QUANTUM STATES OF TOPOLOGICALLY MASSIVE ELECTRODYNAMICS AND GRAVITY

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February 15, 2005

DFUPG-120/96

hep-th 9602127

Abstract

The free quantum states of topologically massive electrodynamics and gravity in 2+1 dimensions, are explicitly found. It is shown that in both theories the states are described by infrared-regular polarization tensors containing a regularization phase which depends on the spin. This is done by explicitly realizing the quantum algebra on a functional Hilbert space and by finding the Wightman function to define the scalar product on such a Hilbert space. The physical properties of the states are analyzed defining creation and annihilation operators.

For both theories, a canonical and covariant quantization procedure is developed. The higher order derivatives in the gravitational lagrangian are treated by means of a preliminary Dirac procedure.

The closure of the Poincaré algebra is guaranteed by the infrared-finiteness of the states which is related to the spin of the excitations through the regularization phase. Such a phase may have interesting physical consequences.

1 Introduction

The topological mass arising from the Chern-Simons term in 2+1-dimensional topologically massive gauge theories [1, 2, 3], provides, at the quantum level, an infrared cut-off which seems to cure the infrared problem without disturbing the ultraviolet or gauge properties of these theories. However, as it was recognized already in the seminal work by Deser, Jackiw and Templeton [3], a few delicate points, concerning the infrared behavior of these theories, need a careful treatment. In particular the closure of the Poincaré algebra, seems to be subordinate to a particular definition of the phase of the field operators [3].

In this paper we shall perform a careful analytical derivation of the free quantum states of topologically massive planar theories devoting particular attention to the treatment of the infrared ambiguities. The exact knowledge of the states of these theories leads to that of the polarization tensors. For electrodynamics, the states are known but infrared-ambiguous [4], whereas for gravity, they have yet not been studied. We shall show that in both theories the states are described by infrared regular polarization tensors containing a regularization phase which depends on the spin. Such a phase was first introduced in Ref.[3].

For electrodynamics, an infrared modification of the polarization vector of physical photons leads to interesting consequences in phenomenological applications of the interacting theory with fermionic matter. In particular, the correct form of the polarization vector might help in proving the conjectured existence of fermion-photon bound-states[5]. In the gravitational case, the calculation of the polarization tensor of physical gravitons might be considered as a first step for the understanding of some questions arising, at the quantum level, in the relation between the first and second order formulations of the theory[6].

In the following we shall use a general procedure [7, 8] to construct the Hilbert space of the physical states and its scalar product. The latter is defined in terms of the pertinent two-points Wightman function which we shall explicitly compute for both topologically massive electrodynamics and gravity. Such a procedure has been succesfully used also in constructing Fock spaces with generalized statistics[9].

We shall perform a canonical analysis keeping into account both the constraints related to gauge invariance and those, arising in the gravitational case, due to the higher order derivatives of the lagrangian. In order to work always with well defined quantities one needs a careful definition of the action of the operators on the quantum states; this is achieved by introducing a suitable set of test functions.

We analyse topologically massive electrodynamics and gravity simultaneously, taking as lagrangians

$$\mathcal{L}_{ELEC}= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{\mu}{2}\mathcal{E}^{\mu\nu\rho}\partial_\mu A_\nu A_\rho \ , \tag{1}$$

$$\mathcal{L}_{GRAV}=\sqrt{g}R+\frac{1}{2\mu}\mathcal{E}^{\mu\lambda\nu}\Gamma^\rho{}_{\lambda\sigma}\left(\partial_\mu\Gamma^\sigma{}_{\nu\rho}+\frac{2}{3}\Gamma^\sigma{}_{\mu\xi}\Gamma^\xi{}_{\nu\rho}\right) \ . \tag{2}$$

In Section 2 we shall define the canonical variables and brackets in electrodynamics and linearized gravity, using in both cases a covariant gauge choice, Lorentz and Landau respectively. For linearized gravity a preliminary Dirac procedure is needed. This allows us to find the canonical brackets that are consistent with the second class constraints introduced when enlarging the configuration space to the time derivatives of some components of the metric [10].

