STABILITY OF NON-SUSY MINKOWSKI VACUA IN SUPERGRAVITY

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- SUSY breaking in SUGRA scenarios.
- SUGRA models with chiral multiplets.
- Flatness and stability constraints.
- Factorizable scalar manifolds.
- Minimal string moduli spaces.
- Homogeneous scalar manifolds.
- Enhanced string moduli spaces.

Gomez-Reino, Scrucca (hep-th/0602246)

Gomez-Reino, Scrucca (work in progress)

SUSY BREAKING AND SUGRA

Direct spontaneous SUSY breaking at the tree-level implies, in a renormalizable and anomaly-free theory with rigid SUSY, a sum rule on the mass spectrum:

$$\mathrm{STr}\,[M^2] = \sum_J (-1)^{2J}(2J+1)\,m_J^2 = 0$$

This predicts rather generically that one of the superparticles is lighter than its ordinary partner, in contradiction with experimental observation.

The standard paradigm to evade this difficulty is to assume that SUSY breaking occurs spontaneously in a hidden sector with fields Φ_i and is transmitted to the visible sector with fields Q_a only indirectly, through some suppressed interactions.

The effect of SUSY breaking on the visible sector can be parametrized through super-renormalizable soft breaking terms, which depend both on the details of the hidden sector theory and on the mediation mechanism.

The relevant effective Lagrangian for phenomenology has then the general form:

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{susy}} + \mathcal{L}_{\mathrm{soft}}$$

A natural mediation mechanism is provided by gravitational interactions. The general setup becomes then that of SUGRA, with local SUSY.

SUSY breaking occurs at some scale M in the hidden sector and is transmitted to the visible sector through gravitational interactions, whose strength is set by $M_{\rm P} \gg M$.

The microscopic theory might be some superstring model. But below $M_{\rm P}$, and in particular at M, this can be effectively described by a non-renormalizable SUGRA theory.

Soft terms originate from higher-dimensional operators that mix visible fields Q_a to hidden fields Φ_i and are suppressed by powers of $M_{\rm P}$, and their energy scale is

$$m_{\rm soft} \sim \frac{M^2}{M_{\rm P}}$$

Chamseddine, Arnowitt, Nath
Barbieri, Ferrara, Savoy
Hall, Lykken, Weinberg

The main delicate features that are needed in order to get a satisfactory situation are:

- Soft terms with $m_{\rm soft} \sim M_{\rm EW}$ and peculiarities.
- Cosmological constant with $M_{\rm CC} \ll M_{\rm EW}$.
- Hidden scalars with $m > M_{\rm EW}$ and stable.

CHIRAL SUGRA MODELS

The two-derivative Lagrangian of a SUGRA theory involving n chiral multiplets Φ_i is entirely specified by a single real function G of these, and one can set $M_{\rm P} = 1$

The function G can be decomposed into of a non-holomorphic Kähler potential K and a holomorphic superpotential W:

$$G(\Phi_i, \Phi_i^\dagger) = K(\Phi_i, \Phi_i^\dagger) + \log W(\Phi_i) + \log \bar{W}(\Phi_i^\dagger)$$

This decomposition is however ambiguous, due to the Kähler symmetry changing $K \to K + F + \bar{F}$ and $W \to e^{-F} W$.

It is very convenient to denote derivatives of G with respect to Φ_i and $\Phi_i^\dagger$ with indices i and $\bar{i}$:

$$G_i \equiv \frac{\partial G}{\partial \Phi_i}\,, \quad G_{\bar{i}} \equiv \frac{\partial G}{\partial \Phi_i^\dagger}$$

$$G_{ij} \equiv \frac{\partial^2 G}{\partial \Phi_i \partial \Phi_j}\,, \quad G_{\bar{i}\bar{j}} \equiv \frac{\partial^2 G}{\partial \Phi_i^\dagger \partial \Phi_j^\dagger}\,, \quad G_{i\bar{j}} \equiv \frac{\partial^2 G}{\partial \Phi_i \partial \Phi_j^\dagger}$$

$$\ldots$$

Mixed holomorphic/antiholomorphic derivatives of G depend only on K and define a Kähler geometry for the manifold parametrized by the complex scalar fields ϕ^i.

