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A note on the torsion dependence of D-brane RR couplings

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Abstract

The dependence on the torsion $H = db$ of the Wess-Zumino couplings of D-branes that are trivially embedded in space-time is studied. We show that even in this simple set-up some torsion components can be turned on, with a non-trivial effect on the RR couplings. In the special cases in which either the tangent or the normal bundle are trivial, the torsion dependence amounts to substitute the standard curvature with its generalization in the presence of torsion, in the usual couplings involving the roof genus $\hat{A}$.

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1. Introduction

By now, it is well known that both D(irichlet)-branes and O(rientifold)-planes must have Wess-Zumino (WZ) couplings to Ramond Ramond (RR) fields. These couplings are required for a consistent cancellation of anomalies, and their general form is well-known in quite arbitrary backgrounds [1]-[12] (see [13] for effects peculiar to multiple branes). However, their dependence on the Neveu-Schwarz Neveu-Schwarz (NSNS) b-field is not completely established, even for a single D-brane, and its relevance has yet to be understood. At leading order in derivatives (*i.e.* for constant b), the dependence of the WZ couplings on b is completely fixed by the observation of [14] that the gauge invariant field strength living on a D-brane is $\mathcal{F} = F - b/2\pi$, and has indeed been explicitly verified. On general grounds, one expects that the presence of torsion ($H = db$) will modify these couplings, much on the same way as the presence of curvature does. For instance, it was pointed out in [13] that the restoration of a T-duality symmetric WZ coupling seems indeed to require a non-trivial dependence on the torsion, but no investigation has yet been done in this direction.

In this paper, we shall try to deduce the torsion-dependence of WZ couplings by factorizing one-loop CP-odd amplitudes, along the lines of [10, 12, 15], although here we will consider only D-branes. More precisely, we will compute the inflow of anomaly arising from amplitudes with external curvature and torsion vertices. The one-loop amplitudes in question can be exponentiated, reducing the problem to the evaluation of a twisted partition function in the odd spin structure. In the case at hand, this will be the partition function of a supersymmetric σ-model in a curved and contorted background.

We consider D-branes that are trivially embedded in space-time. Requiring a trivial embedding, that is the usual Neumann (N) or Dirichlet (D) boundary conditions for the σ-model, puts severe constraints on the possible torsion terms that may appear. In particular the intrinsic torsion on both the tangent and normal bundle of D-branes must vanish, but some mixed components of H are still allowed. They give rise to an antisymmetric part in the so-called second fundamental form defining the brane embedding in space-time (see the appendix). These are the torsion terms that will be studied in this paper.

Unfortunately, we will not be able to derive a simple expression for the exact torsion dependence in the most general gravitational background, which is nevertheless implicitly encoded in certain computable one-loop determinants[1]. However, in the two particular cases of trivial tangent bundle and trivial normal bundle, we find that the torsion dependence simply amounts to the standard generalization of the

[1]Strictly speaking, our results are derived for IIB D-branes but, as in [12], they apply to IIA D-branes as well.

curvature two-form in presence of torsion, as suggested in [13][2]. Whether or not this extends to arbitrary backgrounds is not clear.

Due to the limits of our analysis, the results of this paper should be taken as a first effort to study and understand the torsion dependence of D-brane couplings. It should also be pointed out that within this approach, non-anomalous torsion-dependent couplings cannot be detected. A more complete analysis is therefore needed to better understand these couplings. Also a more direct analysis along the lines of [9, 11] would be very interesting. Finally, D-branes in presence of torsion can be efficiently analysed within a different approach in the special case of group manifolds (see for instance [21] and references therein). This might be another helpful direction of investigation to better understand D-brane couplings.

2. World-sheet theory

As mentioned in the introduction, we analyse WZ couplings of D-branes by factorizing anomalous one-loop amplitudes in the odd spin-structure. Their torsion dependence can be studied by considering diagrams that contain both gravitons and B-fields as external states. In complete analogy to previous cases [10, 12, 15], where the torsion was set to zero, these amplitudes can be exponentiated. In this way one extracts directly the polynomial of the inflow of anomaly that is given by the partition function (in the odd spin-structure) of the resulting σ-model. By factorization, the WZ couplings responsible for this inflow of anomaly can then be extracted without the need of implementing the WZ descent procedure, that gives the actual gravitational or Lorentz anomaly[3].