In Section 3 we shall derive the two-points Wightman functions by solving the Cauchy problem arising from the equations of motion and the equal-time commutators. For linearized gravity the derivation turns out to be quite difficult since it involves a third order Cauchy problem. Nevertheless, for both theories, the explicit solution of the Cauchy problem is provided.

In Section 4 we construct the Hilbert space of the physical states deriving the expression for the one-particle states. The polarization tensors constructed in this section are well defined in the infrared. As a consequence, the states we exhibit are the only possible states allowing for the weak closure of the Poincaré algebra.

In Section 5 we analyse the physical properties of the quantum states, their mass and spin. To achieve this result we perform a normal mode expansion obtained in terms of the Wightman function.

Section 6 is devoted to some concluding remarks.

2 Canonical brackets

Let us start by introducing the canonical variables and by finding the canonical brackets of topologically massive electrodynamics and linearized gravity. We choose, in the framework of an indefinite metric Fock space, covariant gauges to maintain explicit Lorentz invariance.

2.1 Electrodynamics

The gauge-fixed lagrangian is, in the Lorentz gauge $\partial_\mu A^\mu = 0$,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\mu}{2} \mathcal{E}^{\mu\nu\rho} \partial_\mu A_\nu A_\rho - \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2 \quad , \tag{3}$$

leading to the equations of motion

$$\Box A^\mu + (\xi - 1)\, \partial^\mu \partial_\nu A^\nu + \mu \mathcal{E}^{\mu\nu\rho} \partial_\nu A_\rho = 0 \quad . \tag{4}$$

The lagrangian (3) is regular, i.e. $\det \left(\partial^2 \mathcal{L} / \partial \left(\partial^0 A^\mu\right) \partial \left(\partial^0 A^\nu\right)\right) \neq 0$, and the canonical momentum

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 A_\mu\right)} = F^{\mu 0} + \frac{\mu}{2} \mathcal{E}^{0\mu\nu} A_\nu - \xi \eta^{\mu 0} \partial_\nu A^\nu \tag{5}$$

can be uniquely inverted to obtain the velocities. The hamiltonian version of the theory is constructed in the usual way; the only non-vanishing canonical Poisson bracket is

$$\left\{A^\mu\left(\vec{x}, t\right), \pi^\nu\left(\vec{y}, t\right)\right\} = \eta^{\mu\nu} \delta^2 \left(\vec{x} - \vec{y}\right) \quad . \tag{6}$$

The energy-momentum and angular-momentum tensors have the usual form

$$T^{\mu\nu} = \pi^{\rho(\mu)} \partial^\nu A_\rho - \eta^{\mu\nu} \mathcal{L} \quad , \tag{7}$$

$$M^{\alpha\mu\nu} = x^\mu T^{\alpha\nu} - x^\nu T^{\alpha\mu} + \pi^{\rho(\alpha)} \left(\Sigma^{\mu\nu}\right)_{\rho\tau} A^\tau \quad , \tag{8}$$

where

$$\pi^{\rho(\mu)} = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\rho\right)} \quad , \tag{9}$$

and

$$\left(\Sigma^{\mu\nu}\right)_{\rho\tau} = \delta^\mu_\rho \delta^\nu_\tau - \delta^\mu_\tau \delta^\nu_\rho \tag{10}$$

is the spin matrix of the field A^μ.

The quantum version of the theory is obtained using the correspondence principle

$$\{A, B\} \rightarrow -i\, [A, B] \tag{11}$$

and defining the physical states as the ones destroyed by the annihilation part of $\partial \cdot A$

$$\left(\partial_\mu A^\mu\right)^- |\Phi_f\rangle = 0 \quad . \tag{12}$$

2.2 Gravity

To perform a canonical analysis of the topologically massive gravity we consider its linearized version, $g^{\mu\nu} = \eta^{\mu\nu} + kh^{\mu\nu}$, with the Minkowski signature diag $(\eta_{\mu\nu}) = (+, -, -)$. The linearized lagrangian is invariant under the abelian gauge transformations

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \quad . \tag{13}$$

To fix the gauge we then select the Landau gauge $\partial_\mu h^{\mu\nu} = 0$. The linearized lagrangian is [3]

