = \tilde{G}_v^{R-}$ and $W_v = \tilde{F}_v^{R-}$, we explicitly check the relations

$$
\begin{aligned}
\langle \partial X^\mu \bar{X}^\nu\rangle_{osc} &= \frac{1}{2}\langle \psi^\mu\bar{\psi}^\nu\rangle_{osc}^{R-} , \\
\langle \partial X^\mu X^\nu\rangle_{osc} &= \frac{i}{2}\langle \psi^\mu\psi^\nu\rangle_{osc}^{R-} , \\
\langle \bar{\partial}\bar{X}^\mu X^\nu\rangle_{osc} &= \frac{1}{2}\langle \bar{\psi}^\mu\psi^\nu\rangle_{osc}^{R-} , \\
\langle \bar{\partial}\bar{X}^\mu \bar{X}^\nu\rangle_{osc} &= -\frac{i}{2}\langle \bar{\psi}^\mu\bar{\psi}^\nu\rangle_{osc}^{R-} . \qquad (D.79)
\end{aligned}
$$

The periodicities of the fermionic propagators in the four spin structures, which should follow from an involution from the torus to the cylinder [208], can be understood by considering the light-cone combinations $\psi^\pm = (\psi^0 \pm \psi^1)/\sqrt{2}$ and in particular their propagators

$\langle\psi^\pm(z)\bar\psi^\pm(\bar z)\rangle_s = P^s_{\epsilon(\pm)}$, which are given by

$$P^s_{\epsilon(\pm)} = \frac{i}{4}(F^s_\epsilon \pm G^s_\epsilon)\ . \tag{D.80}$$

Using the explicit form of F^s_ϵ and G^s_ϵ, one can then check the transformation around the two cycles of the covering torus with modulus $2il$, $w \to w + m + n2il$ with $w = z - \bar z = 2i\tau$, that is $\tau \to \tau - \frac{i}{2}m + nl$. One finds

$$\begin{aligned}
P^{R+}_{\epsilon(\pm)}(\tau - \frac{i}{2}m + nl, l) &= (-1)^n e^{\pm 2\pi n\epsilon} P^{R+}_{\epsilon(\pm)}(\tau, l)\ ,\\
P^{R-}_{\epsilon(\pm)}(\tau - \frac{i}{2}m + nl, l) &= e^{\pm 2\pi n\epsilon} P^{R-}_{\epsilon(\pm)}(\tau, l)\ ,\\
P^{NS+}_{\epsilon(\pm)}(\tau - \frac{i}{2}m + nl, l) &= (-1)^n(-1)^m e^{\pm 2\pi n\epsilon} P^{NS+}_{\epsilon(\pm)}(\tau, l)\ ,\\
P^{NS-}_{\epsilon(\pm)}(\tau - \frac{i}{2}m + nl, l) &= (-1)^m e^{\pm 2\pi n\epsilon} P^{NS-}_{\epsilon(\pm)}(\tau, l)\ .
\end{aligned} \tag{D.81}$$

These transformation rules for $m = 0$ correspond to the boundary conditions at the two ends of the cylinder for the $\psi^\pm$ which are

$$\begin{aligned}
\psi^\pm(z)|_{\tau=0} &= -i\eta e^{\pm 2\pi\epsilon_2}\bar\psi^\mp(\bar z)|_{\tau=0}\ ,\\
\psi^\pm(z)|_{\tau=l} &= -i\eta e^{\pm 2\pi\epsilon_1}\bar\psi^\mp(\bar z)|_{\tau=l}\ .
\end{aligned} \tag{D.82}$$

where the two possible sign choices $\eta = \pm$ on the r.h.s. correspond to the $\pm$ spin-structures. The local behavior of these functions for $\tau \to 0$ is found to be

$$P^s_{\epsilon(\pm)}(\tau, l) \to \frac{1}{8\pi i\tau} e^{\pm 2\pi\epsilon_2}\ . \tag{D.83}$$