The σ-model in question is the supersymmetric σ-model in presence of a generic gravitational and torsion background. In superspace, the action is given by

$$S(\Phi) = \frac{1}{4}\int d^2x\, d^2\theta \left[g_{MN}(\Phi)\bar{D}\Phi^M D\Phi^N - b_{MN}(\Phi)(\bar{D}\Phi^M \gamma^3 D\Phi^N)\right] , \qquad (2.1)$$

where $\Phi^M(x,\theta) = \phi^M(x) + \bar{\theta}\,\psi^M(x) + 1/2\,\bar{\theta}\theta\, F^M(x)$ denote ten chiral superfields ($M = 0, ..., 9$), $D_\alpha = \partial/\partial\,\theta^\alpha - i(\not{\partial}\,\theta)_\alpha$ and $\bar{D}_\alpha$ is its complex conjugate. We take the following conventions for two-dimensional γ-matrices: in terms of Pauli matrices σ_i, $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_1$, $\gamma^3 = \sigma_3$. These satisfy the property $\gamma^3\gamma^\alpha = \epsilon^\alpha{}_\beta\gamma^\beta$, with $\epsilon^{01} = +1$. Moreover, it is natural to introduce the following connections

$$\Gamma^M{}_{PQ} = \frac{1}{2}\, g^{MN}\left(g_{NP,Q} + g_{NQ,P} - g_{PQ,N}\right) , \qquad (2.2)$$

$$H^M{}_{PQ} = \frac{1}{2}\, g^{MN}\left(b_{NP,Q} + b_{PQ,N} + b_{QN,P}\right) , \qquad (2.3)$$

[2]The simple replacement $R \to \mathcal{R}$ is known to occur for standard chiral fermions in four dimensions [16, 17, 18] (see however [19] for a recent controversy), but a similar result in D dimensions has been obtained only for completely antisymmetric torsion [20].

[3]See in particular section 2 of [15] for further details.

and define the corresponding curvatures as

$$R^M{}_{NPQ} = \Gamma^M{}_{NQ,P} - \Gamma^M{}_{NP,Q} + \Gamma^M{}_{RP}\,\Gamma^R{}_{NQ} - \Gamma^M{}_{RQ}\,\Gamma^R{}_{NP}\ , \tag{2.4}$$

$$G^M{}_{NPQ} = H^M{}_{NQ;P} - H^M{}_{NP;Q} + H^M{}_{RP}\,H^R{}_{NQ} - H^M{}_{RQ}\,H^R{}_{NP}\ . \tag{2.5}$$

Commas and semicolons denote the usual derivatives and covariant derivatives, in terms of the symmetric connection (2.2).

By expanding in $\theta, \bar\theta$ all the terms in (2.1), and eliminating the auxiliary fields F^M, one gets the following action in components [22]:

$$\begin{aligned} S = \frac{1}{2}\int d^2x \Big[& g_{MN}\partial_\alpha\phi^M\partial^\alpha\phi^N + \epsilon^{\alpha\beta}b_{MN}\partial_\alpha\phi^M\partial_\beta\phi^N + i g_{MN}\bar\psi^M\hat{\not{D}}\psi^N \\ & + \frac{i}{2}\partial_\alpha(b_{MN}\bar\psi^M\gamma^\alpha\gamma^3\psi^N) + \frac{1}{8}\mathcal{R}_{MNPQ}\bar\psi^M(1+\gamma^3)\psi^P\bar\psi^N(1+\gamma^3)\psi^Q\Big] \end{aligned} \tag{2.6}$$

in terms of the generalized covariant derivative

$$\hat{\not{D}}\psi^M = \gamma^\alpha[\partial_\alpha\psi^M + (\Gamma^M{}_{PQ} - \gamma^3 H^M{}_{PQ})\,\partial_\alpha\phi^P\psi^Q]\ , \tag{2.7}$$

and the generalized Riemann tensor

$$\mathcal{R}_{MNPQ} = R_{MNPQ} + G_{MNPQ}\ . \tag{2.8}$$

constructed from $\Gamma^M{}_{PQ} + H^M{}_{PQ}$ [23]. The action (2.6) is invariant under the following supersymmetry transformations:

$$\begin{aligned} \delta_\epsilon\phi^M &= \bar\epsilon\psi^M\ , \\ \delta_\epsilon\psi^M &= -i\not{\partial}\phi^M\epsilon + \frac{1}{2}(\Gamma^M{}_{AB}\bar\psi^A\psi^B + H^M{}_{AB}\bar\psi^A\gamma^3\psi^B)\epsilon\ . \end{aligned} \tag{2.9}$$

From a σ-model point of view, D-branes are represented as world-sheet boundaries with suitable Neumann (N) or Dirichlet (D) boundary conditions (b.c.). D-branes in flat spaces or trivially embedded in curved space satisfy the usual b.c.:

$$\begin{aligned} &\partial_\sigma\phi^\mu(0,\tau) = 0\ , && \partial_\tau\phi^i(0,\tau) = 0\ , \\ &\psi_1^\mu(0,\tau) = \psi_2^\mu(0,\tau)\ , && \psi_1^i(0,\tau) = -\psi_2^i(0,\tau)\ , \end{aligned} \tag{2.10}$$

where ψ_1 and ψ_2 are the two components of the Majorana spinor ψ. Here and throughout the paper we use greek indices $\mu, \nu, ... = 0, ..., p$ and latin indices $i, j, ... = p{+}1, ..., 9$ to denote respectively N and D directions of a Dp-brane. We now implement the b.c. (2.10) in the action (2.6) and require, as usual, that all the boundary terms in the variation of (2.6) vanish. It is a simple although laborious exercise to verify that all boundary terms vanish if the following constraints on the background are satisfied:

$$\begin{aligned} &g_{\mu i}\mid_M = \partial_i g_{\mu\nu}\mid_M = 0\ , \\ &b_{\mu\nu}\mid_M = b_{ij}\mid_M = \partial_i\, b_{\mu j}\mid_M = 0\ , \end{aligned} \tag{2.11}$$

where $|_M$ is to remind that these conditions must hold only on the boundary of the world-sheet, that is the D-brane world-volume M. In terms of the field strength H, (2.11) imply that $H_{\mu\nu\rho} = H_{\mu ij} = H_{ijk} = 0$. In other words, the only possible torsion components compatible with the usual b.c. (2.10) are those with two N and one D indices, $H_{\mu\nu i}$[4].

Generically, a world-sheet boundary also breaks both of the two world-sheet supersymmetries. This is indeed the case for the conditions (2.10), in a generic background. Interestingly, the same constraints (2.11) are required to leave a combination of the left and right supersymmetries unbroken. In fact, if (2.11) hold, the combination $\delta = \delta_1 - \delta_2$ of the two original supersymmetry variations $\delta_{1,2}$ is preserved.

Notice finally that (2.11) are not the only possible solution to the boundary conditions (2.10). One could also consider more complicated cases where different terms in (2.11) are non-vanishing and compensate each other to give a total vanishing boundary term in the variation of the action (2.6). We will not consider such cases.

3. Reduction to 0+1 dimension and quantization.

When the world-sheet theory is supersymmetric, the evaluation of the partition function in the odd spin-structure is greatly simplified. Indeed, it is a topological quantity, the Witten index [24], receiving contributions only from zero-energy states. These correspond to field configurations which are constant in the space-like direction of the world-sheet, and one can therefore use a 0+1 dimensional effective theory for the computation of the index.

It is convenient to introduce fermions with flat indices both on the tangent and the normal bundles. Defining then new fermions as $\psi^{\underline{\mu}} = e^{\underline{\mu}}_{\mu}\psi^{\mu}$ and $\psi^{\underline{i}} = e^{\underline{i}}_{i}\psi^{i}$, where $\psi^{\mu} = \psi^{\mu}_1/\sqrt{2} = \psi^{\mu}_2/\sqrt{2}$ and $\psi^{i} = \psi^{i}_1/\sqrt{2} = -\psi^{i}_2/\sqrt{2}$, the $1+0$ dimensional reduction of the action (2.6), with the restrictions (2.11), yields:

$$L = \frac{1}{2}g_{\mu\nu}\dot{\phi}^{\mu}\dot{\phi}^{\nu} + \frac{i}{2}\psi^{\underline{\mu}}\Big(\dot{\psi}_{\underline{\mu}} + \omega^{(0)}_{\rho\,\underline{\mu}\underline{\nu}}\dot{\phi}^{\rho}\psi^{\underline{\nu}}\Big) + \frac{i}{2}\psi^{\underline{i}}\Big(\dot{\psi}_{\underline{i}} + \omega^{(0)}_{\rho\,\underline{i}\underline{j}}\dot{\phi}^{\rho}\psi^{\underline{j}}\Big) + \frac{1}{4}R_{\underline{\mu}\underline{\nu}\underline{i}\underline{j}}\psi^{\underline{\mu}}\psi^{\underline{\nu}}\psi^{\underline{i}}\psi^{\underline{j}}$$
$$- iH_{\rho\underline{\mu}\underline{i}}\dot{\phi}^{\rho}\psi^{\underline{\mu}}\psi^{\underline{i}} - \frac{1}{6}H_{\underline{\mu}\underline{\nu}\underline{\rho};\underline{i}}\psi^{\underline{\mu}}\psi^{\underline{\nu}}\psi^{\underline{\rho}}\psi^{\underline{i}} + \frac{1}{8}H_{\underline{k}\underline{\mu}\underline{\nu}}H^{\underline{k}}{}_{\underline{\rho}\underline{\sigma}}\psi^{\underline{\mu}}\psi^{\underline{\nu}}\psi^{\underline{\rho}}\psi^{\underline{\sigma}}\ , \qquad (3.1)$$

where $\omega^{(0)}$ is the torsion-free connection one-form. The supersymmetry transformations leaving it invariant are ($\epsilon = \epsilon_1/\sqrt{2} = -\epsilon_2/\sqrt{2}$):

$$\delta_\epsilon\phi^{\mu} = i\, e^{\mu}_{\underline{\mu}}\,\psi^{\underline{\mu}}\,\epsilon\ ,$$
$$\delta_\epsilon\psi^{\underline{\mu}} = e^{\underline{\mu}}_{\mu}\,\dot{\phi}^{\mu}\,\epsilon + i\, e^{\nu}_{\underline{\nu}}\,\omega^{(0)\underline{\mu}}_{\nu\ \ \underline{\rho}}\psi^{\underline{\nu}}\psi^{\underline{\rho}}\,\epsilon\ ,$$
$$\delta_\epsilon\psi^{\underline{i}} = i\, e^{\mu}_{\underline{\mu}}\,\omega^{(0)\underline{i}}_{\mu\ \ \underline{j}}\,\psi^{\underline{\mu}}\psi^{\underline{j}}\,\epsilon + \frac{i}{2}H^{\underline{i}}_{\underline{\mu}\underline{\nu}}\psi^{\underline{\mu}}\psi^{\underline{\nu}}\epsilon\ , \qquad (3.2)$$