The metric and its inverse, the Chirstoffel connection and the Riemann tensor are given by:

$$g_{i\bar{j}} = G_{i\bar{j}} \ , \quad g^{i\bar{j}} = G^{-1 i\bar{j}}$$
$$\Gamma^k_{ij} = G^{-1 k\bar{l}} G_{ij\bar{l}} \ , \quad \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = G^{-1 \bar{k}l} G_{\bar{i}\bar{j}l}$$
$$R_{i\bar{j}p\bar{q}} = G_{i\bar{j}p\bar{q}} - G^{-1 \bar{s}r} G_{ip\bar{s}} G_{\bar{j}\bar{q}r}$$

Pure holomorphic or antiholomorphic derivatives of G depend instead both on K and W, and determine the way supersymmetry is possibly broken.

All the quantities can be written with covariant and contravariant indices that are raised and lowered with the metric.

The complex auxiliary fields F^i are fixed by their non-dynamical equation of motion to the values:

$$F^i = e^{G/2} G^i$$

The scalar fields ϕ^i have a wave-function normalization given by $Z_{i\bar{j}} = g_{i\bar{j}}$ and a potential, which determines their vev and mass and controls spontaneous SUSY breaking, of the form:

$$V = e^G \left(G^k G_k - 3 \right)$$

The flatness condition of vanishing cosmological constant is that $V = 0$ on the vacuum and implies that at that point:

$$g_{i\bar{j}} G^i G^{\bar{j}} = 3$$

The first derivatives of the potential controlling its variations can be computed as $\delta_i = \nabla_i V$ and are given by:

$$\delta_i = e^G \Big(G_i + G^k \nabla_i G_k \Big)$$

The stationarity conditions defining extrema of the potential are $\delta_i = 0$ and imply:

$$G_i + G^k \nabla_i G_k = 0$$

The two types of second derivatives of the potential that determine the squared masses can be computed as $m^2_{i\bar{j}} = \nabla_i \nabla_{\bar{j}} V$ and $m^2_{ij} = \nabla_i \nabla_j V$, and one easily finds:

$$m^2_{i\bar{j}} = e^G \Big(g_{i\bar{j}} + \nabla_i G^k \, \nabla_{\bar{j}} G_k - R_{i\bar{j}p\bar{q}} \, G^p G^{\bar{q}} \Big)$$
$$m^2_{ij} = e^G \Big(\nabla_i G_j + \nabla_j G_i + \frac{1}{2} G^k \Big\{ \nabla_i, \nabla_j \Big\} G_k \Big)$$

The stability condition ensuring that the extremum is really a local minimum is that the whole $2n$-dimensional squared-mass matrix is positive definite:

$$m^2 = \begin{pmatrix} m^2_{i\bar{j}} & m^2_{ij} \\ m^2_{\bar{i}\bar{j}} & m^2_{\bar{i}j} \end{pmatrix} > 0$$

The constraints implied by this condition are difficult to work out explicitly. The only systematic approach is to diagonalize it and study the behavior of all the eigenvalues.

The fermion fields ψ^i split into 1 Goldstino linear combination $\psi = G_i \psi^i$ that is absorbed by the gravitino field, and $n-1$ physical orthogonal combinations $\tilde{\psi}^i$. The normalization of their wave-function is $\tilde{Z}_{i\bar{j}} = g_{i\bar{j}}$, and their mass is encoded in:

$$\tilde{m}_{ij} = e^G \Big(\nabla_i G_j + \frac{1}{3} G_i G_j\Big)$$

More precisely, the $2n$-dimensional mass matrix is given by

$$\tilde{m} = \begin{pmatrix} 0 & \tilde{m}_{ij} \\ \tilde{m}_{\bar{i}\bar{j}} & 0 \end{pmatrix}$$

