CLASSICAL VERSUS QUANTUM SYMMETRIES AND ANOMALIES

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- Symmetries in classical field theory.
- Symmetries in quantum field theory.
- Regularization and anomalies.
- Anomalies in gobal chiral symmetries.
- Anomalies in local gauge symmetries.
- General structure of anomalies.
- Implications on particle physics.

SYMMETRIES IN CLASSICAL FIELD THEORY

Canonical formulation

A classical field theory is specified by an action functional:

$$S = \int d^4x\, \mathcal{L}\Big(\phi, \partial_\mu \phi\Big)$$

The equations of motions are given by:

$$\frac{\delta S}{\delta \phi} \equiv \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$$

The corresponding Hamiltonian has the form:

$$H = \int d^3\vec{x}\, \mathcal{H}(\phi, \partial_i \phi, \pi)$$

The equations of motions are then rewritten as

$$\dot{\phi} = \frac{\delta H}{\delta \pi} \equiv \frac{\partial \mathcal{H}}{\partial \pi}, \qquad \dot{\pi} = -\frac{\delta H}{\delta \phi} \equiv -\frac{\partial \mathcal{H}}{\partial \phi} + \partial_i \frac{\partial \mathcal{H}}{\partial \partial_i \phi}$$

In terms of Poisson brackets, these become

$$\dot{\phi} = \left\{\phi, H\right\}_{\mathbf{P}}, \qquad \dot{\pi} = \left\{\pi, H\right\}_{\mathbf{P}}$$

For a generic functional F on phase space, one finds then:

$$\dot{F} = \left\{F, H\right\}_{\mathbf{P}}$$

Conservation laws

A conserved quantity is associated with a current satisfying:

$$\partial_\mu J^\mu = 0$$

The corresponding conserved charge reads:

$$Q = \int d^3\vec{x}\, J^0$$

Assuming that Q does not dependent explicitly on time, one has then:

$$\left\{Q, H\right\}_{\mathbf{P}} = 0$$

Global symmetries and Noether theorem

Conserved charges are associated to continuous global symmetries of S.
Consider indeed an infinitesimal transformation with parameter $\delta\alpha$:

$$\delta_\alpha \phi = \delta\alpha \, \frac{\delta\phi}{\delta\alpha}$$

Using the equations of motion, one finds:

$$\delta_\alpha S = \int d^4x \, \partial_\mu \Big(\delta\alpha \, J^\mu_\alpha \Big)$$

where

$$J^\mu_\alpha = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \frac{\delta\phi}{\delta\alpha}$$

The invariance of S under transformations with constant $\delta\alpha$ implies thus:

$$\partial_\mu J^\mu_\alpha = 0$$

The corresponding conserved charge is

$$Q_\alpha = \int d^3\vec{x} \, \pi \, \frac{\delta\phi}{\delta\alpha}$$

The charge Q_α generates the infinitesimal symmetry transformations in phase space:

$$\frac{\delta\phi}{\delta\alpha} = \Big\{\phi, Q_\alpha\Big\}_{\mathrm{P}}, \qquad \frac{\delta\pi}{\delta\alpha} = \Big\{\pi, Q_\alpha\Big\}_{\mathrm{P}}$$

For a generic functional F on phase space, one finds then:

$$\frac{\delta F}{\delta\alpha} = \Big\{F, Q_\alpha\Big\}_{\mathrm{P}}$$

If there are several symmetries forming some Lie group with structure constants f_{ijk}, the corresponding charges Q_i realize the group algebra through Poisson brackets:

$$\Big\{Q_i, Q_j\Big\}_{\mathrm{P}} = f_{ij}{}^{k} Q_k$$

Note finally that generalized transformations with non-constant $\delta\alpha$ do not represent an invariance, but induce a simple variation of S:

$$\delta_\alpha S = \int d^4x\, \partial_\mu \delta\alpha\, J^\mu_\alpha$$

Local symmetries and gauge fields

A theory with fields ϕ possessing a global symmetry can be promoted to a new theory involving also a gauge field $A_{\alpha\mu}$ with suitable couplings and transformation laws, in which the symmetry becomes local.

We assume that the action splits into two separately gauge-invartiant parts $S_{\mathrm{G}}$and S_{M} as follows:

$$S = \int d^4x \, \mathcal{L}_{\mathrm{G}}\left(A_{\alpha\nu}, \partial_\mu A_{\alpha\nu}\right) + \int d^4x \, \mathcal{L}_{\mathrm{M}}\left(\phi, \partial_\mu \phi, A_{\alpha\nu}, \partial_\mu A_{\alpha\nu}\right)$$

The new field $A_{\alpha\mu}$ is assumed to transform as:

$$\delta_\alpha A_{\alpha\mu} = g^{-1} \partial_\mu \delta\alpha$$

The current of the interacting theory is defined as the source term coming from S_{M} in the equation of motion for $A_{\alpha\mu}$:

$$J_{\alpha\mu} = -g^{-1} \frac{\delta S_{\mathrm{M}}}{\delta A_{\alpha\mu}}$$

The equations of motion for the gauge field take then the form:

$$\frac{\delta S_{\mathrm{G}}}{\delta A_{\alpha\mu}} = g\, J_{\alpha\mu}$$

The current J^{μ}_{α} is conserved, as a consequence of the invariance of S_{M} under local transformations with $\delta\alpha$ vanishing at infinity. Indeed, the only source of non-stationarity of S_{M} comes from $\delta_{\alpha} A_{\alpha\mu}$, and thus:

$$\delta_{\alpha} S_{\mathrm{M}} = \int d^4x\, \frac{\delta S_{\mathrm{M}}}{\delta A_{\alpha\mu}}\, \delta_{\alpha} A_{\alpha\mu} = -\int d^4x\, J^{\mu}_{\alpha}\, \partial_{\mu}\delta\alpha$$

Integrating by parts, we see that the invariance of S_{M} implies:

$$\partial_{\mu} J^{\mu}_{\alpha} = 0$$

This is the same conservation law as in the global case, except that J^{μ}_{α} may now depend also on $A_{\alpha\mu}$.

Example: CED

The Lagrangian for a free fermion is:

$$\mathcal{L} = i\,\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\,\bar{\psi}\psi$$

This has the global invariance $\delta_{\alpha}\psi = i\,\delta\alpha\,\psi$, leading to the conserved current

$$J^{\mu} = \bar{\psi}\gamma^{\mu}\psi$$

The gauged version of this theory involves $D_{\mu} = \partial_{\mu} - i\,g A_{\mu}$ and

$$\mathcal{L} = i\,\bar{\psi}\gamma^{\mu}D_{\mu}\psi - m\,\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

This has the local invariance $\delta_{\alpha}\psi = i\,\delta\alpha\,\psi$, $\delta_{\alpha}A_{\mu} = g^{-1}\partial_{\mu}\delta\alpha$, and the conserved current is again:

$$J^{\mu} = \bar{\psi}\gamma^{\mu}\psi$$

SYMMETRIES IN QUANTUM FIELD THEORY

Operatorial formulation

In the operatorial formulation of quantum field theory, the fields become operators ϕ acting on a Hilbert space of particle states $|n\rangle$, and Poisson brackets become commutators or anticommutators:

$$\Big\{\cdots,\cdots\Big\}_{\mathrm{P}} \to -i\Big[\cdots,\cdots\Big\}$$

The field operators satisfy the differential equations of motion $\delta S/\delta\phi = 0$ as function of the coordinates:

$$\frac{\delta S}{\delta \phi} \equiv \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$$

The time evolution is dictated by the Heisenberg equations of motion, and for any operator O constructed from the fields one has

$$\dot{O} = -i\Big[O, H\Big]$$

The basic objects that one considers at the quantum level are correlation functions of the type:

$$\left\langle \phi(x_1)\cdots\phi(x_n)\right\rangle = \left\langle 0\middle| T\phi(x_1)\cdots\phi(x_n)\middle|0\right\rangle$$

The generating functional of all such correlation functions is constructed by introducing a external source current J for each field ϕ:

$$Z[J] = \left\langle 0\middle| T\exp\left\{i\int d^4x\, J(x)\phi(x)\right\}\middle|0\right\rangle$$

Correlation functions satisfy a slightly different type of equations of motion compared to the field operators.