[4]There is also an alternative and probably faster way to see how the conditions (2.11) arise. By taking the usual vertex operator for B one easily verify, using the b.c. conditions (2.10), that on M only the $H_{\mu\nu i}$ components survive.

and the supercharge is

$$Q = e_{\mu\underline{\nu}}\dot{\phi}^{\mu}\psi^{\underline{\nu}} - \frac{i}{2}H_{\underline{\mu\nu i}}\psi^{\underline{\mu}}\psi^{\underline{\nu}}\psi^{\underline{i}} \ . \tag{3.3}$$

Before starting to evaluate the partition function associated to the action (3.1), it is very useful to quantize the theory to have a more precise understanding of which kind of anomalies we are studying. Indeed, it is well-known that the ill-defined traces encoding anomalies in Fujikawa's approach can be regulated and evaluated as the high temperature limit of the partition functions of suitable supersymmetric theories, corresponding to indices of certain operators. An investigation in this direction is also further motivated by the observation of [20] that the Atiyah-Singer index theorem in presence of torsion is associated to a supersymmetric quantum mechanical model that is *not* the reduction to 0+1 dimension of (2.6) with the constraints (2.10)[5]; the right model is rather the reduction to 0+1 dimension of an heterotic σ-model. This is in agreement with our result that for the purely Neumann case, no torsion is consistent with the conditions (2.10).

The conjugate momenta for ϕ^{μ}, $\psi^{\underline{\mu}}$ and $\psi^{\underline{i}}$ are given by

$$\pi_{\mu} = g_{\mu\nu}\dot{\phi}^{\nu} + \frac{i}{2}\omega^{(0)}_{\mu\,\underline{\alpha\beta}}\psi^{\underline{\alpha}}\psi^{\underline{\beta}} + \frac{i}{2}\omega^{(0)}_{\mu\,\underline{ij}}\psi^{\underline{i}}\psi^{\underline{j}} - iH_{\mu\underline{\nu i}}\psi^{\underline{\nu}}\psi^{\underline{i}} \ , \ \ \tau_{\underline{\mu}} = \frac{i}{2}\psi_{\underline{\mu}} \ , \ \ \tau_{\underline{i}} = \frac{i}{2}\psi_{\underline{i}} \ , \tag{3.4}$$

and the Hamiltonian is

$$H = \frac{1}{2}g_{\mu\nu}\dot{\phi}^{\mu}\dot{\phi}^{\nu} - \frac{1}{4}R_{\underline{\mu\nu ij}}\psi^{\underline{\mu}}\psi^{\underline{\nu}}\psi^{\underline{i}}\psi^{\underline{j}} + \frac{1}{6}H_{\underline{\mu\nu\rho};\underline{i}}\psi^{\underline{\mu}}\psi^{\underline{\nu}}\psi^{\underline{\rho}}\psi^{\underline{i}} - \frac{1}{8}H_{k\underline{\mu\nu}}H^{k}{}_{\underline{\rho\sigma}}\psi^{\underline{\mu}}\psi^{\underline{\nu}}\psi^{\underline{\rho}}\psi^{\underline{\sigma}} \ .$$

The Hamiltonian formulation and the quantization of supersymmetric quantum mechanical models like the one studied here presents some well-known subtleties. Indeed, due to the constraints in the fermionic sector of phase-space, it is necessary to use the Dirac procedure, and replace the standard Poisson brackets with Dirac brackets which are compatible with these constraints. Quantization can then proceed in the usual way, replacing Dirac brackets with commutators or anticommutators (see [26, 27, 28]). We will assume here that the net result of this lengthy procedure is that one can use as canonical variables ϕ^{μ}, π_{μ}, $\psi^{\underline{\mu}}$ and $\psi^{\underline{i}}$, with the following non-vanishing commutations relations:

$$[\phi^{\mu}, \pi_{\nu}] = i\delta^{\mu}_{\nu} \ , \ \ \{\psi^{\underline{\mu}}, \psi^{\underline{\nu}}\} = \eta^{\underline{\mu\nu}} \ , \ \ \{\psi^{\underline{i}}, \psi^{\underline{j}}\} = \delta^{\underline{ij}} \ . \tag{3.5}$$