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}\left[(\partial_\rho h_{\mu\nu})(\partial^\rho h^{\mu\nu}) - (\partial_\mu h)(\partial^\mu h) + 2(\partial^\nu h)(\partial^\mu h_{\mu\nu}) + 2(\partial^\mu h_{\mu\nu})(\partial_\rho h^{\rho\nu})\right] \\ &+\frac{1}{4\mu}\mathcal{E}_{\mu\alpha\beta}\left(\partial^\alpha \partial_\nu h_\rho{}^\beta - \partial^\alpha \partial_\rho h_\nu{}^\beta\right)\partial^\nu h^{\mu\rho} - \frac{1}{2\xi}\left(\partial_\mu h^{\mu\lambda}\right)\left(\partial_\nu h^\nu{}_\lambda\right) \quad . \end{aligned} \tag{14}$$

Since (14) contains second order derivatives, the equations of motion are third order

$$\begin{aligned} &\Big\{\Box h^{\mu\nu} + \partial^\mu \partial^\nu h + (\xi - 1)\left(\partial^\nu \partial_\alpha h^{\alpha\mu} + \partial^\mu \partial_\alpha h^{\alpha\nu}\right) - \eta^{\mu\nu}\left(\Box h - \partial_\alpha \partial_\beta h^{\alpha\beta}\right) \\ &-\frac{1}{2\mu}\left[\mathcal{E}^{\mu\alpha\beta}\partial_\alpha\left(\Box h_\beta{}^\nu - \partial^\nu \partial_\lambda h^\lambda{}_\beta\right) + \mathcal{E}^{\nu\alpha\beta}\partial_\alpha\left(\Box h_\beta{}^\mu - \partial^\mu \partial_\lambda h^\lambda{}_\beta\right)\right]\Big\} = 0 \ . \end{aligned} \tag{15}$$

In general the order of the equations of motion should be twice that of the lagrangian; the fact that this does not occur here is a first evidence of the constrained nature of this theory. In fact, the lagrangian is singular, i.e. $\det\left(\partial^2 \mathcal{L}/\partial\left(\partial^0 \partial^0 A^\mu\right)\partial\left(\partial^0 \partial^0 A^\nu\right)\right) = 0$. In Eq. (14) all the dependence from the second order derivatives lies in the Chern-Simons lagrangian, whose only non-vanishing terms are

$$\mathcal{L}_{CS} = \frac{1}{4\mu}\mathcal{E}_{ij}\left(\partial^k h^{i0} - \partial^0 h^{ik}\right)\left(\partial^0\right)^2 h_k{}^j \ . \tag{16}$$

Therefore, only the space-space components of $h^{\mu\nu}$ have higher order dynamics.

To extend the hamiltonian formulation to this case, one needs to decouple the second order time-derivatives, so that the dynamics becomes first order in time. This is done by defining an additional canonical variable for the time-derivative of each variable having higher order dynamics. One can then extend the definition of the canonical momentum so that the formal structure of the canonical Legendre transformation is maintained [10]. This procedure has been applied to topologically massive gravity in Ref. [11, 12], whereas the canonical formulation of the full nonlinear theory has been analyzed in Ref. [13].

At variance with Ref. [12], it is clear from (16) that it suffices to take as independent canonical variables $h^{\mu\nu}$ and $k^{ij} = \partial^0 h^{ij}$, together with their respective conjugate momenta, defined as

$$\pi^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 h_{\mu\nu}\right)} - 2\partial_\rho \frac{\partial \mathcal{L}}{\partial\left(\partial_\rho \partial_0 h_{\mu\nu}\right)} + \partial_0 \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \partial_0 h_{\mu\nu}\right)} \ , \tag{17}$$

$$s^{ij} = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \partial_0 h_{ij}\right)} \ . \tag{18}$$