Indeed, when the term involving $\partial_0\pi$ in the equation of motion hits the time step-functions defining the T-product, one gets contact terms:

$$\begin{aligned}\partial_0 T\pi(x)\phi(y) &= \Big[\pi(x),\phi(y)\Big\}\delta(t_x-t_y)\\ &= -i\,\delta(x-y)\end{aligned}$$

One finds then that the equations of motion from the fields translate into the following relations between correlation functions:

$$\left\langle \frac{\delta S}{\delta \phi}(x)\phi(x_1)\cdots\phi(x_n)\right\rangle$$

$$= i\sum_k \left\langle \phi(x_1)\cdots\phi(x_{k-1})\phi(x_{k+1})\cdots\phi(x_n)\right\rangle \delta(x-x_k)$$

The S matrix elements can be obtained through reduction as amputated correlation functions. They depend only on the most singular part of the correlation functions, with one pole for each external particle.

S-matrix elements $=$ amputated correlation functions

These satisfy then the classical equations of motion, because the contact terms always miss at least one of the poles and are not enough singular to contribute.

Path integral formulation

The correlation functions can be computed through a functional integral over all possible paths, weighted by a phase given by the action:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\int \mathcal{D}\phi \, \phi(x_1) \cdots \phi(x_n) e^{i\,S}}{\int \mathcal{D}\phi \, e^{i\,S}}$$

The generating functional for correlation functions is then given by

$$Z[J] = \int \mathcal{D}\phi \, e^{i\,S + i \int J\phi}$$

In this formulation, the equations of motion satisfied by correlation functions can be derived more directly, in a way that is similar to the classical case.

One uses a variational approach and considers an arbitrary infinitesimal field variation $\delta\phi$ vanishing at infinity.

One looks then at the path-integral with some field insertions and view this transformation as a shift in the dummy integration variables. This should leave the result unchanged, implying:

$$\delta\Big(\int \mathcal{D}\phi \, \phi(x_1) \cdots \phi(x_n) e^{i\,S} \Big) = 0$$

Since the Jacobian of the field transformation is 1, so that $\delta \mathcal{D}\phi = 0$, this equation implies:

$$\int \mathcal{D}\phi \, \Big(i \, \delta\phi(x) \frac{\delta S}{\delta \phi}(x) \, \phi(x_1) \cdots \phi(x_n) + \sum_k \phi(x_1) \cdots \delta\phi(x_k) \cdots \phi(x_n) \Big) e^{iS} = 0$$

Requiring that this should hold for any $\delta\phi$ and dividing by the path-integral without field insertions, one finally recovers:

$$\Big\langle \frac{\delta S}{\delta \phi}(x) \phi(x_1) \cdots \phi(x_n) \Big\rangle = i \sum_k \big\langle \phi(x_1) \cdots \phi(x_{k-1}) \phi(x_{k+1}) \cdots \phi(x_n) \big\rangle \delta(x - x_k)$$

Conservation laws

Naively, it is expected that conserved quantities of the classical theory should lead to corresponding conserved quantities in the quantum theory.

More precisely, any classical conservation law should turn into an equation for the corresponding current operator:

$$\partial_\mu J^\mu_\alpha = 0$$

The symmetry transformations associated to the conserved charge Q_α are realized through operatorial transformations induced by $U(\alpha) = e^{i\alpha Q_\alpha}$. For an infinitesimal transformation, one finds

$$\frac{\delta O}{\delta \alpha} = i \left[O, Q_\alpha \right]$$

Moreover, conserved charges must commute with H if they do not depend explicitly on time:

$$\left[Q_\alpha, H \right] = 0$$

Ward identities

As for the equation of motions, which imply some identities for correlation functions involving the fields, the conservation equation $\partial_\mu J^\mu_\alpha = 0$ implies some identities for correlation functions involving the current.

Again, the difference with respect to the operatorial equations consists in some contact terms, which are relevant off-shell but not on-shell.

In the operatorial formulation, the contact terms arise from the T-product, as before. Using the form of the current and the canonical commutation relations, one finds:

$$\left\langle \partial_\mu J^\mu_\alpha(x)\phi(x_1)\cdots\phi(x_n)\right\rangle$$

$$= i\sum\nolimits_k \left\langle \phi(x_1)\cdots\frac{\delta\phi}{\delta\alpha}(x_k)\cdots\phi(x_n)\right\rangle \delta(x-x_k)$$

These identities imply again relations for S matrix elements, but the extra terms do not contribute, because they are not enough singular, and these matrix elements satisfy therefore the classical conservation law.

In the path-integral formulation, these Ward identities can be derived by proceeding as for the equations of motion, but considering an infinitesimal symmetry transformation of the type:

$$\delta_\alpha \phi = \delta\alpha \, \frac{\delta\phi}{\delta\alpha}$$

If this corresponds to a classical global symmetry, $\delta_\alpha S = 0$ for constant $\delta\alpha$. However, for non-constant $\delta\alpha$ vanishing at infinity, one finds:

$$\delta_\alpha S = \int d^4x \, \partial_\mu \delta\alpha \, J^\mu_\alpha = -\int d^4x \, \delta\alpha \, \partial_\mu J^\mu_\alpha$$

Consider then the path integral with some field insertions, and view this transformation as a change of the dummy integration variables. This should leave the result unchanged:

$$\delta_\alpha \Big(\int \mathcal{D}\phi \, \phi(x_1) \cdots \phi(x_n) e^{i\,S} \Big) = 0$$

Assuming that the Jacobian of the transformation is 1, as it turns out to be for most of the relevant symmetries, so that $\delta_\alpha \mathcal{D}\phi = 0$, this relation implies:

$$\int \mathcal{D}\phi \left(-i \int d^4x \, \delta\alpha(x) \partial_\mu J^\mu_\alpha(x) \, \phi(x_1) \cdots \phi(x_n) \right.$$
$$\left. + \sum_k \phi(x_1) \cdots \delta\alpha(x_k) \frac{\delta\phi}{\delta\alpha}(x_k) \cdots \phi(x_n) \right) e^{iS} = 0$$

Requiring this to hold for any $\delta\alpha$ and dividing by the path-integral without insertions, one recovers finally the same Ward identity as in the operator formalism:

$$\left\langle \partial_\mu J^\mu_\alpha(x) \phi(x_1) \cdots \phi(x_n) \right\rangle$$
$$= i \sum_k \left\langle \phi(x_1) \cdots \frac{\delta\phi}{\delta\alpha}(x_k) \cdots \phi(x_n) \right\rangle \delta(x - x_k)$$

Example: QED

Consider the special correlation function $\langle J^\mu(x)\psi(y_1)\bar{\psi}(y_2)\rangle$ in spinor electrodynamics. The ward identity gives in this case:

$$\begin{aligned}\big\langle \partial_\mu J^\mu(x)\psi(y_1)\bar{\psi}(y_2)\big\rangle = &-\big\langle\psi(y_1)\bar{\psi}(y_2)\big\rangle\delta(x-y_1)\\ &+\big\langle\psi(y_1)\bar{\psi}(y_2)\big\rangle\delta(x-y_2)\end{aligned}$$