In terms of these variables, the supercharge becomes:

$$Q = -ie^{\mu}_{\underline{\mu}}\psi^{\underline{\mu}}\Big(i\pi_{\mu} + \frac{1}{2}\omega^{(0)}_{\mu\,\underline{\rho\sigma}}\psi^{\underline{\rho}}\psi^{\underline{\sigma}} + \frac{1}{2}\omega^{(0)}_{\mu\,\underline{ij}}\psi^{\underline{i}}\psi^{\underline{j}} - \frac{1}{2}H_{\mu\underline{\rho i}}\psi^{\underline{\rho}}\psi^{\underline{i}}\Big) \ , \tag{3.6}$$

[5]When the torsion vanishes, these are the standard constraints to compute pure gravitational anomalies [25] (see [12] for the relevance of Dirichlet boundary conditions for non-trivial normal bundles) whose form is also given by the Atiyah-Singer index theorem.

and it is straightforward to check that Q does indeed generate the correct supersymmetry transformations (3.2): $\delta_\epsilon = [\epsilon Q, \quad]$. Also, the supersymmetry algebra guarantees that $H = 1/2\{Q, Q\}$.

According to (3.5), the canonical operators ϕ^μ, π_μ, $\psi^{\underline{\mu}}$ and $\psi^{\underline{i}}$ can be realized on the target space as

$$\phi^\mu \to x^\mu \ , \quad \pi_\mu \to -i\partial_\mu \ , \quad \psi^{\underline{\mu}} \to \gamma^{\underline{\mu}}/\sqrt{2} \ , \quad \psi^{\underline{i}} \to \gamma^{\underline{i}}/\sqrt{2} \ , \tag{3.7}$$

where $\gamma^{\underline{\mu}}, \gamma^{\underline{i}}$ are the space-time γ-matrices in the directions which are respectively parallel and transverse to the brane. The supercharge (3.6) is then finally given by $Q = -i\not{D}/\sqrt{2}$, where $\not{D}$ is the following world-volume Dirac operator:

$$\not{D} = e_{\underline{\mu}}^{\mu}\gamma^{\underline{\mu}}\Big(\partial_\mu + \frac{1}{4}\omega^{(0)}_{\mu\,\underline{\rho}\underline{\sigma}}\gamma^{\underline{\rho}\underline{\sigma}} + \frac{1}{4}\omega^{(0)}_{\mu\,\underline{i}\underline{j}}\gamma^{\underline{i}\underline{j}} - \frac{1}{4}H_{\mu\underline{\rho}\underline{i}}\gamma^{\underline{\rho}}\gamma^{\underline{i}}\Big) \ . \tag{3.8}$$

The operator (3.8) contains as expected a mixed torsion connection, beside the usual tangent and normal bundle spin connections. This shows that the index computed here encodes the anomaly of a chiral spinor in a curved and contorted background.

4. Inflow of anomaly and WZ couplings

We now turn to the computation of the inflow of anomaly. This section follows closely the analysis reported in [12], with some modifications due to the presence of torsion. According to the previous considerations, the inflow of anomaly is given by the high temperature limit of the partition function

$$Z = \mathrm{Tr}\,[\Gamma^{D+1}e^{-t(i\not{D})^2}] \ , \tag{4.1}$$

where Γ^{D+1} is the chiral matrix in D dimensions and $\not{D}$ is the Dirac operator (3.8).

The functional integral representation for (4.1) is

$$Z = \int_P \mathcal{D}\phi^\mu(\tau) \int_P \mathcal{D}\psi^{\underline{\mu}}(\tau) \int_P \mathcal{D}\psi^{\underline{i}}(\tau) \exp\left\{-\int_0^t d\tau\, L\left(\phi^\mu(\tau), \psi^{\underline{\mu}}(\tau), \psi^{\underline{i}}(\tau)\right)\right\} \ , \tag{4.2}$$

with L as in (3.1). All the fields are periodic (P) in the odd spin-structure. In order to evaluate this path-integral in the high-temperature limit $t \to 0$, it is convenient to expand the fields in normal coordinates around constant paths $\phi^\mu = \phi_0^\mu + \xi^\mu$, $\psi^{\underline{\mu}} = \psi_0^{\underline{\mu}} + \chi^{\underline{\mu}}$ and $\psi^{\underline{i}} = \psi_0^{\underline{i}} + \chi^{\underline{i}}$.