In terms of the above variables, everything works essentially as usual, except for the fact that the phase space has been extended. In particular, the only non-vanishing canonical Poisson brackets are

$$\left\{h^{\mu\nu}(\vec{x}, t), \pi^{\alpha\beta}(\vec{y}, t)\right\} = \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}\right)\delta^2(\vec{x} - \vec{y}) \ , \tag{19}$$

$$\left\{k^{ij}\left(\vec{x},t\right),s^{mn}\left(\vec{y},t\right)\right\} = \frac{1}{2}\left(\eta^{im}\eta^{jn}+\eta^{in}\eta^{jm}\right)\delta^{2}\left(\vec{x}-\vec{y}\right) \ , \tag{20}$$

and the energy-momentum and angular-momentum tensors become

$$T^{\mu\nu} = \pi^{\rho\tau(\mu)}\partial^{\nu}h_{\rho\tau} + s^{ij(\mu)}\partial^{\nu}k_{ij} - \eta^{\mu\nu}\mathcal{L} \ , \tag{21}$$

$$M^{\mu\alpha\beta} = x^{\alpha}T^{\mu\beta} - x^{\beta}T^{\mu\alpha} + \pi^{\rho\tau(\mu)}\left(\Sigma^{\alpha\beta}\right)_{\rho\tau}{}^{\lambda\sigma}h_{\lambda\sigma} + s^{ij(\mu)}\left(\Sigma^{\alpha\beta 0\lambda}\right)_{ij}{}^{\rho\tau}\partial_{\lambda}h_{\rho\tau} \ , \tag{22}$$

where

$$\pi^{\mu\nu(\alpha)} = \frac{\partial\mathcal{L}}{\partial\left(\partial_{\alpha}h_{\mu\nu}\right)} - 2\partial_{\rho}\frac{\partial\mathcal{L}}{\partial\left(\partial_{\rho}\partial_{\alpha}h_{\mu\nu}\right)} + \partial_{0}\frac{\partial\mathcal{L}}{\partial\left(\partial_{0}\partial_{\alpha}h_{\mu\nu}\right)} \ , \tag{23}$$

$$s^{ij(\alpha)} = \frac{\partial\mathcal{L}}{\partial\left(\partial_{\alpha}\partial_{0}h_{ij}\right)} \ , \tag{24}$$

and

$$\left(\Sigma^{\mu\nu}\right)^{\rho\tau\alpha\beta} = \frac{1}{2}\left[\eta^{\mu\rho}\eta^{\nu\alpha}\eta^{\tau\beta} + \eta^{\mu\tau}\eta^{\nu\beta}\eta^{\rho\alpha} + (\alpha\leftrightarrow\beta) - (\mu\leftrightarrow\nu)\right] \ , \tag{25}$$

$$\left(\Sigma^{\mu\nu 0\lambda}\right)^{ij\alpha\beta} = \left(\Sigma^{\mu\nu}\right)^{ij\alpha\beta}\eta^{\lambda 0} - \frac{1}{2}\left(\eta^{\mu 0}\eta^{\nu\lambda} - \eta^{\nu 0}\eta^{\mu\lambda}\right)\left(\eta^{i\alpha}\eta^{j\beta} + \eta^{i\beta}\eta^{j\alpha}\right) \tag{26}$$

are the spin matrices of the fields $h^{\mu\nu}$ and k^{ij}.

A straightforward calculation gives for the canonical momenta

$$\pi^{00} = -\xi\partial_{0}h^{00} + \left(\frac{1}{2}-\xi\right)\partial_{i}h^{i0} \ , \tag{27}$$

$$\begin{aligned}\pi^{0i} = & -\frac{1}{2}\left[\xi\partial_{0}h^{0i} + (\xi-1)\,\partial_{j}h^{ji} + \tfrac{1}{2}\partial^{i}h\right] - \frac{1}{4\mu}\mathcal{E}_{mn}\left[\partial^{m}\partial^{i}h^{0n} - \partial^{m}k^{in}\right] \\ & -\frac{1}{8\mu}\mathcal{E}_{m}{}^{i}\left[\partial_{j}k^{mj} - \partial^{j}\partial_{j}h^{m0}\right] \ ,\end{aligned} \tag{28}$$