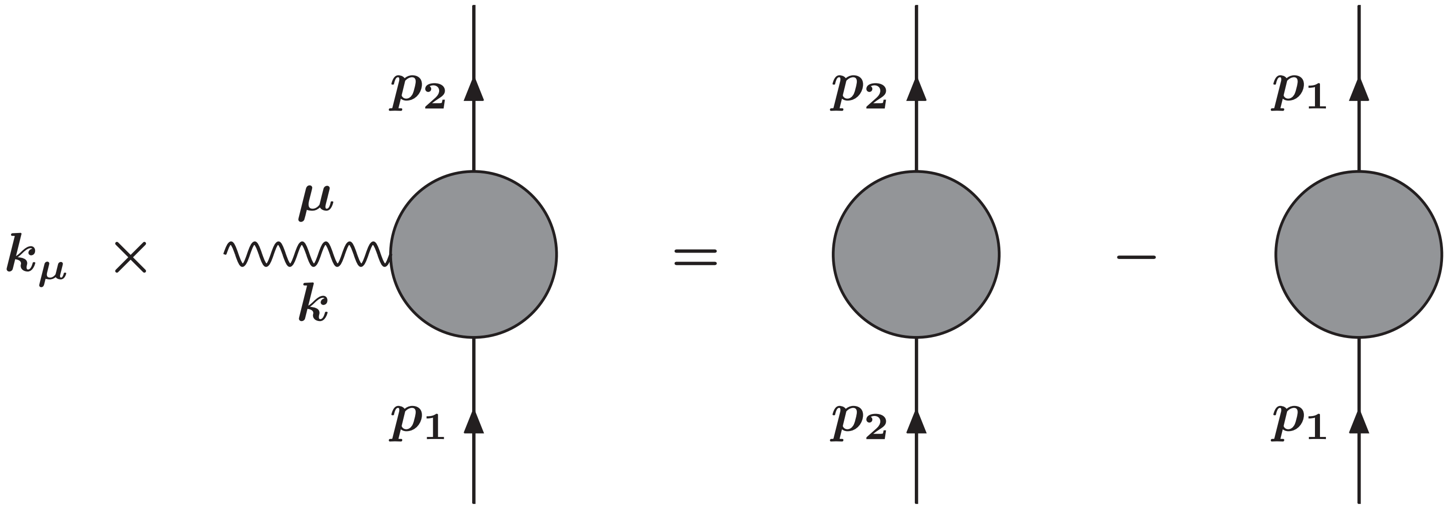
The left hand side corresponds to the coupling between one unphysical gauge boson and two fermions, whereas the right hand side contains the fermion propagator.

Taking the Fourier transform of this relation, with momenta k, p_1, p_2 which satisfy $k = p_2 - p_1$ by translational invariance, one finds:

$$k_\mu M^\mu(p_1, p_2) = S(p_2) - S(p_1)$$

This represents a relation between the cubic vertex between a longitudinal photon and two fermions, and the propagator of the fermions.

Diagrammatically, this means the following relation between the connected 3-point and 2-point correlation functions:



In order to get the corresponding relation between S matrix elements, one needs to amputate the two external fermion lines, by writing

$$M^\mu(p_1, p_2) = S(p_1)\big(-i\,\Gamma^\mu(p_1, p_2)\big)S(p_2)$$

The ward identity becomes then

$$- i \, k_\mu \Gamma^\mu(p_1, p_2) = S^{-1}(p_1) - S^{-1}(p_2)$$

Finally, one can use the 1PI decomposition of the propagator

$$S(p) = \frac{i}{\not{p} - m - \Sigma(\not{p})}$$

and decompose the vertex as

$$\Gamma^\mu(p_1, p_2) = \gamma^\mu + \Lambda^\mu(p_1, p_2)$$

One arrives then at the following relation:

$$(p_2 - p_1)_\mu \Lambda^\mu(p_1, p_2) = \Sigma(p_2) - \Sigma(p_1)$$

This implies that unphysical longitudinal photons are decoupled on-shell, and have a simple and rigidly determined effect off-shell. This property is crucial for the consistency of the quantum theory.

REGULARIZATION AND ANOMALIES

Regularization and anomalies

In quantum field theory, there are UV divergences. One needs therefore to regularize the theory with some finite cut-off, renormalize it, and finally remove the cut-off.

Due to this complication, the formal derivation of the Ward identities can happen to be invalidated, with the appearence of so-called anomalies. The classical symmetry is then broken by quantum effects.

Quantum anomalies in a classical symmetry can appear only if there does not exist any UV regularization of the theory which manifestly preserves that symmetry.

However, anomalies are actually finite IR effects. They do not dependent on the regularization method, but only on which symmetries this respects. They represent thus genuine physical effects.

The precise way anomalies show up varies with the formalism, but always rests on some subtlety related to regularization.

Operatorial formalism

In the operatorial formalism, the subtlety is that the current J_α^μ associated to a classical symmetry is a composite field, involving products of fields at the same point, which gives a singular behavior.

Path integral formalism

In the path integral formalism, the subtlety is that the measure $\mathcal{D}\phi$ is a formal infinite-dimensional product which needs to be properly defined, and which can give rise to unexpected jacobians under transformations.

Diagrammatic expansion

In a perturbative diagrammatic expansion, the subtlety is that certain loop diagrams are linearly divergent, and shifting the momentum integration variables is not trivially allowed.

Physical effects of anomalies

The effect of an anomaly in a symmetry existing in the classical limit is that the symmetry disappears at the quantum level. More precisely, it is violated by specific and computable effects.

For global symmetries, which correspond to true restrictions on the theory, this is perfectly consistent. It may happen and means that the classical selection rules are violated at the quantum level in a specific way.

The predicted violation of selection rules associated to these effects has been verified experimentally in several situations.

For local symmetries, which correspond to fake redundancies of the theory, this is inconsistent, because unphysical states do not decouple and ruin unitarity. It must therefore be excluded.

The consistency requirements implied by the absence of such catastrophic effects are satisfied in a non-trivial way by relevant physical models.

ANOMALIES IN GLOBAL CHIRAL SYMMETRIES

Chiral symmetry for fermions

Consider the theory of a massless Dirac fermion interacting with an Abelian gauge field in the standard minimal way through $D_\mu = \partial_\mu - igA_\mu$:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

This has a $U(1)$ local gauge symmetry associated to group elements of the form $e^{i\alpha}$ and acting as:

$$\delta\psi = i\delta\alpha\psi \, , \quad \delta\bar{\psi} = -i\delta\alpha\bar{\psi}$$

$$\delta A_\mu = g^{-1}\partial_\mu \delta\alpha$$

The corresponding conserved current is:

$$J^\mu = \bar{\psi}\gamma^\mu\psi$$

It also has a $U(1)_5$ global chiral symmetry associated to group elements of the form $e^{i\alpha_5\gamma_5}$ and acting as:

$$\delta_5\psi = i\delta\alpha_5\gamma_5\psi\ , \quad \delta_5\bar{\psi} = i\delta\alpha_5\bar{\psi}\gamma_5$$
$$\delta_5 A_\mu = 0$$

The corresponding conserved current is:

$$J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$$

At the classical level, both symmetries are present and:

$$\partial_\mu J^\mu = 0\,, \quad \partial_\mu J_5^\mu = 0$$

At the quantum level, however, it is impossible to regularize the theory while preserving both of these symmetries, and one of the two is broken. Using a regularization preserving Q but not Q_5, one finds for instance:

$$\langle\partial_\mu J^\mu\rangle = 0\,, \quad \langle\partial_\mu J_5^\mu\rangle \neq 0$$

Operatorial computation

In the operatorial formalism, we need to regularize the current operator in a gauge invariant way. This can be done by the so-called point-splitting method, defining:

$$J_5^\mu(x,\epsilon) = \bar\psi(x+\frac{\epsilon}{2})\gamma^\mu\gamma_5\psi(x-\frac{\epsilon}{2})\exp\left\{ig\int_{x-\epsilon/2}^{x+\epsilon/2} dy^\mu A_\mu(y)\right\}$$

The Wilson line factor is needed in order for the regularized current to be invariant under local gauge transformations.