It is convenient to rescale the fermions according to $\psi^\pm \to \hat\psi^\pm = e^{\mp v_2}\psi^\pm$, their propagators becomming $\hat P^s_{\epsilon(\pm)} = e^{\mp 2\pi\epsilon_2} P^s_{\epsilon(\pm)}$. The monodromy properties do not change, but the boundary conditions now become

$$\begin{aligned}
\hat\psi^\pm(z) &= -i\eta\hat{\bar\psi}^\mp(\bar z)\ ,\quad z = \bar z\ ,\\
\hat\psi^\pm(z) &= -i\eta e^{\pm 2\pi(\epsilon_1 - \epsilon_2)}\hat{\bar\psi}^\mp(\bar z)\ ,\quad z = \bar z + 2il\ ,
\end{aligned} \tag{D.84}$$

and the local behavior for $\tau \to 0$ simplifies to the conventional one

$$\hat P^s_{\epsilon(\pm)}(\tau, l) \to \frac{1}{4\pi w}\ . \tag{D.85}$$

It is now clear how to do the twisted involution to pass from the covering torus to the cylinder: the twisted boundary conditions on the cylinder are obtained from a non-trivial phase transformation around the long cycle of the torus with imaginary angle $i\epsilon$. Actually, the monodromy properties of the functions $\hat P^s_{\epsilon(\pm)}$, together with their local behavior, imply them to be combinations of twisted ϑ-functions, with argument $w = 2i\tau$, modulus $2il$ and imaginary twist $i\epsilon$. In fact, one can check that

$$\hat P^{R+}_{\epsilon(\pm)}(w, l) = \frac{1}{4\pi}\frac{\vartheta_2(w \pm i\epsilon|2il)\vartheta_1'(0|2il)}{\vartheta_1(w|2il)\vartheta_2(\pm i\epsilon|2il)}\ , \tag{D.86}$$

$$\hat P^{NS+}_{\epsilon(\pm)}(w, l) = \frac{1}{4\pi}\frac{\vartheta_3(w \pm i\epsilon|2il)\vartheta_1'(0|2il)}{\vartheta_1(w|2il)\vartheta_3(\pm i\epsilon|2il)}\ , \tag{D.87}$$

$$\hat P^{NS-}_{\epsilon(\pm)}(w, l) = \frac{1}{4\pi}\frac{\vartheta_4(w \pm i\epsilon|2il)\vartheta_1'(0|2il)}{\vartheta_1(w|2il)\vartheta_4(\pm i\epsilon|2il)}\ . \tag{D.88}$$

In the odd spin-structure case, the propagator is not analytic and can therefore not be unambiguously determined following this procedure.

In order to study the amplitudes in the large distance limit, we need the $l \to \infty$ asymptotics of the correlations. For the bosonic correlations with one derivative, one finds

$$\tilde{K}_\epsilon \underset{l\to\infty}{\longrightarrow} -\cosh 2\pi\epsilon_2 \frac{e^{-4\pi\tau}}{1-e^{-4\pi\tau}} - \cosh 2\pi\epsilon_1 \frac{e^{-4\pi l'}}{1-e^{-4\pi l'}} \ , \tag{D.89}$$

$$\tilde{L}_\epsilon \underset{l\to\infty}{\longrightarrow} -\sinh 2\pi\epsilon_2 \frac{e^{-4\pi\tau}}{1-e^{-4\pi\tau}} - \sinh 2\pi\epsilon_1 \frac{e^{-4\pi l'}}{1-e^{-4\pi l'}} \ , \tag{D.90}$$

$$\tilde{W}_\epsilon \underset{l\to\infty}{\longrightarrow} -\frac{\pi(\epsilon_1-\epsilon_2)}{2\pi l} - 2\sinh 2\pi(\epsilon_1-\epsilon_2)\, e^{-4\pi l} \ , \tag{D.91}$$

and the bosonic exponential becomes

$$\langle e^{ip\cdot X}\rangle_{osc} \underset{l\to\infty}{\longrightarrow} \left[1-e^{-4\pi\tau}\right]^{-\frac{p^{(2)2}}{2\pi}} \left[1-e^{-4\pi l'}\right]^{-\frac{p^{(1)2}}{2\pi}} \ . \tag{D.92}$$