Finally, the graviton and gravitino fields $h^{\mu\nu}$ and ψ^μ have canonical wave-function normalizations $Z_{\rm gra} = 1$ and $Z_{3/2} = 1$, and masses given by:

$$m^2_{\rm gra} = 0 \ , \ \ m_{3/2} = e^{G/2}$$

The mass matrices that emerge have a rich structure that depends in a quite involved way on G, or K and W. However, they satisfy a quite simple sum rule, fixing the supertrace of the squared mass matrix for the whole theory:

$$\begin{aligned} {\rm STr} M^2 \ &= \ 2\, g^{i\bar{j}}\, m^2_{i\bar{j}} - 2\, g^{i\bar{j}} g^{p\bar{q}}\, \tilde{m}_{ip}\, \tilde{m}_{\bar{j}\bar{q}} - 4\, m^2_{3/2} \\ &= \ 2\, e^G \Big(n - 1 - R_{i\bar{j}}\, G^i G^{\bar{j}}\Big) \end{aligned}$$

Cremmer, Ferrara, Girardello, Van Proeyen

Cremmer, Julia, Scherk, Ferrara, Girardello, Van Nieuwenhuizen

Bagger, Witten

FLATNESS AND STABILITY CONSTRAINTS

It would be interesting to have some simple criterium for the conditions of flatness and stability to hold, to discriminate between non-suitable and potentially suitable models.

More precisely, it would be useful to have a simple condition depending on K but not W, to constrain the type of geometry independently of the mechanism of SUSY breaking.

The supertrace sum rule does not help much. It gives an information about the trace of the scalar's m^2, but only with respect to the trace of the fermion's $\tilde{m}\tilde{m}^\dagger$. This is useful only for the visible sector, not the hidden one.

Our strategy is to look for some simple necessary condition for having $m^2 > 0$ for the scalars, that relies just on its structure and does not invoke the fermion's $\tilde{m}$.

The crucial fact that we exploit is that all the upper-left submatrices of a positive definite matrix must be positive definite. In our case, this implies in particular that the n-dimensional submatrix $m^2_{i\bar{j}}$ should be positive definite:

$$m^2_{i\bar{j}} > 0$$

This condition means that $\forall z^i$ one must have $m^2_{i\bar{j}}\, z^i \bar{z}^{\bar{j}} > 0$. One can then look for a specific z^i that leads to a particularly simple condition. The right choice is $z^i = G^i$, for which:

$$m^2_{i\bar{j}}\, G^i G^{\bar{j}} = e^G \Big(6 - R_{i\bar{j}p\bar{q}}\, G^i G^{\bar{j}} G^p G^{\bar{q}}\Big)$$

The corresponding necessary condition $m^2_{i\bar{j}}\, G^i G^{\bar{j}} > 0$ reduces then to the extremely simple curvature constraint:

$$R_{i\bar{j}p\bar{q}}\, G^i G^{\bar{j}} G^p G^{\bar{q}} < 6$$

Notice that the special direction $z^i = G^i$ considered to derive the necessary condition $m^2_{i\bar{j}}\, G^i G^{\bar{j}} > 0$ concerning the scalars corresponds to the Goldstino direction for the fermions, and correspondingly $\tilde{m}_{ij}\, G^i G^j = 0$.

Summarizing, we conclude that at any stationary point of the potential the following two conditions must be satisfied in order to have a chance to reach a satisfactory situation:

Flatness: $g_{i\bar{j}}\, G^i G^{\bar{j}} = 3$ (necessary & sufficient)

Stability: $R_{i\bar{j}p\bar{q}}\, G^i G^{\bar{j}} G^p G^{\bar{q}} < 6$ (necessary)

The metric $g_{i\bar{j}}$ and the curvature $R_{i\bar{j}p\bar{q}}$ depend only on K and characterize the geometry. The quantities G^i depend also on W, and control the SUSY breaking direction, in view of the relation $G^i = F^i/m_{3/2}$.

For a given underlying geometry, the above two relations constrain respectively the overall amount and the direction of SUSY breaking that is compatible with the requirements of flatness and stability.