Using the equations of motion $\gamma^\mu\partial_\mu\psi = ig\gamma^\mu\psi A_\mu$, one computes then the divergence of the current:

$$\partial_\mu J_5^\mu(x,\epsilon) = -igJ_5^\mu(x,\epsilon)\left[A_\mu(y)\Big|_{x-\epsilon/2}^{x+\epsilon/2} - \partial_\mu\int_{x-\epsilon/2}^{x+\epsilon/2} dy^\mu A_\mu(y)\right]$$

At leading order in ϵ this yields:

$$\partial_\mu J_5^\mu(x,\epsilon) = igJ_5^\mu(x,\epsilon)F_{\mu\nu}(x)\epsilon^\nu$$

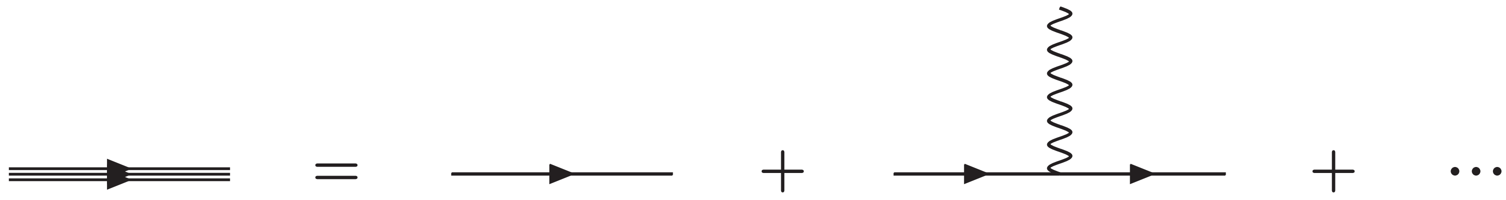
Taking the vacuum expectation value of this object, and treating $A_\mu(x)$ as an external field, one finds:

$$\big\langle \partial_\mu J_5^\mu(x, \epsilon)\big\rangle = ig\big\langle J_5^\mu(x, \epsilon)\big\rangle F_{\mu\nu}(x)\epsilon^\nu$$

The quantity appearing on the right hand side is essentially the fermion propagator in the background field $A_\mu(x)$. For small ϵ, one finds indeed:

$$\big\langle J_5^\mu(x, \epsilon)\big\rangle = i\,\mathrm{tr}\Big[\gamma_5\gamma^\mu S(A(x), \epsilon)\Big]$$

This has a singularity, which can be computed by expanding the fermion propagator in power of the external gauge field $A_\mu(x)$. Diagrammatically, this corresponds to the following series:



The term without insertion is singular, but does not contribute to the trace.
The term with just one insertion is still singular and contributes to the trace.
The terms with more insertions are regular and negligible.

Computing the relevant diagram with one insertion, one finds:

$$\begin{aligned}\left\langle J_5^\mu(x,\epsilon)\right\rangle &= ig\int\frac{d^4k}{(2\pi)^4}\,e^{-ikx}\int\frac{d^4p}{(2\pi)^4}\,e^{ip\epsilon}\,\mathrm{tr}\left[\gamma_5\gamma^\mu\frac{1}{\not p}\not A(k)\frac{1}{\not p-\not k}\right]\\ &= -4g\int\frac{d^4k}{(2\pi)^4}\,e^{-ikx}\int\frac{d^4p}{(2\pi)^4}\,e^{ip\epsilon}\,\frac{\epsilon^{\mu\alpha\beta\gamma}p_\alpha A_\beta(k)(p-k)_\gamma}{p^2(p-k)^2}\\ &= -4g\int\frac{d^4k}{(2\pi)^4}\,e^{-ikx}\epsilon^{\mu\alpha\beta\gamma}k_\beta A_\gamma(k)\int\frac{d^4p}{(2\pi)^4}\,e^{ip\epsilon}\,\frac{p_\alpha}{p^2(p-k)^2}\end{aligned}$$

In the limit of small ϵ, the integral in p becomes linearly divergent, and one easily computes, by analytic continuation:

$$\int\frac{d^4p}{(2\pi)^4}e^{ip\epsilon}\,\frac{p_\alpha}{p^4} = -\frac{i}{8\pi^2}\frac{\epsilon_\alpha}{\epsilon^2}$$

This leaves:

$$\langle J_5^\mu(x,\epsilon)\rangle = \frac{ig}{4\pi^2}\frac{\epsilon_\alpha}{\epsilon^2}\epsilon^{\mu\alpha\beta\gamma}F_{\beta\gamma}(x)$$

For the divergence of the current, this implies:

$$\langle \partial_\mu J_5^\mu(x,\epsilon)\rangle = -\frac{g^2}{4\pi^2}\frac{\epsilon_\alpha\epsilon^\nu}{\epsilon^2}\epsilon^{\mu\alpha\beta\gamma}F_{\beta\gamma}(x)F_{\mu\nu}(x)$$

Finally, one can take the limit $\epsilon \longrightarrow 0$ in a symmetric way, with

$$\lim_{\epsilon\to 0}\frac{\epsilon^\alpha\epsilon^\beta}{\epsilon^2} = \frac{1}{4}\eta^{\alpha\beta}$$

This yields a non-trivial finite result:

$$\langle \partial_\mu J_5^\mu(x)\rangle = -\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x)$$

On the other hand, proceeding in the same way one finds

$$\langle \partial_\mu J^\mu(x)\rangle = 0$$

Path integral computation

In the path-integral formalism, we need to suitably define and regularize the integration measure. This can be done by expanding the fermion fields in a basis of eigenmodes of the kinetic operator $i\not{D}$:

$$i\not{D}\psi_n(x) = \lambda_n \psi_n(x)$$

It is useful to temporarily make an analytic continuation to Euclidean space. Then $i\not{D}$ is Hermitian, and the basis is orthonormal and complete:

$$\int d^4x\, \psi_m^\dagger(x)\psi_n(x) = \delta_{mn} \; , \quad \sum\nolimits_n \psi_n(x)\psi_n^\dagger(y) = \delta(x-y)$$

The fields ψ and $\bar{\psi}$, which must be treated as independent, can now be expanded as follows:

$$\psi(x) = \sum\nolimits_n a_n \psi_n(x) \; , \quad \bar{\psi}(x) = \sum\nolimits_n \hat{a}_n \psi_n(x)$$

The path integral measure is then defined as:

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \prod\nolimits_n da_n d\hat{a}_n$$

The local version of the chiral transformation, namely $\delta_5\psi = i\delta\alpha_5\gamma_5\psi$ and $\delta_5\bar{\psi} = i\delta\alpha_5\bar{\psi}\gamma_5$ with non-constant $\delta\alpha_5$, acts on the modes as:

$$\delta a_m = \sum_n \Big(\delta_{mn} + \delta C_{mn}\Big) a_n \,, \qquad \delta\hat{a}_m = \sum_n \Big(\delta_{mn} + \delta C_{mn}\Big)\hat{a}_n$$

where:

$$\delta C_{mn} = i\int d^4x\, \delta\alpha_5(x)\psi_m^\dagger(x)\gamma_5\psi_n(x)$$

The Jacobian associated to each of these transformations has the form:

$$J = \det(1 + \delta C) = \exp\Big\{\mathrm{tr}\log(1 + \delta C)\Big\} = \exp\Big\{\mathrm{tr}\,\delta C\Big\}$$

This gives a result of the form:

$$J = \exp\Big\{ i\int d^4x\, \delta\alpha_5(x)\mathcal{A}(x)\Big\}$$

where:

$$\mathcal{A}(x) = \sum_n \psi_n^\dagger(x)\gamma_5\psi_n(x)$$

If $\mathcal{A}(x) \neq 0$, then $J \neq 1$ and an additional term appears in the derivation of the Ward identity. Indeed, viewing the transformation as a change of variables, one should have:

$$\delta_5 \left(\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{iS} \right) = 0$$

The action transforms as before, but the measure gets rescaled by J^{-2}:

$$e^{i\,S} \longrightarrow e^{i\,S} \exp\left\{ i \int d^4x\, \delta\alpha_5(x)\, \partial_\mu J^\mu(x) \right\}$$