The fermionic propagators in the four spin-structures reduce to

$$\tilde{F}^{R\pm}_\epsilon \underset{l\to\infty}{\longrightarrow} -\cosh 2\pi\epsilon_2 \frac{e^{-4\pi\tau}}{1-e^{-4\pi\tau}} \mp \cosh 2\pi\epsilon_1 \frac{e^{-4\pi l'}}{1-e^{-4\pi l'}} \ , \tag{D.93}$$

$$\tilde{G}^{R\pm}_\epsilon \underset{l\to\infty}{\longrightarrow} -\sinh 2\pi\epsilon_2 \frac{e^{-4\pi\tau}}{1-e^{-4\pi\tau}} \mp \sinh 2\pi\epsilon_1 \frac{e^{-4\pi l'}}{1-e^{-4\pi l'}} \ , \tag{D.94}$$

$$\tilde{U}^{R\pm}_\epsilon \underset{l\to\infty}{\longrightarrow} -\frac{\pi(\epsilon_1-\epsilon_2)}{2\pi l} \pm 2\sinh 2\pi(\epsilon_1-\epsilon_2)\, e^{-4\pi l} \ , \tag{D.95}$$

and

$$\begin{aligned} F^{NS\pm}_\epsilon \underset{l\to\infty}{\longrightarrow} & -\cosh 2\pi\epsilon_2 \frac{e^{-2\pi\tau}}{1-e^{-4\pi\tau}} \mp \cosh 2\epsilon_1 \frac{e^{-2\pi l'}}{1-e^{-4\pi l'}} \\ & + e^{-2\pi l}\left[\pm\cosh 2\pi(\epsilon_1-2\epsilon_2)\, e^{-2\pi\tau} + \cosh 2\pi(\epsilon_2-2\epsilon_1)\, e^{-2\pi l'}\right] \ , \end{aligned} \tag{D.96}$$

$$\begin{aligned} G^{NS\pm}_\epsilon \underset{l\to\infty}{\longrightarrow} & -\sinh 2\pi\epsilon_2 \frac{e^{-2\pi\tau}}{1-e^{-4\pi\tau}} \mp \sinh 2\pi\epsilon_1 \frac{e^{-2\pi l'}}{1-e^{-4\pi l'}} \\ & + e^{-2\pi l}\left[\mp\sinh 2\pi(\epsilon_1-2\epsilon_2)\, e^{-2\pi\tau} - \sinh 2\pi(\epsilon_2-2\epsilon_1)\, e^{2\pi l'}\right] \ , \end{aligned} \tag{D.97}$$

$$U^{NS\pm}_\epsilon \underset{l\to\infty}{\longrightarrow} -\frac{\pi(\epsilon_1-\epsilon_2)}{2\pi l} \pm 2\sinh 2\pi(\epsilon_1-\epsilon_2)\, e^{-2\pi l} \ . \tag{D.98}$$

Bibliography

[1] A. Einstein, *Ann. Phys.* **49** (1916) 769.

[2] T. Kaluza, *Sitz. Ber. Preuss. Akad.*, *Phys. Rev. Math. K.* **1** (1921) 966.

[3] O. Klein, *Z. Phys.* **37** (1929) 895.

[4] J. Bagger and J. Wess, *"Supersymmetry and supergravity"*, Princeton Series in Physics (1992).

[5] M.F. Sohnius, *Phys. Rep.* **128** (1985) 39.

[6] P. Van Nieuwenhuizen, *Phys. Rep.* **68** (1981) 189.

[7] L. Castellani, R. D'Auria e P. Fré, *"Superstrings and supergravity: a geometric perspective"*, World-Scientific, Singapore (1991).

[8] J.H. Schwarz, *Phys. Rep.* **89** (1982) 223.

[9] M.B. Green, J.H. Schwarz e E. Witten, *"Superstring theory"*, Cambridge University Press (1987).

[10] D. Lüst and S. Theisen, *"Lectures in string theory"*, Lecture Notes in Physics **346**, Springer Verlag (1989).

[11] H. Ooguri and Z. Yin, *"Tasi lectures on perturbative string theories"*, hep-th 9612254.

[12] E. Kiritsis, *"Introduction to superstring theory"*, hep-th/9709062.

[13] P. Ginsparg, *"Applied conformal field theory"*, Proceedings of Les Houches 88, North Holland (1989) 1.

[14] B.R. Greene, *"String theory on Calabi-Yau manifolds"*, hep-th/9702155.

[15] L. Dixon, J. Harvey, C. Vafa and E. Witten, *Nucl. Phys.* **B261** (1985) 651.

[16] L. Dixon, J. Harvey, C. Vafa and E. Witten, *Nucl. Phys.* **B274** (1986) 285.

[17] J.H. Schwarz, *Nucl. Phys. Proc. Suppl.* **B55** (1997) 1.