One can imagine fixed tensors $g_{i\bar{j}}$ and $R_{i\bar{j}p\bar{q}}$ and scan over all the possible vectors G^i. The length of this vector is then fixed by the metric, whereas its orientation is constrained to lie within a certain critical cone specified by the curvature.

The strategy to derive these constraints is to first determine the direction that minimizes the quartic form $R_{i\bar{j}p\bar{q}}G^iG^{\bar{j}}G^pG^{\bar{q}}$ for fixed value of the quadratic form $g_{i\bar{j}}G^iG^{\bar{j}}$, and then check how far apart from it the former stays small enough.

This variational problem is hard to solve in full generality. However, it is possible to obtain very simple and strong results for the subclass of models based on spaces that are factorized or homogeneous.

FACTORIZABLE SPACES

Suppose that the n-dimensional scalar manifold is the product of n 1-dimensional manifolds. The function K splits then into a sum of terms depending on a single field, while W can instead still be arbitrary:

$$K = \sum_{k=1}^{n} K^{(k)}(\Phi_k, \Phi_k^\dagger)$$

$$W = W(\Phi_1, \ldots, \Phi_n)$$

This assumption represents a Kähler-invariant constraint on G, implying that all its mixed derivatives vanish unless they are purely holomorphic or antiholomorphic:

$$G_{i\bar{j}} = 0\,, \qquad i,j \text{ not equal}$$

$$G_{ij\bar{k}} = G_{i\bar{j}\bar{k}} = 0\,, \qquad i,j,k \text{ not equal}$$

$$G_{ijk\bar{l}} = G_{ij\bar{k}\bar{l}} = G_{i\bar{j}\bar{k}\bar{l}} = 0\,, \qquad i,j,k,l \text{ not equal}$$

$$\ldots$$

In this situation, the metric and curvature tensors become both diagonal and have only n non-vanishing components. This simplifies the problem sufficiently much to be able to solve it exactly.

The non-vanishing components of the metric are $g_{i\bar{i}} = G_{i\bar{i}}$, and those of the curvature tensor are related to these by:

$$R_{i\bar{i}i\bar{i}} = R_i \, g_{i\bar{i}}^2$$

where the crucial parameters are the n curvature scalars R_i associated to each complex scalar field:

$$R_i = \frac{G_{ii\bar{i}\bar{i}}}{G_{i\bar{i}}^2} - \frac{G_{ii\bar{i}}G_{i\bar{i}\bar{i}}}{G_{i\bar{i}}^3}$$

The two flatness and stability conditions derived before then simplify to the following expressions:

$$\text{Flatness:} \;\; \sum_k \Theta_k^2 = 1$$

$$\text{Stability:} \;\; \sum_k R_k \, \Theta_k^4 < \frac{2}{3}$$

in terms of the new positive real variables

$$\Theta_i^2 = \frac{G_i G^i}{3} = \frac{G_i G_{\bar{i}}}{3G_{i\bar{i}}} \quad \text{(no sum)}$$

The variational problem defined by the two conditions thus simplifies from a quartic to a quadratic problem, and can be solved exactly.

It is straightforward to show that when $R_i > 0$ these constraints can admit solutions only if the following curvature bound is satisfied:

$$\sum_k R_k^{-1} > \frac{3}{2}$$

If this necessary condition is satisfied, the solutions correspond to the following domain in the space of variables:

$$\Theta_i^2 \in \left[\Theta_i^{2-}, \Theta_i^{2+}\right]$$

where:

$$\Theta_i^{2+} = \begin{cases} \dfrac{R_i^{-1} + \sqrt{\frac{2}{3} R_i^{-1} \Big(\sum_{k \neq i} R_k^{-1}\Big)\Big(\sum_k R_k^{-1} - \frac{3}{2}\Big)}}{\sum_k R_k^{-1}}, & R_i^{-1} < \dfrac{3}{2} \\ 1\,, & R_i^{-1} > \dfrac{3}{2} \end{cases}$$

$$\Theta_i^{2-} = \begin{cases} \dfrac{R_i^{-1} - \sqrt{\frac{2}{3} R_i^{-1} \Big(\sum_{k \neq i} R_k^{-1}\Big)\Big(\sum_k R_k^{-1} - \frac{3}{2}\Big)}}{\sum_k R_k^{-1}}, & \sum_{k \neq i} R_k^{-1} < \dfrac{3}{2} \\ 0\,, & \sum_{k \neq i} R_k^{-1} > \dfrac{3}{2} \end{cases}$$