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} \longrightarrow \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \exp\left\{ -2i \int d^4x\, \delta\alpha_5(x) \mathcal{A}(x) \right\}$$

We conclude then that the Ward identity becomes in this case:

$$\left\langle \partial_\mu J_5^\mu(x) \right\rangle = 2\, \mathcal{A}(x)$$

The anomalous exponent $\mathcal{A}$ is ambiguous and needs to be regularized, since formally it takes the form:

$$\mathcal{A}(x) = \operatorname{tr}\left[\gamma_5\right] \delta(0) = 0 \cdot \infty$$

One can introduce for this a cut-off Λ and define:

$$\mathcal{A}(x,\Lambda) = \sum_n \psi_n^\dagger(x)\gamma_5 e^{-\lambda_n^2/\Lambda^2}\psi_n(x)$$

This corresponds to the following trace over the spectrum of states:

$$\mathcal{A}(x,\Lambda) = \mathrm{Tr}'\left[\gamma_5\, e^{-(i\not{D})^2/\Lambda^2}\right]$$

To evaluated this, we first use the identity:

$$\begin{aligned}(i\not{D})^2 &= -\frac{1}{4}\left\{\gamma^\mu,\gamma^\nu\right\}\left\{D_\mu,D_\nu\right\} - \frac{1}{4}\left[\gamma^\mu,\gamma^\nu\right]\left[D_\mu,D_\nu\right]\\ &= -D^2 + \frac{g}{2}\sigma^{\mu\nu}F_{\mu\nu}\end{aligned}$$

We next observe that what matters for large Λ is the asymptotic tail of large eigenvalues. We can then expand in powers of A_μ and keep only as many as needed to get a non-zero trace over spinor indices:

$$\mathcal{A}(x,\Lambda) = \frac{g^2}{8\,\Lambda^4}\mathrm{Tr}'\left[e^{-\Box/\Lambda^2}\right]\mathrm{tr}\left[\gamma_5\,\sigma^{\mu\nu}\sigma^{\alpha\beta}\right]F_{\mu\nu}(x)F_{\alpha\beta}(x)$$

The trace over spinor indices gives $\mathrm{tr}[\gamma_5\,\sigma^{\mu\nu}\sigma^{\alpha\beta}] = 4i\epsilon^{\mu\nu\alpha\beta}$, whereas the remaining trace over free states gives:

$$\mathrm{Tr}'\left[e^{-\Box/\Lambda^2}\right] = \int\frac{d^4p}{(2\pi)^4}\,e^{p^2/\Lambda^2} = \frac{i}{16\pi^2}\Lambda^4$$

We are finally left with a finite result when $\Lambda \to \infty$:

$$\mathcal{A}(x) = -\frac{g^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x)$$

This reproduces the same result as before for the anomalous Ward identity:

$$\left\langle\partial_\mu J_5^\mu(x)\right\rangle = -\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x)$$

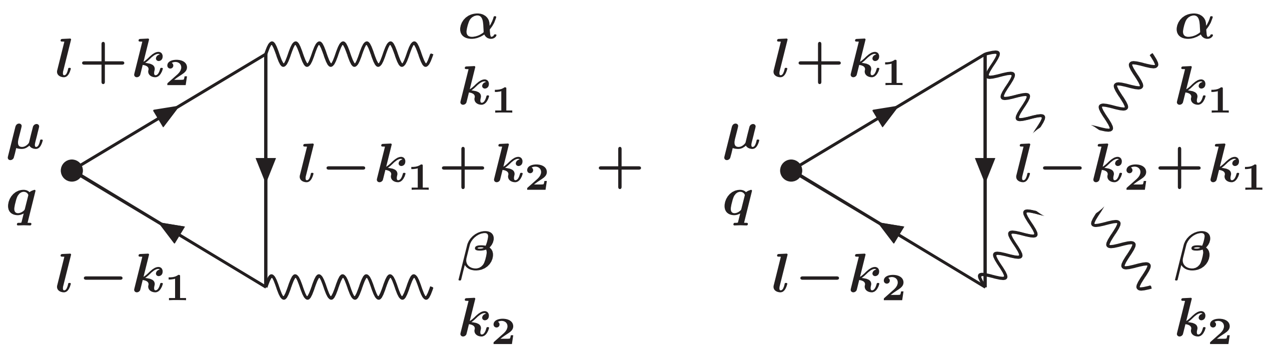
One the other hand, proceeding similarly one finds:

$$\left\langle\partial_\mu J^\mu(x)\right\rangle = 0$$

Diagrammatic approach

It is instructive to study how the anomaly emerges within a perturbative approach in terms of Feynmann diagrams. It turns out that it entirely comes from a linearly divergent one-loop triangle diagram.

Consider the matrix element of J_5^μ between the vacuum and a 2-photon state, at the one-loop level. This receives contributions from two similar diagrams:



Note that the second diagram is identical to the first one but with crossed external photons with $(\alpha, k_1) \leftrightarrow (\beta, k_2)$.

One finds:

$$T^{\mu\alpha\beta} = ig^2\int\frac{d^4l}{(2\pi)^4}\left\{\mathrm{tr}\left[\gamma^\mu\gamma_5\frac{1}{\not{l}-\not{k}_1}\gamma^\beta\frac{1}{\not{l}-\not{k}_1+\not{k}_2}\gamma^\alpha\frac{1}{\not{l}+\not{k}_2}\right]\right.$$
$$\left. + \,\mathrm{tr}\left[\gamma^\mu\gamma_5\frac{1}{\not{l}-\not{k}_2}\gamma^\alpha\frac{1}{\not{l}-\not{k}_2+\not{k}_1}\gamma^\beta\frac{1}{\not{l}+\not{k}_1}\right]\right\}$$

Taking the divergence of the current corresponds to contract this result with $q_\mu = (k_1+k_2)_\mu$. One can then decompose

$$q_\mu\gamma^\mu\gamma_5 = (\not{l}+\not{k}_2)\gamma_5 + \gamma_5(\not{l}-\not{k}_1) = (\not{l}+\not{k}_1)\gamma_5 + \gamma_5(\not{l}-\not{k}_2)$$

In each term of $\Delta^{\alpha\beta} = q_\mu T^{\mu\alpha\beta}$ there is then one propagator denominator that cancels, and finally one finds:

$$\Delta^{\alpha\beta} = ig^2\int\frac{d^4l}{(2\pi)^4}\left\{\mathrm{tr}\left[\gamma_5\frac{1}{\not{l}-\not{k}_1}\gamma^\beta\frac{1}{\not{l}-\not{k}_1+\not{k}_2}\gamma^\alpha - \gamma_5\frac{1}{\not{l}-\not{k}_1+\not{k}_2}\gamma^\alpha\frac{1}{\not{l}+\not{k}_2}\gamma^\beta\right]\right.$$
$$\left. + \,\mathrm{tr}\left[\gamma_5\frac{1}{\not{l}-\not{k}_2}\gamma^\alpha\frac{1}{\not{l}-\not{k}_2+\not{k}_1}\gamma^\beta - \gamma_5\frac{1}{\not{l}-\not{k}_2+\not{k}_1}\gamma^\beta\frac{1}{\not{l}+\not{k}_1}\gamma^\alpha\right]\right\}$$

If one could freely shift l, each diagram would be antisymmetric under $(\alpha, k_1) \leftrightarrow (\beta, k_2)$ and the result would cancel.