[18] M.J. Duff, *Int. J. Mod. Phys.* **A11** (1996) 5623.

[19] S. Förste, J. Louis, *"Duality in String Theory"*, hep-th/9612192.

[20] C. Vafa, *"Lectures on Strings and Dualities"*, hep-th/9702201.

[21] A. Sen, *"An introduction to non-perturbative string theory"*, hep-th/9802051.

[22] P.A.M. Dirac, *Proc. Roy. Soc.* **A133** (1931) 60.

[23] G. 't Hooft, *Nucl. Phys.* **B79** (1974) 276.

[24] M. Polyakov, *JETP Lett.* **20** (1974) 194.

[25] C. Montonen and D. Olive, *Phys. Lett.* **B72** (1977) 117.

[26] E. Witten, *Nucl. Phys.* **B443** (1995) 85.

[27] A. Dabholkar, *Phys. Lett.* **B357** (1995) 307.

[28] C.M. Hull, *Phys. Lett.* **B357** (1995) 545.

[29] J. Polchinski and E. Witten, *Nucl. Phys.* **B460** (1996) 525.

[30] C.M. Hull and P.K. Townsend, *Nucl. Phys.* **B438** (1995) 109.

[31] M.J. Duff, *Nucl. Phys.* **B442** (1995) 47.

[32] M.J. Duff and R. Khuri, *Nucl. Phys.* **B411** (1994) 473.

[33] J. Harvey and A. Strominger, *Nucl. Phys.* **B449** (1995) 535.

[34] A. Sen, *Nucl. Phys.* **B450** (1995) 103.

[35] M.B. Green and J.H. Schwarz, *Phys. Lett.* **B122** (1983) 143.

[36] J.H. Schwarz and P.C. West, *Phys. Lett.* **B126** (1983) 301.

[37] J.H. Schwarz, *Nucl. Phys.* **B226** (1983) 269.

[38] P. Howe and P.C. West, *Nucl. Phys.* **B238** (1984) 181.

[39] A. Giveon, M. Porrati and E. Rabinovic, *Phys. Rep.* **244** (1994) 77.

[40] E. Cremmer, B. Julia, *Nucl. Phys.* **B159** (1979) 141.

[41] M.K. Gaillard and B. Zumino, *Nucl. Phys.* **B193** (1981) 221.

[42] J. Scherk and J.H. Schwarz, *Nucl. Phys.* **B153** (1979) 61.

[43] I.C.G. Campbell and P.C. West, *Nucl. Phys.* **B243** (1984) 112.

[44] F. Giani and M. Pernici, *Phys. Rev.* **D30** (1984) 325.

[45] M. Huq and M.A. Namazie, *Class. Quant. Grav.* **2** (1985) 293.

[46] E. Cremmer, B. Julia and J. Scherk, *Phys. Lett.* **B76** (1978) 409.

[47] P.K. Townsend, *Phys. Lett.* **B350** (1995) 184.

[48] C.M. Hull, *Nucl. Phys.* **B468** (1996) 113.

[49] H. Lü, C.N. Pope, E. Sezgin and K.S. Stelle, *Nucl. Phys.* **B456** (1995) 669.

[50] H. Lü, C.N. Pope, E. Sezgin and K.S. Stelle, *Phys. Lett.* **B371** (1996) 46.

[51] H. Lü, C.N. Pope and K.S. Stelle, *Nucl. Phys.* **B481** (1996) 313.

[52] K.S. Stelle, *"Lectures on supergravity p-branes"*, hep-th/9701088.

[53] M.J. Duff, *"Supermembranes"*, hep-th/9611203.

[54] R.I. Nepomechie, *Phys. Rev.* **D31** (1985) 1921.

[55] C. Teitelboim, *Phys. Lett.* **B167** 63.

[56] J. Polchinski, *Phys. Rev. Lett.* **75** (1995) 4724.

[57] J. Dai, R.G. Leigh and J. Polchinski, *Mod. Phys. Lett.* **A4** (1989) 2073.

[58] R.G. Leigh, *Mod. Phys. Lett.* **A4** (1989) 2767.

[59] J. Polchinski, S. Chaudhuri and C.V. Johnson, *"Notes on D-branes"*, hep-th/9602052.

[60] J. Polchinski, *"Tasi lectures on D-branes"*, hep-th/9611050.

[61] C.P. Bachas, *"(Half) a lecture on D-branes"*, hep-th/9701019.