The SUSY breaking direction must therefore lie in a certain Goldstino cone specified by the curvature scalars. Its axis is the preferred direction minimizing the quartic curvature form:

$$\Theta_i^{20} = \frac{R_i^{-1}}{\sum_k R_k^{-1}}$$

and its solid angle grows with the excess of effective inverse curvature $\sum_k R_k^{-1}$ with respect to the threshold $3/2$.

The most important qualitative result is that the direction of SUSY breaking must align more along the directions of low curvature than those of high curvature.

More precisely, a given Θ_i can become as large as 1 only if its curvature satisfies $R_i^{-1} > 3/2$ on its own, and as low as 0 only if the curvature of the remaining fields satisfy $\sum_{k \neq i} R_k^{-1} > 3/2$ on their own.

This means that whether the relevance of a particular chiral multiplet Φ_i for SUSY breaking is high or low depends on whether the corresponding inverse curvature R_i^{-1} is large or small with respect to the threshold value $3/2$.

This also implies that the mass/auxiliary m_i/F^i of a chiral multiplet Φ_i can be small/large only if $R_i^{-1} > 3/2$ and on the contrary large/small only if $\sum_{k \neq i} R_k^{-1} > 3/2$.

MINIMAL STRING MODULI SPACES

In string models, a good candidate for the hidden sector is the universal sector containing the neutral moduli.

The vev of the scalars in these superfields control the coupling constant and the geometry of the manifold of extra dimensions, and are of order 1. The corresponding auxiliary fields could then represent the original seed for SUSY breaking.

Kaplunovky, Louis

At leading order in the derivative and loop expansion, the Kähler potential K in this sector has the general form

$$K = -\sum_k n_k \ln\left(\Phi_k + \Phi_k^\dagger\right)$$

Witten

The form that the superpotential W can possibly take is less universal and is mainly related to effects like gaugino condensation and fluxes. We therefore keep it arbitrary.

The above situation is of the separable type, and we can therefore apply the results we derived. The curvature scalars are constant and related to the numerical coefficients n_i as:

$$R_i = \frac{2}{n_i} \quad \Rightarrow \quad R_i^{-1} = \frac{n_i}{2}$$

The necessary condition $\sum_k R_k^{-1} > 3/2$ on the curvatures then implies the simple numerical constraint:

$$\sum_k n_k > 3$$

The most relevant moduli are the dilaton S, controlling the coupling, and the global Kähler modulus T, controlling the volume of the compact dimensions. In the limit of small coupling and large volume where S and T are large, one finds:

$$n_S = 1\ ,\ \ n_T = 3$$

It follows that S does not satisfy the curvature bound on its own, whereas T saturates it marginally. This implies that S cannot dominate SUSY breaking, unless large corrections arise, whereas T could do so, even with small corrections.

Keeping both fields in the low-energy effective action, one finds that m_S cannot be much smaller than m_T, and F_S cannot be too large compared to F_T. More precisely, the Goldstino angle θ must be between 0 and $\pi/4$, and correspondingly:

$$0 < \frac{|F_S|/\mathrm{Re}S}{|F_T|/\mathrm{Re}T} < \frac{1}{\sqrt{3}}$$

This demonstrates that the scenario where S dominates over T cannot be realized, at least in the controllable limit where both are large.