But the integral is linearly divergent and must be regularized. A finite shift in the integration variable leaves then a finite surface term:

$$\Delta(a) = \int \frac{d^4l}{(2\pi)^4} \Big[f(l+a) - f(l) \Big] = \int \frac{d^4l}{(2\pi)^4} \Big[a^\mu \partial_\mu f(l) + \cdots \Big]$$
$$= \frac{i}{8\pi^2} \lim_{l\to\infty} a^\mu l_\mu \, l^2 f(l)$$

Applying this to the expression for $\Delta^{\alpha\beta}$ with the appropriate shifts, and evaluating the spinorial traces, one finds two identical terms adding up:

$$\Delta^{\alpha\beta} = \frac{g^2}{2\pi^2} \, \epsilon^{\alpha\beta\rho\sigma} k_{1\rho} k_{2\tau}$$

This result implies finally the anomalous Ward identity:

$$\Big\langle \partial_\mu J_5^\mu(x) \Big\rangle = -\frac{g^2}{16\pi^2} \, \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}(x) F_{\alpha\beta}(x)$$

A similar computation for the gauge current yields instead:

$$\Big\langle \partial_\mu J^\mu(x) \Big\rangle = 0$$

Generalization to non-Abelian symmetries

The analysis of chiral anomalies can be extended to theories with several fermions and more general symmetries, forming a group G with generators satisfying $[T_a, T_b] = i f_{abc} T_c$.

Consider for instance a theory with massless Dirac fermions interacting with non-Abelian gauge fields with $D_\mu = \partial_\mu - i g A_{a\mu} T_a$:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu \psi - \frac{1}{4} F_{a\mu\nu} F_a^{\mu\nu}$$

This has a G local gauge symmetry associated to group elements of the form $e^{i\alpha_a T_a}$ and acting as:

$$\delta\psi = i\delta\alpha_a T_a \psi \ , \quad \delta\bar{\psi} = -i\delta\alpha_a T_a \bar{\psi}$$

$$\delta A_{a\mu} = g^{-1}\partial_\mu \delta\alpha_a + f_{abc} A_{b\mu} \delta\alpha_c$$

The corresponding covariantly conserved currents are given by:

$$J_a^\mu = \bar{\psi}\gamma^\mu T_a \psi$$

It also has a G_5 global chiral symmetry associated to group elements of the form $e^{i\alpha_{5a}T_a\gamma_5}$ and acting as:

$$\delta_5\psi = i\delta\alpha_{5a}T_a\gamma_5\psi\;, \quad \delta_5\bar\psi = i\delta\alpha_{5a}\bar\psi\gamma_5 T_a$$

$$\delta_5 A_{a\mu} = f_{abc}A_{b\mu}\delta\alpha_{5c}$$

The corresponding covariantly conserved currents are given by:

$$J^\mu_{5a} = \bar\psi\gamma^\mu\gamma_5 T_a\psi$$

At the classical level, both symmetries are present and:

$$D_\mu J^\mu_a = 0\,, \quad D_\mu J^\mu_{5a} = 0$$

At the quantum level, however, one finds:

$$\big\langle D_\mu J^\mu_a\big\rangle = 0\,, \quad \big\langle D_\mu J^\mu_{5a}\big\rangle = -\frac{g^2}{16\pi^2}\,d_{abc}\,\epsilon^{\mu\nu\alpha\beta}F_{b\mu\nu}F_{c\alpha\beta}$$

in terms of the symmetric constants:

$$d_{abc} = \mathrm{symtr}\Big[T_aT_bT_c\Big]$$

Different regularizations

In the previous computations, we have used regularizations preserving manifestly the local gauge symmetry, and found an anomaly in the global chiral symmetry.

One can generalize these computations by using families of regularizations that depend on a continuous parameter ξ, and which preserve the gauge symmetry only for $\xi = 0$.

The deformation concerns respectively the phase of the Wilson line factor, the gauge field dependence of the operator used to regulate the trace and a shift in the loop momentum in the three methods that have been used.

It turns out that such regularizations preserve then the chiral symmetry for some other non-zero value of the parameter, which we can conventionally take to be $\xi = 1$.

One finds in this way that the two Ward identities become:

$$\langle D_\mu J_a^\mu \rangle = 2\,\xi\,\mathcal{A}_a\,, \quad \langle D_\mu J_{5a}^\mu \rangle = 2\,(1-\xi)\,\mathcal{A}_a$$

where

$$\mathcal{A}_a = -\frac{g^2}{32\pi^2}\,d_{abc}\,\epsilon^{\mu\nu\alpha\beta}F_{b\mu\nu}F_{c\alpha\beta}$$

This shows that it is possible to preserve either the gauge symmetry, for $\xi = 0$, or the chiral symmetry, for $\xi = 1$, but not both simultaneously.

Since the gauge symmetry is local and the chiral symmetry is global, we are forced by consistency to choose the option where gauge invariance is preserved and chiral symmetry is sacrificed.

ANOMALIES IN LOCAL GAUGE SYMMETRIES

Gauged chiral symmetry

Much as an ordinary gauge symmetry is local thanks to a vector gauge field A_μ, a global chiral symmetry can be made local by introducing an additional axial gauge field $A_{5\mu}$.

Consider then a theory with massless Dirac fermions interacting vectorially with gauge fields $A_{a\mu}$ and axially with gauge fields $A_{5a\mu}$, with couplings determined by $D_\mu = \partial_\mu - igA_{a\mu}T_a - ig\gamma_5 A_{5a\mu}T_a$:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - \frac{1}{4}F_{a\mu\nu}F_a^{\mu\nu} - \frac{1}{4}F_{5a\mu\nu}F_{5a}^{\mu\nu}$$

This theory has two independent groups G and G_5 of local symmetries, associated to the currents:

$$J_a^\mu = \bar{\psi}\gamma^\mu T_A\psi$$
$$J_{5a}^\mu = \bar{\psi}\gamma^\mu\gamma_5 T_a\psi$$

The occurrence of an anomaly implies now a breakdown of gauge-invariance for the quantum effective action of the gauge fields:

$$\Gamma[A, A_5] = -i \log \langle e^{iS[\psi, A, A_5]} \rangle_\psi$$

Indeed, under infinitesimal gauge transformation one gets:

$$\delta\Gamma[A, A_5] = \int d^4x \, \delta\alpha_a(x) \big\langle D_\mu J^\mu_a(x) \big\rangle$$

$$\delta_5\Gamma[A, A_5] = \int d^4x \, \delta\alpha_{5a}(x) \big\langle D_\mu J^\mu_{5a}(x) \big\rangle$$

In the presence of a non-trivial anomaly, these cannot be made both zero, and the theory becomes therefore unavoidably inconsistent.

A change in regularization allows to shift but not to eliminate the anomaly. It corresponds to adding to the action a local non-invariant counter-term:

$$\Delta\Gamma[\xi, A, A_5] = \text{local counter-term}$$

The anomalous correlations are those with an odd number of γ_5: $\langle J_5 J J \rangle$ and $\langle J_5 J_5 J_5 \rangle$. The standard choice it to preserve G and give up G_5.

One can reformulate this theory in a more symmetric way in terms of two chiral sectors, with:

$$\psi_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\psi \;, \qquad A^{\mu}_{aL,R} = A^{\mu}_{a} \pm A^{\mu}_{5a}$$

The symmetries get then reshuffled to G_L and G_R, with currents

$$J^{\mu}_{aL,R} = \frac{1}{2}\left(J^{\mu}_{a} \pm J^{\mu}_{5a}\right)$$

In this language, the anomalous correlations are those with only L or R fields: $\langle J_L J_L J_L \rangle$ and $\langle J_R J_R J_R \rangle$.

Notice now that $\psi_{L,R}$ is equivalent to $\psi^c_{R,L}$, and one can thus reinterpret this theory as a chiral fermion interacting with a gauge field with group $G_L \times G_R$ in a representation of the type $(R,1) \oplus (1,R^c)$.

In this formulation, it becomes clear that this theory can be generalize to less symmetric situations, with an arbitrary gauge group and chiral fields in arbitrary representations.