By doing the expansion described above, one should pay attention to dangerous terms involving four Neumann fermionic zero modes and leading to divergences. A term of this type, involving the standard Riemann tensor, has already been dropped because it vanishes thanks to the Bianchi identity, but the last term in (3.1) remains, essentially as a consequence of the fact that the torsion part of the curvature (2.5) does not satisfy the same Bianchi identity as the geometric part (2.4). Here, we shall assume a very safe approach and restrict to the particular case in which

$$H_{\underline{i}[\underline{\mu}\underline{\nu}}H^{\underline{i}}{}_{\underline{\rho}\underline{\sigma}]} = 0 \ . \tag{4.3}$$

In this case, for the same arguments explained in subsection 2.1.1 of [12], it is sufficient to keep only interaction terms up to quadratic order in the fluctuations and, among these, only those involving fermionic zero modes $\psi_0^{\underline{\mu}}$ in the Neumann directions. One gets then the following effective Lagrangian:

$$L^{eff} = \frac{1}{2}\Big[\dot{\xi}_{\underline{\mu}}\dot{\xi}^{\underline{\mu}} + i\chi_{\underline{\mu}}\dot{\chi}^{\underline{\mu}} + i\chi_{\underline{i}}\dot{\chi}^{\underline{i}} + iR_{\underline{\mu\nu}}\xi^{\underline{\mu}}\dot{\xi}^{\underline{\nu}} + R_{\underline{ij}}\chi^{\underline{i}}\chi^{\underline{j}} + 2iH_{\underline{\mu i}}\dot{\xi}^{\underline{\mu}}\chi^{\underline{i}} + R_{\underline{ij}}\psi_0^{\underline{i}}\psi_0^{\underline{j}}\Big] \qquad (4.4)$$

with[6]

$$R_{\underline{\mu\nu}} = \frac{1}{2}R_{\underline{\mu\nu\rho\sigma}}(\phi_0)\psi_0^{\underline{\rho}}\psi_0^{\underline{\sigma}} \ , \quad R_{\underline{ij}} = \frac{1}{2}R_{\underline{ij\rho\sigma}}(\phi_0)\psi_0^{\underline{\rho}}\psi_0^{\underline{\sigma}} \ , \quad H_{\underline{\mu i}} = H_{\underline{\mu i\rho}}(\phi_0)\psi_0^{\underline{\rho}} \ . \qquad (4.5)$$

Since the effective Lagrangian (4.4) is quadratic, it is in principle straightforward at this point to compute the partition function (4.2). As expected, the result is independent of t, as can be seen by rescaling the Neumann fermionic zero modes. As in the torsionless case, the last term in (4.4) produces the Euler class of the normal bundle $e(R')$. The remaining pieces of the Lagrangian give a bunch of determinants, which for the time being we denote by $Y^2(R, R', H)$. The result is therefore:

$$Z = \int_M Y^2(R, R', H)\, e(R') \ . \qquad (4.6)$$

As noted in [7], the Euler class term in (4.6) is due to a topological property of currents and it appears only in the inflow of anomaly, and must not be taken into account when factorizing (4.6) to extract the WZ couplings for D-branes[7]. Once Y is known, the D-brane WZ couplings we are looking for are given by

$$S_{D_p} = \frac{\mu_p}{2}\int_M \ C \wedge Y|_{(p+1)-form} \ , \qquad (4.7)$$

using the same conventions of [12] to normalize the D_p brane charge μ_p and denoting with C the formal sum of all the RR forms.

In spite of the fact that the effective Lagrangian (4.4) is quadratic, the evaluation of the corresponding determinants is difficult, due to the presence of a mixing between bosons and fermions. The general result for Y does not seem to lead to any simple combination of characteristic classes. This is quite disappointing but actually perfectly sensible, and probably just reflects the well-known difficulties in incorporating unambiguously the effects of torsion in characteristic classes. Rather than insisting on the exact result, we will limit ourselves to the two particular cases in which either the tangent or the normal bundle is trivial, which lead to simple results.

[6] In the following, we shall distinguish through a prime the normal bundle curvature from the tangent bundle curvature.

[7] There is a subtlety here for the D3-brane [7]. In this case the anomaly $\mathcal{A} = 2\pi i Z^{(1)} = 2\pi i \int_M e(R')^{(1)}$ does not vanish, whereas the inflow of anomaly does, since the descent procedure for the inflow has not to be taken to the Euler class term. We do not have a resolution to this issue, that seems to be due to additional subtleties in the definition of the WZ coupling of the self-dual D3-brane [7].

In the case of generic tangent bundle but trivial normal bundle, $R_{\underline{ij}} = 0$, it is convenient to redefine the fermionic fluctuation as $\chi_{\underline{i}} \to \chi_{\underline{i}} + iH_{\underline{i\mu}}\xi^{\underline{\mu}}$. By doing so, the torsion term in (4.4) gets reabsorbed and an effective bosonic interaction is generated:

$$L^{eff} = \frac{1}{2}\left[\dot{\xi}_{\underline{\mu}}\dot{\xi}^{\underline{\mu}} + i\chi_{\underline{\mu}}\dot{\chi}^{\underline{\mu}} + i\chi_{\underline{i}}\dot{\chi}^{\underline{i}} + i\left(R_{\underline{\mu\nu}} - H_{\underline{i\mu}}H^{\underline{i}}{}_{\underline{\nu}}\right)\xi^{\underline{\mu}}\dot{\xi}^{\underline{\nu}}\right] . \tag{4.8}$$

The evaluation of the determinants is then straightforward, and one finds:

$$Y = \sqrt{\widehat{A}(\mathcal{R})} , \tag{4.9}$$

in terms of the generalized curvature of the tangent bundle, eq.(A.6).