[62] C.P. Bachas, *"Lectures on D-branes"*, hep-th/9806199.

[63] S. Sethi and M. Stern, *"D-brane bound states redux*, hep-th/9705046.

[64] M. Porrati and A. Rozenberg, *Nucl. Phys.* **B515** (1998) 184.

[65] S. Shenker, *"Random surfaces and quantum gravity"*, Proceedings of Cargese 90, Plenum (1991) 91.

[66] J. Polchinski, *Phys. Rev.* **D50** (1994) 6041.

[67] S.W. Hawking, *Comm. Math. Phys.* **43** (1975) 199.

[68] J. Beckenstein, *Phys. Rev.* **D7** (1973) 2333.

[69] J. Beckenstein, *Phys. Rev.* **D9** (1974) 3292.

[70] S.W. Hawking, *Phys. Rev.* **D13** (1976) 191.

[71] J.M. Maldacena, *"Black holes in string theory"*, hep-th/9607235.

[72] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Phys. Lett.* **B189** (1987) 75.

[73] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Ann. Phys. (N.Y.)* **185** (1988) 330.

[74] M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, *Phys. Lett.* **B191** (1987) 70.

[75] B. De Wit, J. Hoppe and H. Nicolai, *Nucl. Phys.* **B305** (1988) 545.

[76] B. De Wit, M. Lüscher and H. Nicolai, *Nucl. Phys.* **B320** (1989) 135.

[77] B. De Wit, *Nucl. Phys. Proc. Suppl.* **B56** (1997) 76.

[78] H. Nicolai and R. Helling, "Supermembranes and m(atrix) theory", hep-th/9809103.

[79] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *Phys. Rev.* **D55** (1997) 5112.

[80] A. Bilal, *"M(atrix) theory: a pedagogical introduction"*, hep-th/9710136.

[81] T. Banks, *"Matrix theory"*, hep-th/9710131.

[82] D. Bigatti and L. Suskind, *"Review of matrix theory"*, hep-th/9712072.

[83] L. Suskind, *"Another conjecture about m(atrix) theory"*, het-th/9704080.

[84] N. Seiberg, *Phys. Rev. Lett.* **79** (1997) 3577.

[85] E. Witten, *Nucl. Phys.* **B460** (1996) 335.

[86] D. Kabat and and P. Pouliot, *Phys. Rev. Lett.* **77** (1996) 1004.

[87] U.H. Danielsson, G. Ferretti and B. Sundborg, *Int. J. Mod. Phys.* **A11** (1996) 5463.

[88] U.H. Danielsson and G. Ferretti, *Int. J. Mod. Phys.* **A12** (1997) 4581.

[89] M.R. Douglas, D. Kabat, P. Pouliot and S.H. Shenker, *Nucl. Phys.* **B485** (1997) 85.

[90] C.P. Bachas, *Phys. Lett.* **B374** (1996) 37.

[91] C. Lovelace, *Phys. Lett.* **B34** (1971) 500.

[92] L. Clavelli and J. Shapiro, *Nucl. Phys.* **B57** (1973) 490.

[93] M. Ademollo, R. D'Auria, F. Gliozzi, E. Napolitano, S. Sciuto and P. Di Vecchia, *Nucl. Phys.* **B94** (1975) 221.

[94] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, *Nucl. Phys.* **B293** (1987) 83.

[95] J. Polchinski and Y. Cai, *Nucl Phys.* **B296** (1988) 91.

[96] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, *Nucl. Phys.* **B308** (1988) 221.

[97] S.S. Yost, *Nucl. Phys.* **B321** (1989) 629.

[98] G. Lifschytz, *Phys. Lett.* **B388** (1996) 720.

[99] F. Hussain, R. Iengo, C. Núñez and C.A. Scrucca, *Phys. Lett.* **B409** (1997) 101.

[100] F. Hussain, R. Iengo, C. Núñez and C.A. Scrucca, *Nucl. Phys.* **B517** (1998) 92.

[101] M. Bertolini, R. Iengo and C.A. Scrucca, *Nucl. Phys.* **B522** (1998) 193.

[102] M. Bertolini, P. Fré, R. Iengo and C.A. Scrucca, *Phys. Lett.* **B431** (1998) 22.

[103] J.F. Morales, C.A. Scrucca and M. Serone, *Phys. Lett.* **B417** (1998) 233.