$$\begin{aligned}\pi^{ij} = & -\frac{1}{2}\left[k^{ij} - \eta^{ij}k^{l}{}_{l} + \eta^{ij}\partial_{k}h^{k0}\right] \\ & +\frac{1}{8\mu}\left[\mathcal{E}_{k}{}^{j}\left(2\partial_{0}k^{ki} - 2\partial^{i}\partial_{0}h^{k0} - 2\partial^{k}\partial_{0}h^{i0}\right.\right. \\ & \left.\left. +2\partial^{k}\partial^{i}h^{00} + \partial^{l}\partial_{l}h^{ki} - \partial^{i}\partial_{l}h^{kl}\right) + (i\leftrightarrow j)\right] \ ,\end{aligned} \tag{29}$$

$$s^{ij} = -\frac{1}{8\mu}\left[\mathcal{E}_{k}{}^{j}\left(k^{ki} - \partial^{i}h^{k0}\right) + (i\leftrightarrow j)\right] \ . \tag{30}$$

From the trace of Eq. (29) and Eq. (30), follow the constraints

$$\Lambda = \pi^{l}{}_{l} - \frac{1}{2}k^{l}{}_{l} + \partial_{k}h^{k0} + \frac{1}{4\mu}\mathcal{E}_{ki}\partial^{i}\partial_{l}h^{kl} \approx 0 \ , \tag{31}$$

$$\mathcal{O}^{ij} = s^{ij} + \frac{1}{8\mu}\left[\mathcal{E}_{k}{}^{j}\left(k^{ki} - \partial^{i}h^{k0}\right) + (i\leftrightarrow j)\right] \approx 0 \ . \tag{32}$$

This shows that the lagrangian remains singular even after the gauge-fixing. This is due to the fact that the constraints (31), (32), are not related to the gauge invariance (13), but to the enlargement of the phase space produced by the introduction of the new variables k_{ij}. These constraints are in fact second class. The canonical Poisson brackets are incompatible with these constraints [1]. With this type of constraints we found more convenient to apply the Dirac procedure [14, 15], instead of the quicker method proposed in Ref. [16].

We have the four constraints

$$\varphi^{[1]} = \mathcal{O}^{11} \ , \ \varphi^{[2]} = \mathcal{O}^{22} \ , \ \varphi^{[3]} = \mathcal{O}^{12} \ , \ \varphi^{[4]} = \Lambda \ . \tag{33}$$

No secondary constraints are produced by the dynamical compatibility condition; thus, the complete set of constraints to deal with is given only by Eq. (33). The matrix of the Poisson brackets of these constraints

$$\begin{aligned} M^{[a][b]}\left(\vec{x},\vec{y},t\right) &= \left\{\varphi^{[a]}\left(\vec{x},t\right),\varphi^{[b]}\left(\vec{y},t\right)\right\} \\ &= \frac{1}{2}\begin{pmatrix} 0 & 0 & \frac{1}{2\mu} & -1 \\ 0 & 0 & -\frac{1}{2\mu} & -1 \\ -\frac{1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}\delta^2\left(\vec{x}-\vec{y}\right) \end{aligned} \tag{34}$$

is non-singular.

The Dirac brackets, defined by

$$\begin{aligned} &\left\{A\left(\vec{x},t\right),B\left(\vec{y},t\right)\right\}^* = \\ &\quad = \left\{A\left(\vec{x},t\right),B\left(\vec{y},t\right)\right\} \\ &\quad - \int d^2\vec{z}\int d^2\vec{w}\left\{A\left(\vec{x},t\right),\varphi^{[a]}\left(\vec{z},t\right)\right\}M^{-1}_{[a][b]}\left(\vec{z},\vec{w},t\right)\left\{\varphi^{[b]}\left(\vec{w},t\right),B\left(\vec{y},t\right)\right\} \ , \end{aligned} \tag{35}$$

can be computed. After a lengthy but straightforward computation, we obtain

$$\left\{k^{ij}\left(\vec{x},t\right),k^{mn}\left(\vec{y},t\right)\right\}^* = \frac{\mu}{2}\left[\mathcal{E}^{im}\eta^{jn} + (i\leftrightarrow j) + (m\leftrightarrow n)\right]\delta^2\left(\vec{x}-\vec{y}\right) \ , \tag{36}$$

$$\left\{h^{ij}\left(\vec{x},t\right),k^{mn}\left(\vec{y},t\right)\right\}^* = \eta^{ij}\eta^{mn}\delta^2\left(\vec{x}-\vec{y}\right) \ , \tag{37}$$