In the case of trivial tangent bundle, $R_{\underline{\mu\nu}} = 0$, but generic normal bundle, it is instead more convenient to redefine the bosonic fluctuations $\xi^{\underline{\mu}}$ in such a way that $\dot{\xi}_{\underline{\mu}} \to \dot{\xi}_{\underline{\mu}} + iH_{\underline{\mu i}}\chi^{\underline{i}}$. By doing this, the torsion term in (4.4) gets again reabsorbed and this time an effective fermionic interaction is generated:

$$L^{eff} = \frac{1}{2}\left[\dot{\xi}_{\underline{\mu}}\dot{\xi}^{\underline{\mu}} + i\chi_{\underline{\mu}}\dot{\chi}^{\underline{\mu}} + i\chi_{\underline{i}}\dot{\chi}^{\underline{i}} + \left(R_{\underline{ij}} - H_{\underline{\mu i}}H^{\underline{\mu}}{}_{\underline{j}}\right)\chi^{\underline{i}}\chi^{\underline{j}} + R_{\underline{ij}}\psi_0^{\underline{i}}\psi_0^{\underline{j}}\right] . \tag{4.10}$$

One finds then:

$$Y = \sqrt{\frac{1}{\widehat{A}(\mathcal{R}')}} , \tag{4.11}$$

in terms of the generalized curvature of the normal bundle, eq.(A.7).

5. Conclusions

In this note, we have shown that D-brane anomalous couplings do have a non-trivial dependence on torsion. The precise form of this dependence is implicitly encoded in certain one-loop determinants, but does apparently not admit any simple expression in terms of standard characteristic classes. Specialising to the two cases of trivial tangent and normal bundles, we were however able to prove that the torsion dependence amounts simply to the replacement $R \to \mathcal{R}$ and $R' \to \mathcal{R}'$ in the usual torsion-free RR couplings. One might then wonder whether, as guessed in [13], the result for generic curvature and torsion is miraculously:

$$Y = \sqrt{\frac{\widehat{A}(\mathcal{R})}{\widehat{A}(\mathcal{R}')}} . \tag{5.1}$$

Unfortunately, we are not able to answer unambiguously to this question since in this general case we did not find a sensible way to compute the determinants arising from the Lagrangian (4.4). Furthermore, we will not adress here the issue of whether the couplings we find restore T-duality, as proposed in [13], because the restricted region of validity of our results does not allow to make any precise statement in this direction.

We also find a bit problematic to relax the condition (4.3) on the torsion. We did not analyze in full detail the consequence of the presence of such a term in (3.1), but it is not excluded that the potential divergence we found is linked to similar divergent terms appearing in the literature for the computation of the four-dimensional chiral anomaly in presence of torsion[8].

An important observation is now in order. As shown in [29], torsion does not give rise to new true anomalies, as long as it appears just through generalized curvatures in the usual characteristic classes. Indeed, the addition of a covariant term to the spin-connection gives rise to a modification to the gravitational anomaly that may be reabsorbed by adding to the action a local counter-term. This suggests that there could be some ambiguity in deriving results in this context using anomaly arguments. However, the quantity we compute can be interpreted as the inflow of anomaly arising from the RR interaction of D-branes with couplings (4.7). As such, we believe that our results are not affected by this ambiguity. We stress this important point because, differently from the torsion-free case, the supersymmetric quantum mechanical models relevant to compute the inflow of anomaly on D-branes and generic anomalies with torsion are different, as already mentioned in section 3.

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A. Geometry of sub-manifolds with torsion

In this appendix we give a brief review of the geometry of sub-manifolds, along the lines of standard books [30]. A good reference is also [31]. A similar analysis for the torsion-free case, and applied again to D-brane physics, can also be found in appendix A of [32].

Let X be the ten-dimensional space-time endowed with a generic connection and let M be a $p+1$-dimensional sub-manifold of X, corresponding to the embedding of the D-brane world-volume into space-time. The embedding is defined by the equations $\phi^M = \phi^M(\sigma^\mu)$ $(\mu = 0, ..., p)$. Cartan's structure equations on X read then

$$T^{\underline{M}} = d\theta^{\underline{M}} + \omega^{\underline{M}}{}_{\underline{N}} \wedge \theta^{\underline{N}} \,, \tag{A.1}$$

$$\mathcal{R}^{\underline{M}}{}_{\underline{N}} = d\omega^{\underline{M}}{}_{\underline{N}} + \omega^{\underline{M}}{}_{\underline{P}} \wedge \omega^{\underline{P}}{}_{\underline{N}} \,, \tag{A.2}$$

[8]It might be that this divergent term is irrelevant upon integration over the world-volume.

where underlined indices represent flat indices and we denoted the two-form curvature with $\mathcal{R}$ to distinguish it from the geometric curvature R, constructed in terms of the torsion-free connection form $\omega^{(0)}$.