[104] J.F. Morales, C.A. Scrucca and M. Serone, *"Scale independent spin effects in D-brane dynamics"*, hep-th/9801183, to appear on *Nucl. Phys.* **B**.

[105] E.S. Fradkin and A.A. Tseytlin, *Phys. Lett.* **B163** (1985) 123.

[106] O.D. Andreev and A.A. Tseytlin, *Nucl. Phys.* **B311** (1988) 205.

[107] C.G. Callan and L. Thorlacius, *Nucl. Phys.* **B319** (1989) 133.

[108] C.G. Callan and L. Thorlacius, *Nucl. Phys.* **B329** (1990) 117.

[109] R. Iengo and C.A. Scrucca, *Phys. Rev.* **B57** (1998) 6046.

[110] R. Iengo and C. Acatrinei, *"Pair production of open strings - relativistic versus dissipative dynamics "*, hep-th/9806048.

[111] A.O. Caldeira and A.J. Leggett, *Ann. Phys.* **149** (1983) 374.

[112] A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, *Nucl. Phys.* **B280** (1987) 599.

[113] W. Taylor, *"Lectures on D-branes, gauge theory and m(atrices)"*, hep-th/9801182.

[114] D.J. Gross and E. Witten, *Nucl. Phys.* **B277** (1986) 1.

[115] C.G. Callan, D. Friedan, E. Martinec and M. Perry, *Nucl. Phys.* **B262** (1985) 593.

[116] A. Sen, *Phys. Rev. Lett.* **55** (1985) 1846.

[117] E.S. Fradkin and A.A. Tseytlin, *Nucl. Phys.* **B261** (1985) 1.

[118] K. Kikkawa and M. Yamanaka, *Phys. Lett.* **B149** (1984) 357.

[119] N. Sakai and I. Senda, *Prog. Theor. Phys.* **75** (1986) 692.

[120] V.P. Nair, A. Shapire, A. Strominger and F. Wilczek, *Nucl. Phys.* **B287** (1987) 402.

[121] D. Firedan, E. Martinec and S. Shenker, *Nucl. Phys.* **B271** (1986) 93.

[122] M. Li, *Nucl. Phys.* **B460** (1996) 351.

[123] M.R. Douglas, *"Branes within branes"*, het-th/9512077.

[124] M. Bershadsky, V. Sadov and C. Vafa, *Nucl Phys.* **B463** (1996) 398, 420.

[125] M.B. Green, J.A. Harvey and G. Moore, *Class. Quant. Grav.* **14** (1997) 47.

[126] C.P. Bachas and M. Porrati, *Phys. Lett.* **B296** (1992) 77.

[127] C.P. Bachas and E. Kiritsis, *Nucl. Phys. Proc. Suppl.* **B55** 194.

[128] C.P. Bachas and C. Fabre, *Nucl. Phys.* **B476** (1996) 418.

[129] M. Frau, A. Lerda, I. Pesando, R. Russo and S. Sciuto, *Phys. Lett.* **B400** (1997) 52.

[130] P. Di Vecchia, M. Frau, A. Lerda, I. Pesando, R. Russo and S. Sciuto, *Nucl. Phys.* **B507** (1997) 259.

[131] M. Billò, P. Di Vecchia, M. Frau, A. Lerda, I. Pesando, R. Russo and S. Sciuto, *Nucl. Phys.* **B526** (1998) 199.

[132] M. Billò, D. Cangemi and P. Di Vecchia, *Phys. Lett.* **B400** (1997) 63.

[133] C.G. Callan and I.R. Klebanov, *Nucl. Phys.* **B465** (1996) 473.

[134] M. Billò, P. Di Vecchia, M. Frau, A. Lerda, R. Russo and S. Sciuto, *"The Lorentz force between D0 and D6 branes in string and m(atrix) theory"*, hep-th/9805091.

[135] F. Hussain, R. Iengo and C. Núñez, *Nucl. Phys.* **B497** (1997) 205.

[136] F. Hussain, R. Iengo, C. Núñez and C.A. Scrucca, *"Aspects of D-brane dynamics on orbifolds"*, proceedings of the conference "Quantum aspects of gauge theories, supergravity and unification", Neuchâtel (1997), hep-th/9711020.