Gauge symmetry for chiral fermions

Consider first the theory of a massless Weyl fermion interacting with an Abelian gauge field in the standard way with $D_\mu = \partial_\mu - igA_\mu$:

$$\mathcal{L} = i\bar{\chi}\gamma^\mu D_\mu \chi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

This has a $U(1)$ local gauge symmetry associated to group elements of the form $e^{i\alpha}$ and acting as:

$$\delta\chi = i\delta\alpha\chi \; , \qquad \delta\bar{\chi} = -i\delta\alpha\bar{\chi}$$

$$\delta A_\mu = g^{-1}\partial_\mu \delta\alpha$$

The corresponding conserved current is:

$$J^\mu = \bar{\chi}\gamma^\mu\chi$$

There are no extra global symmetries in this case

At the classical level, the symmetry is present and:

$$\partial_\mu J^\mu = 0$$

At the quantum level, however, it is impossible to regularize the theory while preserving this symmetry, because of the chiral nature of the fermion field. One finds:

$$\langle \partial_\mu J^\mu \rangle \neq 0$$

Computation of the anomaly

The precise form of the gauge anomaly can be computed in the same way as for the chiral anomaly, in all the three different approaches that we have seen.

The fact that χ is a Weyl fermion of some definite chirality $\eta = \pm 1$ implies that one can rewrite it as a projection of a Dirac fermion ψ:

$$\chi = \frac{1}{2}(1 + \eta\gamma_5)\psi$$

One can then rewrite the current as:

$$J^\mu = \frac{1}{2}\left(\bar{\psi}\gamma^\mu\psi + \eta\,\bar{\psi}\gamma^\mu\gamma_5\psi\right)$$

From the form of this expression, we see that the computation is almost identical, except for a factor $\eta/2$, and the result is:

$$\langle\partial_\mu J^\mu\rangle = -\frac{\eta\, g^2}{32\pi^2}\,\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x)$$

Generalization to non-Abelian symmetries

Consider now similarly a more general theory with massless Weyl fermions interacting with a non-Abelian gauge field with $D_\mu = \partial_\mu - igA_{a\mu}T_a$:

$$\mathcal{L} = i\bar{\chi}\gamma^\mu D_\mu\chi - \frac{1}{4}F_{a\mu\nu}F_a^{\mu\nu}$$

This possesses a local gauge symmetry associated to group elements of the form $e^{i\alpha_a T_a}$ and acting as:

$$\delta\chi = i\delta\alpha_a T_a\chi\ , \qquad \delta\bar{\chi} = -i\delta\alpha_a T_a\bar{\chi}$$

$$\delta A_{a\mu} = g^{-1}\partial_\mu\delta\alpha_a + f_{abc}A_{b\mu}\delta\alpha_c$$

The corresponding covariantly conserved currents are given by:

$$J^{\mu}_{a} = \bar{\chi}\gamma^{\mu}T_{a}\chi$$

As before, the theory does not posses any additional and independent global symmetries.

At the classical level, the symmetry is present and:

$$D_{\mu}J^{\mu}_{a} = 0$$

At the quantum level, however, one finds:

$$\langle D_{\mu}J^{\mu}_{a}\rangle = -\frac{\eta\, g^{2}}{32\pi^{2}}\, d_{abc}\, \epsilon^{\mu\nu\alpha\beta} F_{b\mu\nu} F_{c\alpha\beta}$$

in terms of the symmetric constants:

$$d_{abc} = \mathrm{symtr}\left[T_{a}T_{b}T_{c}\right]$$

Conditions for anomaly cancellation

Gauge anomalies must be avoided, since they ruin the consistency of the theory. Chiral gauge theories can thus be consistent only if the various contributions to gauge anomalies cancel out.

A fermion of chirality η in a representation R is equivalent to a fermion of opposite chiralilty $-\eta$ in the conjugate representation R^c. These give the same contribution to the anomaly, because $d_{abc}(R) = -d_{abc}(R^c)$.

A chiral fermion in a real representation does not contribute to the anomaly, because in this case $d_{abc} = 0$. As a consequence, only massless fermions can contribute to the anomaly.

Anomaly cancellation implies then a non-trivial restriction on the allowed spectrum of massless chiral fermions:

$$\sum_R \eta_R \, d_{abc}(R) = 0$$

GENERAL STRUCTURE OF ANOMALIES

Gauge theories and differential forms

Using differential forms, and rescaling the coupling g, one can describe a Yang-mills theory with arbitrary group G in terms of the following forms:

$$A = A_{a\mu} T_a \, dx^\mu$$
$$F = \frac{1}{2} F_{a\mu\nu} T_a \, dx^\mu dx^\nu$$
$$J = J_{a\mu} T_a \, dx^\mu$$

The covariant derivative can be represented with the help of the exterior derivative d, which when applied to a p-form produces a $(p+1)$-form:

$$D = d + \Big[A, \cdots\Big] \, , \quad D^2 = \Big[F, \cdots\Big]$$

The relation between F and A implies that:

$$F = dA + A^2$$

Using the Hodge dual operation $*$, which converts any p-form into a dual $(4-p)$-form, the Bianchi identity, the equations of motion and the classical conservation law can be written as:

$$DF = 0\ , \quad D^*F = -^*J\ , \quad D^*J = 0$$

For the particular case of an Abelian theory with group $U(1)$, the above formulae simplify, because:

$$D = d\ , \quad D^2 = 0$$

The field strength is then just:

$$F = dA$$

and the Bianchi identity, equations of motion and classical conservation law become:

$$dF = 0\ , \quad d^*F = -^*J\ , \quad d^*J = 0$$

Mathematical interpretation of the chiral anomaly

Consider now the theory of a massless Dirac fermion with a generic G local gauge symmetry and the minimal $U(1)_5$ global chiral symmetry. This is the prototypical example where anomalies arise.

The anomaly is encoded in the following gauge-invariant 4-form constructed out of the 2-form F:

$$\mathcal{A} = -\frac{1}{8\pi^2}\,\mathrm{tr}\,F^2$$

The local version of the conservation law takes the form of a deformed continuity equation ruling the flow of charge and can be written as:

$$d^*J_5 = 2\,\mathcal{A}$$

The integrated version of this conservation law defines instead the total variation of charge between asymptotic past and future:

$$\Delta Q_5 = 2\int \mathcal{A}$$

It turns out that the anomaly is a closed form, by the Bianchi identity:

$$d\mathcal{A} = 0$$

It is also exact, in the sense that if F can be expressed in terms of A, then it can be rewritten as

$$\mathcal{A} = -\frac{1}{8\pi^2}\, dC$$

in terms of a Chern-Simons 3-form depending on the 1-form A:

$$C = A\, dA + \frac{2}{3} A^3$$

The local conservation law is violated by any non-zero F. Note that one can define a new current $\tilde{J}_5 = J_5 - 2{}^*C$ which is conserved: $d{}^*\tilde{J}_5 = 0$, but this is not gauge invariant.

The global conservation law is preserved for any F that can be written in terms of an A. But, it is violated by an integer for topologically non-trivial fiber bundles, where F cannot be described by a globally defined A.

More precisely, the Atyah-Singer index theorem states that the integral of $\mathcal{A}$ is the index of the Dirac operator, which counts the difference between the numbers n_L and n_R of its L-handed and R-handed zero-modes:

$$\begin{aligned} \int A &= \dim\ker\big(i\not{D} P_L\big) - \dim\ker\big(i\not{D} P_R\big) \\ &= n_L - n_R \end{aligned}$$

This difference can be non-zero only in a topologically non-trivial background F, which somehow distinguishes the two chiralities.

The eigenmodes of $i\not{D}$ with non-zero eigenvalues occur in pairs of opposite chirality and eigenvalues. The index can then be written also as:

$$\begin{aligned} \int A &= \int d^4x \sum_n \psi_n^\dagger(x)\gamma_5\psi_n(x) \\ &= \mathrm{Tr}\left[\gamma_5\right] \end{aligned}$$

From this writing, we see that the path-integral evaluation of the anomaly represents a physicist proof of the Atyah-Singer theorem.

IMPLICATIONS OF ANOMALIES IN PARTICLE PHYSICS

Global anomalies in the standard model

Consider the $SU(3) \times U(1)$ gauge theory of strong and electromagnetic interactions, in the limit where the $SU(2)$ week interactions are neglected. This is a vectorial theory, with no axial couplings distinguishing chiralities.