$$\left\{\pi^{0i}\left(\vec{x},t\right),\pi^{0j}\left(\vec{y},t\right)\right\}^* = -\frac{5}{64\mu}\mathcal{E}^{ij}\partial^k\partial_k\delta^2\left(\vec{x}-\vec{y}\right) \ , \tag{38}$$

$$\left\{\pi^{0k}\left(\vec{x},t\right),\pi^{mn}\left(\vec{y},t\right)\right\}^* = \frac{1}{64\mu^2}\mathcal{E}^{ik}\left(\mathcal{E}^{mj}\partial^n + \mathcal{E}^{nj}\partial^m\right)\partial_i\partial_j\delta^2\left(\vec{x}-\vec{y}\right) \ , \tag{39}$$

$$\left\{s^{ij}\left(\vec{x},t\right),s^{mn}\left(\vec{y},t\right)\right\}^* = \frac{1}{32\mu}\left[\mathcal{E}^{im}\eta^{jn} + (i\leftrightarrow j) + (m\leftrightarrow n)\right]\delta^2\left(\vec{x}-\vec{y}\right) \ , \tag{40}$$

$$\left\{\pi^{0k}\left(\vec{x},t\right),s^{mn}\left(\vec{y},t\right)\right\}^* = \frac{1}{32\mu}\left(\mathcal{E}^{km}\partial^n + \mathcal{E}^{kn}\partial^m + \eta^{mn}\mathcal{E}^{kj}\partial_j\right)\delta^2\left(\vec{x}-\vec{y}\right) \ , \tag{41}$$

$$\left\{h^{\mu\nu}\left(\vec{x},t\right),\pi^{ij}\left(\vec{y},t\right)\right\}^* = \frac{1}{2}\left(\eta^{\mu i}\eta^{\nu j} + \eta^{\mu j}\eta^{\nu i}\right)\delta^2\left(\vec{x}-\vec{y}\right) \ , \tag{42}$$

[1]The meaning of the weak equivalence sign $\approx$ appearing in the constraints is to remember that they are incompatible with the canonical Poisson brackets.

$$\left\{ h^{ij}\left(\vec{x},t\right), \pi^{0k}\left(\vec{y},t\right)\right\}^* = \frac{1}{8\mu}\eta^{ij}\mathcal{E}^{km}\partial_m\delta^2\left(\vec{x}-\vec{y}\right) \ , \tag{43}$$

$$\left\{ h^{0\mu}\left(\vec{x},t\right), \pi^{0\nu}\left(\vec{y},t\right)\right\}^* = \frac{1}{2}\left(\eta^{\mu\nu}+\eta^{\mu 0}\eta^{\nu 0}\right)\delta^2\left(\vec{x}-\vec{y}\right) \ , \tag{44}$$

$$\left\{ k^{ij}\left(\vec{x},t\right), s^{mn}\left(\vec{y},t\right)\right\}^* = \frac{1}{4}\left(\eta^{im}\eta^{jn}+\eta^{in}\eta^{jm}-\eta^{ij}\eta^{mn}\right)\delta^2\left(\vec{x}-\vec{y}\right) \ , \tag{45}$$

$$\left\{ k^{ij}\left(\vec{x},t\right), \pi^{0\mu}\left(\vec{y},t\right)\right\}^* = \frac{1}{8}\left(\eta^{\mu i}\partial^j+\eta^{\mu j}\partial^i+3\eta^{ij}\partial^\mu\right)\delta^2\left(\vec{x}-\vec{y}\right) \ , \tag{46}$$

$$\left\{ k^{ij}\left(\vec{x},t\right), \pi^{mn}\left(\vec{y},t\right)\right\}^* = -\frac{1}{8\mu}\eta^{ij}\left(\mathcal{E}^{km}\partial^n+\mathcal{E}^{kn}\partial^m\right)\partial_k\delta^2\left(\vec{x}-\vec{y}\right) \ . \tag{47}$$

All the other brackets vanish.