HOMOGENEOUS SPACES

Suppose that the n-dimensional scalar manifold has the form of a coset space G/H, where G is the global isometry group and H the local stability group. The function K has then some special form, but W can be arbitrary:

$$K = K^{(G/H)}(\Phi_1, \Phi_1^\dagger, \ldots, \Phi_n, \Phi_n^\dagger)$$

$$W = W(\Phi_1, \ldots, \Phi_n)$$

There exists finitely many such coset Kähler manifolds for each given dimensionality n, and they have been classified.

In this situation, the metric and curvature tensors are invariant under the global transformations of the group G, and their components are actually related. The problem simplifies then again sufficiently much to be able to solve it exactly.

To study non-factorizable cases it is convenient to rewrite the flatness and stability conditions derived before, by considering the square of the first one and reorganizing also the second one, in the form:

$$\text{Flatness:} \quad g_{i\bar{j}}\, g_{p\bar{q}}\, t^{ip}\, t^{\bar{j}\bar{q}} = 1$$

$$\text{Stability:} \quad R_{i\bar{j}p\bar{q}}\, t^{ip}\, t^{\bar{j}\bar{q}} < \frac{2}{3}$$

We have introduced the holomorphic symmetric tensors

$$t^{ij} = \frac{1}{3} G^i G^j$$

This is however not a regular change of variables. Indeed, the $n(n+1)/2$ components of t^{ij} become equivalent to the n independent G^i only by imposing the $n(n-1)/2$ quadratic constraints:

$$(t^{ij})^2 = t^{ii} t^{jj} \ , \quad i \neq j$$

The problem does therefore not trivially reduce from a quartic to a quadratic problem, but becomes rather cubic.

The proceed we consider the map $t^{ij} \to R^{i\,j}_{\,p\,q}\, t^{pq}$ on symmetric tensors and diagonalize it. For homogeneous spaces, this is easy to do since distinct proper subspaces must be associated to distinct irreducible representations of H.

The generic tensor t^{ij} can then be decomposed as

$$t^{ij} = \sum_r t^{ij}_r$$

in terms of orthogonal components t^{ij}_r satisfying:

$$R^{i\,j}_{\,p\,q}\, t^{pq}_r = R_r\, t^{ij}_r$$

The form of these eigentensors and their eigenvalues are know in all the cases. Note that $R = \sum_r R_r$.

Calabi, Vesentini

We can then finally rewrite our conditions as:

$$\text{Flatness:}\ \sum_r \Xi_r^4 = 1$$

$$\text{Stability:}\ \sum_r R_r\, \Xi_r^4 < \frac{2}{3}$$

where now

$$\Xi_r^4 = g_{i\bar{j}}\, g_{p\bar{q}}\, t_r^{ip}\, t_r^{\bar{j}\bar{q}}$$

These new variables also suffer from additional constraints. The form of these constraints depends however on the model, and one has to perform a case by case analysis.

Treating the Ξ_r^4 as independent results in milder necessary conditions where some part of the original information has been lost. When $R_r > 0$, one finds then the restriction:

$$\max\left\{R_r^{-1}\right\} > \frac{3}{2}$$

For factorizable cases only n of the R_r are non-zero, and the other $n(n-1)/2$ vanish. The condition is then trivial.

For maximally symmetric cases where all the R_r are identical, the constraints clearly coincide with that of a single field.

For less symmetric cases where some of the R_r are distinct, the constraint could get milder.

ENHANCED STRING MODULI SPACES

The simplest geometry that can occur for n moduli fields Φ_i in a string model is described by

$$K = -\sum_k n_k \ln\left(\Phi_k + \Phi_k^\dagger\right)$$

This corresponds to a product of isomorphic one-dimensional spaces with a coset structure of the type:

$$\mathcal{M} = \bigotimes_k \frac{SU(1,1)}{U(1)}$$

We found in this case the necessary condition:

$$\sum_k n_k > 3$$

There exist several generalizations of this kind of geometry that can occur in string models. They are also coset spaces, and can therefore be studied efficiently as well.