Neglecting the masses and the electromagnetic coupling of the first family of quarks $q = (u, d)$, the model has an approximate $SU(2)_L \times SU(2)_R$ symmetry, rotating independently $q_L = (u_L, d_L)$ and $q_R = (u_R, d_R)$.

In the axial-vector nomenclature, we have then the following symmetries and currents:

$$SU(2):\; J_a^\mu = \bar{q}\gamma^\mu \tau_a q \qquad \text{(isospin symmetry)}$$

$$SU(2)_5:\; J_a^\mu = \bar{q}\gamma^\mu \tau_a \gamma_5 q \qquad \text{(chiral symmetry)}$$

The isospin symmetry is observed to approximately hold true, whereas the chiral symmetry is not observed at all.

The interpretation is that the vacuum contains a quark condensate $\langle \bar{q} q \rangle$ and spontaneously breaks this symmetry. The Goldstone bosons are $\bar{q} q'$ bound states identified with the pion triplet $\pi = (\pi^+, \pi^-, \pi^0)$.

The chiral symmetry currents J^μ_{5a} have non-zero matrix elements between the vacuum and the pion states, which is parametrized as:

$$\langle 0 | J^\mu_{5a}(0) | \pi_b(p) \rangle = -i f_\pi p^\mu \delta_{ab}$$

This implies:

$$\langle 0 | \partial_\mu J^\mu_{5a}(0) | \pi_a(p) \rangle = f_\pi m^2_\pi \delta_{ab}$$

The approximate operatorial conservation law for the chiral symmetry must therefore be of the type:

$$\partial_\mu J^\mu_{5a}(x) = m^2_\pi f_\pi \pi_a(x) + 2\, \mathcal{A}_a(x)$$

The first term is proportional to m_π and encodes the small explicit breaking of the symmetry. The second is a possible anomaly, to be computed.

To compute the anomalous contribution, we just need to apply the general results, for the case of an $SU(2)_5$ global symmetry interfering with an $SU(3) \times U(1)$ local gauge symmetry. The result is:

$$\mathcal{A}_a(x) = -\sum_{\text{groups}} \frac{g^2}{32\pi^2} d_{aBC}\, \epsilon^{\mu\nu\alpha\beta} F_{B\mu\nu} F_{C\alpha\beta}$$

where in this case:

$$d_{aBC} = \text{tr}\left[\tau_a T_B T_C\right]$$

For the $SU(3)$ part, the matrices T_A act on a different space than the matrices τ_a, and the trace factorizes and vanishes: $d_{aAB} = 0$.

For the $U(1)$ part, one has $T = \sqrt{2}\,\text{diag}(2/3, -1/3)$ for each color, and this gives a non-zero trace with $\tau^3 = \text{diag}(1/2, -1/2)$: $d_{aQQ} = \delta_{a3}$.

Finally, one finds:

$$\mathcal{A}_a(x) = -\delta_{a3} \frac{\alpha_{\text{em}}}{8\pi} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

This means that the conservation law of J^{μ}_{53} corresponding to the π^0 is anomalous, whereas those of $J^{\mu}_{51,2}$ corresponding to the $\pi^{\pm}$ are regular.

A remarkable implication of this anomaly is that it is the dominant reason for the observed decay $\pi^0 \to \gamma\gamma$. To see this, we can consider the low-energy effective theory for the pions π_a and the photon γ.

According to the derived Ward identities, the effective action $\Gamma[\pi, A]$ must behave under infinitesimal $SU(2)$ and $SU(2)_5$ transformations as:

$$\delta\Gamma[\pi, A] = 0$$

$$\delta_5\Gamma[\pi, A] = -\int d^4x\, \delta\alpha_{5a}(x)\Big[m_\pi^2 f_\pi \pi_a(x) + 2\,\mathcal{A}_a(x)\Big]$$

At the linearized level, the π_a and A_μ fields transforms as follows:

$$\delta\pi_a = \epsilon_{abc}\delta\alpha_b\pi_c\,, \qquad \delta A_\mu = 0$$

$$\delta_5\pi_a = \delta\alpha_a f_\pi\,, \qquad \delta_5 A_\mu = 0$$

In order to reproduce the violations of the chiral symmetry by the mass and anomaly terms, $\Gamma[\pi, A]$ must contain the following two breaking terms:

$$\Gamma[\pi, A] \supset \int d^4x \Big[-\frac{1}{2} m_\pi^2 \, \vec{\pi} \cdot \vec{\pi} + \frac{\alpha_{\rm em}}{4\pi} \frac{\pi_3}{f_\pi} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \Big]$$

The second term gives a contribution to the $\pi^0 \longrightarrow \gamma\gamma$ decay rate, which is equal to:

$$\Gamma = \frac{\alpha_{\rm em}^2}{64\pi^3} \frac{m_\pi^3}{f_\pi^2}$$

This turns out to be in very good agreement with experiment, within a few percent of error. This was also one of the first pieces of evidence for the fact that there are 3 colors of quarks.

Gauge anomalies in the standard model

Consider now the full $SU(3) \times SU(2) \times U(1)$ theory of strong, week and electromagnetic interactions, with the 3 generations of all the known quarks and leptons.

This is a chiral gauge theory, and it therefore potentially suffers from gauge anomalies. However, it turns out that the contributions from the various matter fermions cancel, within each family.

The quantum numbers of a family of quarks and leptons are the following:

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ d_1 & d_2 & d_3 \end{pmatrix}_L \;:\; (3,2)_{1/6}\;, \qquad \begin{pmatrix} \nu_l \\ l \end{pmatrix}_L \;:\; (1,2)_{-1/2}$$

$$\begin{matrix} \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}^c_R \;:\; (\bar{3},1)_{-2/3} \\ \begin{pmatrix} d_1 & d_2 & d_3 \end{pmatrix}^c_R \;:\; (\bar{3},1)_{1/3} \end{matrix}\;, \qquad \begin{matrix} \nu^c_R \;:\; (1,1)_0 \\ l^c_R \;:\; (1,1)_1 \end{matrix}$$

The coefficients of the various anomalies are given by:

$$C_{abc} = \sum_R d_{abc}(R) = \sum_R \mathrm{symtr}\Big[T_a T_b T_c\Big]_R$$

A non-trivial anomaly can arise only when there are $0, 2$ or 3 non-Abelian generators, and any number of Abelian generators. This leaves 5 potential types of anomalies, which all cancel.

- $SU(3) - SU(3) - SU(3)$

  Only triplets contribute: $C \sim 2 - 2 = 0$.

- $SU(2) - SU(2) - SU(2)$

  Even doublets do not contribute, because they are real: $C = 0$.

- $U(1) - U(1) - U(1)$

  All fields contribute proportionally to the cube of they hypercharge: $C \sim 6{\cdot}(1/6)^3 + 3{\cdot}(-2/3)^3 + 3{\cdot}(1/3)^3 + 2{\cdot}(-1/2)^3 + (1)^3 = 0$.

- $SU(3) - SU(3) - U(1)$

  Only triplets contribute proportionally to their hypercharge, so that $C \sim 2{\cdot}(1/6) + (-2/3) + (1/3) = 0$.

- $SU(2) - SU(2) - U(1)$

  Only doublets contribute proportionally to their hypercharge, and $C \sim 3{\cdot}(1/6) + (-1/2) = 0$.

Thus all the dangerous gauge anomalies cancel in the standard model. The need for this cancellation led to the prediction of the existence of the t quark, before its discovery.

Another remarkable fact is that the gauge group $SU(3) \times SU(2) \times U(1)$ of the standard model can be unified in the larger simple group $SO(10)$, with all the $15+1$ fermions of a family becoming a single $\mathbf{16}$ representation.

Anomaly cancellation follows then simply from the fact that the $\mathbf{16}$ spinorial representation is real, and can thus not give anomalies.

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