The quantum theory is defined using the generalized correspondence principle

$$\{A,B\}^* \to -i\left[A,B\right] \tag{48}$$

and defining the physical states as the ones satisfying

$$\left(\partial_\mu h^{\mu\nu}\right)^- \left|\Phi_f\right\rangle = 0 \ . \tag{49}$$

3 Two-points functions

We are particularly interested in the Wightman function, since this is the two point function that enters in the definition of the scalar product of the physical Hilbert space. In terms of the Wightman function $W^{(i)(j)}\left(x-y\right)$ [2] the classical Pauli-Jordan function $\Delta^{(i)(j)}\left(x-y\right)$, giving the canonical brackets among the canonical variables, is expressed as

$$\Delta^{(i)(j)}\left(x-y\right) = -i\left(W^{(i)(j)}\left(x-y\right)-W^{(j)(i)}\left(y-x\right)\right) \ , \tag{50}$$

while the propagator is given by

$$S^{(i)(j)}\left(x-y\right) = \theta\left(x_0-y_0\right)W^{(i)(j)}\left(x-y\right)+\theta\left(y_0-x_0\right)W^{(j)(i)}\left(y-x\right) \ . \tag{51}$$

3.1 Electrodynamics

Consider the covariant Poisson bracket of two fields

$$\Delta^{\mu\nu}\left(x-y\right) = \left\{A^\mu\left(x\right), A^\nu\left(y\right)\right\} \ . \tag{52}$$

By definition, the Δ function is a solution of the second order equations of motion

$$\Box\Delta^{\mu\nu}\left(z\right)+\left(\xi-1\right)\partial^\mu\partial_\alpha\Delta^{\alpha\nu}\left(z\right)+\mu\mathcal{E}^{\mu\alpha\beta}\partial_\alpha{\Delta_\beta}^\nu\left(z\right) = 0 \ . \tag{53}$$

The boundary conditions necessary for the uniqueness of the solution stem from the canonical Poisson brackets; the equal-time Poisson brackets correspond to the zero-time conditions for the Δ function

$$\Delta^{\mu\nu}\left(\vec{z},0\right) = 0 \ , \tag{54}$$

[2]Henceforth, we shall use simplified notations. We indicate with (i) the set of all indices needed to describe the theory; for electrodynamics, $\phi^{(i)} = A^\mu$ and $\eta^{(i)(j)} = \eta^{\mu\nu}$, and for gravity, $\phi^{(i)} = h^{\mu\nu}$ and $\eta^{(i)(j)} = \eta^{\mu\nu}$.

$$\partial^0 \Delta^{\mu\nu}(\vec{z},0) = \left(\eta^{\mu\nu} + \frac{1-\xi}{\xi}\eta^{\mu 0}\eta^{\nu 0}\right)\delta^2(\vec{z}) \ . \tag{55}$$

The Δ function is then the unique solution of the second order Cauchy problem (53), (54) and (55). Because of Lorentz covariance and of the symmetry properties of the Poisson brackets, the most general form allowed for the Δ function is

$$\Delta^{\mu\nu}(z) = \eta^{\mu\nu} f_1(z) + \mathcal{E}^{\mu\nu\rho}\partial_\rho f_2(z) + \partial^\mu\partial^\nu f_3(z) \ , \tag{56}$$

where f_1, f_2 and f_3 are scalar functions. Inserting (56) into the Cauchy problem (53) - (55), one finds for the Wightman function

$$\tilde{W}^{\mu\nu}(p) = 2\pi\theta(p_0)\left[\delta\left(p^2\right)M^{(0)\alpha\beta}(p) + \delta\left(p^2-\mu^2\right)M^{(\mu)\alpha\beta}(p)\right] \ , \tag{57}$$

with

$$M^{(0)\alpha\beta}(p) = \frac{i}{\mu}\mathcal{E}^{\alpha\beta\rho}p_\rho - \left(\frac{1}{\mu^2} - \frac{1}{\xi}\frac{1}{p^2}\right)p^\alpha p^\beta \ , \tag{58}$$

$$M^{(\mu)\alpha\beta}(p) = -\left(\eta^{\alpha\beta} - \frac{p^\alpha p^\beta}{\mu^2}\right) - \frac{i}{\mu}\mathcal{E}^{\alpha\beta\rho}p_\rho \ . \tag{59}$$