For instance, standard moduli can get mixed to extra moduli as well as matter fields, and one can have situations with n^2 moduli fields ϕ_{ij} and nm matter fields X_{ia} where:

$$K = -\, n_{\rm all} \ln \det\left(\Phi_{ij} + \Phi_{ij}^\dagger - \sum_a X_{ia}^\dagger X_{ja}\right)$$

Ellis, Kounnas, Nanopoulos

Ferrara, Kounnas, Porrati

This corresponds to an $n(n+m)$-dimensional space that is a coset manifold of the type:

$$\mathcal{M} = \frac{SU(n,n+m)}{U(1)\times SU(n)\times SU(n+m)}$$

The question we want to adress is:

$$n_{\rm all} \text{ restricted ?}$$

It is convenient to perform a holomorphic field redefinition and a Kähler transformation to describe the theory in terms of $n(n+m)$ new superfields $\Phi_{i\alpha}$ with

$$K = -\,n_{\rm all}\ln\det\Big(\delta_{ij} - \sum_\alpha \Phi_{i\alpha}\Phi^\dagger_{\alpha j}\Big)$$

The corresponding metric and Rieman tensors are given by:

$$g_{i\alpha\,\bar j\bar\beta} = h_{i\bar j}\,\tilde h_{\alpha\bar\beta}$$

$$R_{i\alpha\,\bar j\bar\beta\,p\gamma\,\bar q\bar\delta} = \frac{R_{\rm all}}{2}\Big(h_{i\bar j}\,h_{p\bar q}\,\tilde h_{\alpha\bar\delta}\,\tilde h_{\gamma\bar\beta} + h_{i\bar q}\,h_{p\bar j}\,\tilde h_{\alpha\bar\beta}\,\tilde h_{\gamma\bar\delta}\Big)$$

in terms of the two submetrics

$$h_{i\bar j} = n_{\rm all}^{1/2}(\mathbb{1}-\phi\phi^\dagger)^{-1}_{i\bar j}\ ,\quad \tilde h_{\alpha\bar\beta} = n_{\rm all}^{1/2}(\mathbb{1}-\phi^\dagger\phi)^{-1}_{\alpha\bar\beta}$$

and the overall curvature scale

$$R_{\rm all} = \frac{2}{n_{\rm all}} \quad\Rightarrow\quad R_{\rm all}^{-1} = \frac{n_{\rm all}}{2}$$

Zumino
Van Holten

Consider now the action of the holomorphic Rieman tensor

$$R^{i\alpha}{}_{j\beta}{}^{p\gamma}{}_{q\delta} = \frac{R_{\rm all}}{2}\Big(\delta^i_j\delta^p_q\delta^\alpha_\delta\delta^\gamma_\beta + \delta^i_q\delta^p_j\delta^\alpha_\beta\delta^\gamma_\delta\Big)$$

on holomorphic tensors of the form:

$$t^{i\alpha\, j\beta} = \frac{1}{3}\,G^{i\alpha}G^{j\beta}$$

These tensors are symmetric under the exchange of the first and second pairs of indices, and can be decomposed into two irreducible components that are respectively symmetric and antisymmetric in the exchange of each type of indices:

$$t_\pm^{i\alpha j\beta} = \frac{1}{6}\Big(G^{i\alpha}G^{j\beta} \pm G^{i\beta}G^{j\alpha}\Big)$$

These are eigentensors of the Rieman tensor with eigenvalues $R_\pm$ and degeneracies $d_\pm$ given by:

$$R_\pm = \pm R_{\rm all}\;,\quad d_\pm = \frac{n(n\pm 1)}{2}\,\frac{(n+m)(n+m\pm 1)}{2}$$

The flatness and stability conditions read then:

$$\text{Flatness:}\;\; \Xi_+^4 + \Xi_-^4 = 1$$

$$\text{Stability:}\;\; \Xi_+^4 - \Xi_-^4 < \frac{2}{3}\,R_{\rm all}^{-1}$$

where

$$\Xi_\pm = \Big(t_\pm^{i\alpha\, j\beta}\, t_{\pm\, i\alpha\, j\beta}\Big)^{1/4}$$
$$= \frac{1}{18^{1/4}}\Big[\Big(G^{i\alpha}G_{i\alpha}\Big)^2 \pm \Big(G^{i\alpha}G_{j\alpha}G^{j\beta}G_{\beta i}\Big)\Big]^{1/4}$$

If Ξ_+ and Ξ_- were independent, there would always exist solutions. But this is not the case, and one finds again some constraints on the model.