[137] F. Hussain, R. Iengo, C. Núñez and C.A. Scrucca, *"Interaction of D-branes on orbifolds and massless particle emission"*, proceedings of the conference "Physics beyond the standard model: from theory to experiment", Valencia (1997), hep-th/9711021.

[138] M.R. Douglas, H. Ooguri and S.H. Shenker, *Phys. Lett.* **B402** (1997) 36.

[139] I.R. Klebanov and L. Thorlacius, *Phys. Lett.* **B371** (1996) 51.

[140] A. Hashimoto and I.R. Klebanov, *Phys. Lett.* **B381** (1996) 437.

[141] S.S. Gubser, A. Hashimoto. I.R. Klebanov and J.M. Maldacena, *Nucl. Phys.* **B472** (1996) 231.

[142] M.R. Garousi and R.C. Myers, *Nucl. Phys.* **B475** (1996) 193.

[143] B.S. De Witt, *Phys. Rev.* **162** (1967) 1239.

[144] F.A. Berends and R. Gastmans, *Nucl. Phys.* **B88** (1975) 99.

[145] M. Bertolini, P. Fré, F. Hussain, R. Iengo and C.A. Scrucca, *"Black hole - D-brane correspondence: an example"*, proceedings of the conference "Quantum Gravity in the Southern Cone", Bariloche (1998), hep-th/9807209.

[146] J. Schwinger, *Science* **165** (1969) 757.

[147] D. Zwanziger, *Phys. Rev.* **176** (1968) 1480.

[148] B. Julia and A. Zee, *Phys. Rev.* **D11** (1975) 2227.

[149] G. Calucci, R. Jengo and M.T. Vallon, *Nucl. Phys.* **B211** (1983) 77.

[150] G. Calucci and R. Jengo, *Nucl. Phys.* **B223** (1983) 501.

[151] S. Deser, A. Gomberoff, M. Hennaux and C. Teitelboim, *Nucl. Phys.* **B520** (1998) 179.

[152] M.S. Bremer, H. Lu, C.N. Pope and K.S. Stelle. *"Dirac Quantisation Conditions and Kaluza-Klein Reduction"*, hep-th/9710244.

[153] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara and P. Fré, *Nucl. Phys.* **B476** (1996) 397.

[154] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, *J. of Geom. Phys.* **23** (1997) 111.

[155] E. Verlinde and H. Verlinde, *Phys. Lett.* **B192** (1987) 95.

[156] L. Andrianopoli, R. D'Auria, S. Ferrara, P. Fré, M. Trigiante, *Nucl. Phys.* **B509** (1998) 463.

[157] V. Balasubramanian and F. Larsen, *Nucl. Phys.* **B478** (1996) 199-208.

[158] V. Balasubramanian, F. Larsen and R. Leigh, *"Branes at angles and black holes"*, hep-th/9704143.

[159] S. Ferrara, R. Kallosh, A. Strominger, *Phys. Rev.* **D52** (1995) 5412.

[160] A. Strominger, *Phys. Lett.* **B383**(1996) 39.

[161] S. Ferrara and R. Kallosh, *Phys. Rev.* **D54** (1996) 1525.

[162] K. Behrndt, D. Lüst and W. A. Sabra. *Phys. Lett.* **B418** (1998) 303.

[163] R. Kallosh, M. Shmakova and W.K. Wong, *Phys. Rev.* **D54** (1996) 6284.

[164] P. Fré, *"U-duality, Solvable Lie algebras and extremal black-holes"*, Proc. 12th "Italian Conference on General Relativity and Gravitational Physics", M. Bassan et al, editors, World Scientific (1997), hep-th/9702167.

[165] S. Ferrara, *"Bertotti Robinson geometry and supersymmetry"*, Proc. 12th "Italian Conference on General Relativity and Gravitational Physics", M. Bassan et al, editors, World Scientific (1997), hep-th/9701163.

[166] W.A. Sabra, *Nucl. Phys.* **B510** (1998) 247.

[167] P. Fré and P. Soriani, *"The N=2 Wonderworld"*, World Scientific (1995).

[168] S.T. Yau, *Proc. Nat. Acad. Sci.* **74** (1977) 1798.

[169] I. Bandos, D. Sorokin and M. Tonin, *Nucl. Phys.* **B497** (1997) 275.

[170] J.H. Schwarz, *Nucl. Phys.* **B226** (1983) 269.

[171] L. Castellani, *Nucl. Phys.* **B294** (1987) 877.