It is always possible to choose an orthonormal frame, the so-called "adapted frame" [30], in which $\theta^{\underline{i}} \,|_M = 0$ ($\underline{i} = p+1, ..., 9$). In such a frame, (A.1) with $\underline{\mathrm{M}} = \underline{i}$ yields $\omega^{\underline{i}}{}_{\underline{\nu}} \wedge \theta^{\underline{\nu}} = T^{\underline{i}}$. By writing $\omega^{\underline{i}}{}_{\underline{\nu}} = \Omega^{\underline{i}}{}_{\underline{\nu}\underline{\mu}}\theta^{\underline{\mu}}$ and using the explicit form in components for the torsion form, $T^{\underline{i}} = H^{\underline{i}}{}_{\underline{\mu}\underline{\nu}}\theta^{\underline{\mu}} \wedge \theta^{\underline{\nu}}$, this can be rewritten as

$$\left(\Omega^{\underline{i}}{}_{\underline{\nu}\underline{\mu}} + H^{\underline{i}}{}_{\underline{\mu}\underline{\nu}}\right)\theta^{\underline{\mu}} \wedge \theta^{\underline{\nu}} = 0 \,. \tag{A.3}$$

The most generic solution for (A.3) is $\Omega^{\underline{i}}{}_{\underline{\mu}\underline{\nu}} = \Omega^{\underline{i}}{}_{(\underline{\mu}\underline{\nu})} + H^{\underline{i}}{}_{\underline{\mu}\underline{\nu}}$. The tensor $\Omega^{\underline{i}}{}_{\underline{\mu}\underline{\nu}}$ is called the second fundamental form and plays an important role in relating tensors on a manifold with those defined in its sub-manifolds.

From (A.2) with $\underline{\mathrm{M}} = \underline{\mu}, \underline{\mathrm{N}} = \underline{\nu}$, one gets instead generalized Gauss equations,

$$(\mathcal{R}_T)_{\underline{\mu}\underline{\nu}} = \mathcal{R}_{\underline{\mu}\underline{\nu}} + \Omega_{\underline{i}\underline{\mu}\underline{\rho}}\Omega^{\underline{i}}{}_{\underline{\nu}\underline{\sigma}}\theta^{\underline{\rho}} \wedge \theta^{\underline{\sigma}} \,, \tag{A.4}$$

relating the intrinsic curvature $\mathcal{R}_T$ on M to the space-time curvature $\mathcal{R}$. Analogously, taking $\underline{\mathrm{M}} = \underline{i}, \underline{\mathrm{N}} = \underline{j}$ in (A.2), gives generalized Ricci equations,

$$(\mathcal{R}_N)_{\underline{i}\underline{j}} = \mathcal{R}_{\underline{i}\underline{j}} + \Omega_{\underline{\mu}\underline{\rho}\underline{i}}\Omega^{\underline{\mu}}{}_{\underline{\sigma}\underline{j}}\theta^{\underline{\rho}} \wedge \theta^{\underline{\sigma}} \,, \tag{A.5}$$

relating the intrinsic curvature $\mathcal{R}_N$ on the normal bundle of M to the space-time curvature $\mathcal{R}$.

The considerations discussed so far are general. We may now apply them to our particular situation, that is a trivial embedding ($\phi^i = \text{const.}$, $\phi^\mu = \sigma^\mu$) and backgrounds satisfying the constraints (2.11). In this case, it can be shown (for instance going in normal coordinates) that the symmetric part of the second fundamental form $\Omega^{\underline{i}}{}_{(\underline{\mu}\underline{\nu})}$ vanishes. Moreover, since $H_{\underline{\mu}\underline{\nu}\underline{\rho}} = H_{\underline{\mu}\underline{i}\underline{j}} = 0$, one finds that $(\mathcal{R}_T)_{\underline{\mu}\underline{\nu}} = (R_T)_{\underline{\mu}\underline{\nu}} = R_{\underline{\mu}\underline{\nu}}$ and $(\mathcal{R}_N)_{\underline{i}\underline{j}} = (R_N)_{\underline{i}\underline{j}} = R_{\underline{i}\underline{j}}$. Therefore, defining the one-forms $H_{\underline{\mu}\underline{i}} = H_{\underline{\mu}\underline{i}\underline{\rho}}\theta^{\underline{\rho}}$, equations (A.4) and (A.5) yield the following simple expressions for the generalized curvatures of the tangent and the normal bundles:

$$\mathcal{R}_{\underline{\mu}\underline{\nu}} = R_{\underline{\mu}\underline{\nu}} - H_{\underline{i}\underline{\mu}} \wedge H^{\underline{i}}{}_{\underline{\nu}} \,, \tag{A.6}$$

$$\mathcal{R}_{\underline{i}\underline{j}} = R_{\underline{i}\underline{j}} - H_{\underline{\mu}\underline{i}} \wedge H^{\underline{\mu}}{}_{\underline{j}} \,. \tag{A.7}$$

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