To derive explicitly these constraints, it is convenient to go back to the original form of the two conditions in terms of the unconstrained variables $G^{i\alpha}$, which read:

$$\text{Flatness:}\ \ G^{i\alpha}G_{i\alpha} = 3$$
$$\text{Stability:}\ \ G^{i\alpha}G_{j\alpha}G^{j\beta}G_{i\beta} < 6\,R_{\rm all}^{-1}$$

The problem can be further simplified by switching to tangent space indices, which are contracted with $\delta_{I\bar J}$ and $\tilde\delta_{A\bar B}$ rather than $h_{i\bar j}$ and $\tilde h_{\alpha\bar\beta}$. This is done with the subvielbeins:

$$e_{iJ} = n_{\rm all}^{1/4}(\mathbb{1}-\phi\phi^\dagger)_{iJ}^{-1/2}\ ,\ \ \tilde e_{\alpha B} = n_{\rm all}^{1/4}(\mathbb{1}-\phi^\dagger\phi)_{\alpha B}^{-1/2}$$
$$e_{Ij} = n_{\rm all}^{1/4}(\mathbb{1}-\phi\phi^\dagger)_{Ij}^{-1/2}\ ,\ \ \tilde e_{A\beta} = n_{\rm all}^{1/4}(\mathbb{1}-\phi^\dagger\phi)_{A\beta}^{-1/2}$$

The two conditions can then be rewritten as

$$\text{Flatness:}\ \ {\rm Tr}\Big(\Theta^2_{I\bar J}\Big) = 1$$
$$\text{Stability:}\ \ {\rm Tr}\Big(\Theta^4_{I\bar J}\Big) < \frac{2}{3}\,R_{\rm all}^{-1}$$

by introducing the new positive-definite Hermitean matrix of variables

$$\Theta^2_{I\bar J} = \frac{G_I^\alpha G_{\alpha\bar J}}{3} = \frac{e_{Ii}G^{i\alpha}\tilde h_{\alpha\bar\beta}G^{\bar\beta\bar j}e_{\bar j\bar J}}{3}$$

Finally, one can consider the unique matrix $Z_{I\bar{J}} \in SU(n)$ that allows to diagonalize this Hermitian matrix of variables $\Theta^2_{I\bar{J}}$ and use it to rewrite:

$$\Theta^2_{I\bar{J}} = \left(Z\,\Theta^{2D} Z^\dagger\right)_{I\bar{J}}$$

The two conditions reduce then simply to

$$\text{Flatness: } \sum_K \Theta^{2D}_{K\bar{K}} = 1$$
$$\text{Stability: } \sum_K R_{\rm all}\,\Theta^{4D}_{K\bar{K}} < \frac{2}{3}$$

One obtains therefore exactely the same type of conditions as for n standard moduli Φ_i with identical curvatures $R_i = R_{\rm all}$, implying $n_i = n_{\rm all}$.

The SUSY breaking direction is constrained to lie within a certain cone, whose orientation depends on the location of the stationary point, and the curvature condition yields:

$$n\,n_{\rm all} > 3$$

We conclude that extra off-diagonal moduli or matter fields that enhance the space do not allow to evade the restrictions occurring for standard factorized moduli.

CONCLUSIONS AND OUTLOOK

- In models with only chiral multiplets, there exist a very simple and strong necessary condition for stability that constrains the curvature of the geometry and the SUSY breaking direction.
- The consequences of these constraints can be worked out in detail for factorizable and homogeneous geometries, as those occurring for instance in the moduli sector of string models.
- It would be of great interest to generalize this study to models involving also vector multiplets gauging isometries of the scalar manifold.