[172] L. Castellani and I. Pesando, *Int. J. Mod. Phys.* **A8** (1993) 1125.

[173] A. Recknagel and V. Schomerus, *"D-branes in Gepner models"*, hep-th/9712186.

[174] J. Fuchs and C. Schweigert, *"Branes: from free fields to general backgrounds"*, hep-th/9712257.

[175] P.K. Towsend, *"M-branes at angles"*, hep-th/9708074.

[176] M.M.S. Jabbari, *Phys. Lett.* **B420** (1998) 279.

[177] M.B. Green, *Phys. Lett.* **B329** (1994) 435.

[178] M.B. Green and M. Gutperle, *Nucl. Phys.* **B476** (1996) 484.

[179] M.B. Green and M. Gutperle, *Nucl. Phys.* **B498** (1997) 195.

[180] J.L.F. Barbón, *Phys. Lett.* **B404** (1997) 33.

[181] P.C. Aichelburg and F. Embacher, *Phys. Rev.* **D34** (1986) 3006.

[182] M.J. Duff, J.T. Liu and J. Rahmfeld, *Nucl. Phys.* **B494** (1997) 161.

[183] M.J. Duff, J.T. Liu and J. Rahmfeld, *Nucl. Phys.* **B524** (1998) 129.

[184] G.T. Horowitz and A. Strominger, *Nucl. Phys.* **B360** (1991) 197.

[185] R.C. Myers and M.J. Perry, *Ann. Phys.* **172** (1986) 304.

[186] A. Hosoya, K. Ishikawa, Y. Ohkuwa and K. Yamagishi, *Phys. Lett.* **B134** (1984) 44.

[187] S. Weinberg, *"Dynamic and algebraic symmetries"*, in *"Lectures on Elementary Particles and Quantum Field Theory"*, MIT Press, Cambridge, (1970).

[188] S. Ferrara, M. Porrati and V.L. Telegdi, *Phys. Rev.* **D46** (1992) 3529.

[189] R. Jackiw, *Phys. Rev.* **D57** (1998) 2635.

[190] S. Paban, S. Sethi and M. Stern, *"Constraints from extended supersymmetry in quantum mechanics"*, hep-th/9805018.

[191] J.C. Plefka, M. Serone and A.K. Waldron, *"The matrix theory S-matrix"*, hep-th/9806081, to appear on *Phys. Rev. Lett.*.

[192] J.C. Plefka, M. Serone and A.K. Waldron, *"D=11 SUGRA as the low energy effective action of matrix theory: three form scattering"*, hep-th/9809070.

[193] P. Kraus, *Phys. Lett.* **B419** (1998) 73.

[194] I.N. McArthur, *Higher order spin-dependent terms in D0-brane scattering from the matrix model*, hep-th/9806082.

[195] M. Barrio, R. Helling and G. Polhemus, *JHEP* **05** (1998) 012.

[196] J.A. Harvey, *Nucl. Phys. Proc. Suppl.* **B68** (1998) 113.

[197] K. Becker and M. Becker, *Nucl. Phys.* **B506** (1997) 48.

[198] K. Becker, M. Becker, J. Polchinski and A.A. Tseytlin, *Phys. Rev.* **D56** (1997) 3174.

[199] S. Paban, S. Sethi and M. Stern, *"Supersymmetry and higher derivative terms in the effective action of super Yang-Mills theories"*, hep-th/9806028.

[200] Y. Okawa and T. Yoneya, *"Multi-body interactions of D-particles in supergravity and matrix theory"*, hep-th/9806108.

[201] O. Aharony and M. Berkooz, *Nucl. Phys.* **B491** (1997) 184.

[202] G. Lifschytz, *Phys. Lett.* **B409** (1997) 124.

[203] G. Lifschytz and S. D. Mathur, *Nucl. Phys.* **B507** (1997) 621.

[204] G. Lifschytz, *Nucl. Phys.* **B520** (1998) 105.

[205] I. Chepelev and A.A. Tseytlin, *Phys. Rev.* **D56** (1997) 3672.

[206] I. Chepelev and A.A. Tseytlin, *Nucl. Phys.* **B511** (1998) 629.

[207] J.M. Pierre, *Phys. Rev.* **D57** (1998) 1250.

[208] C.P. Burgess and T.R. Morris, *Nucl. Phys.* **B291** (1987) 256, 285.