The propagator reads

$$\begin{aligned}\tilde{S}^{\mu\nu}(p) = &-\frac{i}{p^2+i\epsilon}\left[-\frac{1}{\xi}\frac{p^\mu p^\nu}{p^2+i\epsilon}\right] \\ &-\frac{i}{p^2-\mu^2+i\epsilon}\left[P^{\mu\nu} + i\frac{\mu}{p^2+i\epsilon}\mathcal{E}^{\mu\nu\rho}p_\rho\right] \ ,\end{aligned} \tag{60}$$

with

$$P^{\mu\nu} = \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2+i\epsilon} \ . \tag{61}$$

3.2 Gravity

For gravity one can proceed in the same way introducing the Pauli-Jordan function in terms of the covariant Dirac bracket

$$\Delta^{\mu\nu\alpha\beta}(x-y) = \left\{h^{\mu\nu}(x), h^{\alpha\beta}(y)\right\}^* \ . \tag{62}$$

By definition, the Δ function is a solution of the equations of motion, which are in this case third order

$$\begin{aligned}&\Big\{\Box\Delta^{\mu\nu\alpha\beta}(z) + \partial^\mu\partial^\nu\Delta^\rho{}_\rho{}^{\alpha\beta} - \eta^{\mu\nu}\left(\Box\Delta^\rho{}_\rho{}^{\alpha\beta}(z) - \partial_\rho\partial_\tau\Delta^{\rho\tau\alpha\beta}(z)\right) \\ &+(\xi-1)\left(\partial^\nu\partial_\rho\Delta^{\mu\rho\alpha\beta}(z) + \partial^\mu\partial_\rho\Delta^{\rho\nu\alpha\beta}(z)\right) \\ &-\frac{1}{2\mu}\left[\mathcal{E}^{\mu\rho\tau}\partial_\rho\left(\Box\Delta_\tau{}^{\nu\alpha\beta}(z) - \partial^\nu\partial_\lambda\Delta^\lambda{}_\tau{}^{\alpha\beta}(z)\right) + \mathcal{E}^{\nu\rho\tau}\partial_\rho\left(\Box\Delta^\mu{}_\tau{}^{\alpha\beta}(z) - \partial^\mu\partial_\lambda\Delta^\lambda{}_\tau{}^{\alpha\beta}(z)\right)\right]\Big\} \\ &\quad = 0 \ .\end{aligned} \tag{63}$$

The boundary conditions necessary to define the Cauchy problem arise from the canonical equal-time Dirac brackets. They read

$$\Delta^{\mu\nu\alpha\beta}(\vec{z},0) = 0 \ , \tag{64}$$

$$\partial^0 \Delta^{ijmn}(\vec{z},0) = -\eta^{ij}\eta^{mn}\delta^2(\vec{z}) \ , \tag{65}$$

$$\partial^0 \Delta^{0\mu 0\nu}(\vec{z},0) = \frac{1}{\xi}\eta^{\mu\nu}\delta^2(\vec{z}) \ , \tag{66}$$

$$\partial^0 \Delta^{ij0\mu}(\vec{z},0) = 0 \ , \tag{67}$$

$$\left(\partial^0\right)^2 \Delta^{ijmn}(\vec{z},0) = -\frac{\mu}{2}\left[\mathcal{E}^{im}\eta^{jn} + (i \leftrightarrow j) + (m \leftrightarrow n)\right]\delta^2(\vec{z}) \ , \tag{68}$$

$$\left(\partial^0\right)^2 \Delta^{ij0\mu}(\vec{z},0) = \frac{1}{\xi}\left(\eta^{i\mu}\partial^j + \eta^{j\mu}\partial^i + \xi\eta^{ij}\partial^\mu\right)\delta^2(\vec{z}) \ , \tag{69}$$

$$\left(\partial^0\right)^2 \Delta^{000\mu}(\vec{z},0) = -\frac{1}{\xi}\partial^\mu\delta^2(\vec{z}) \ , \tag{70}$$

$$\left(\partial^0\right)^2 \Delta^{0i0j}(\vec{z},0) = 0 \ . \tag{71}$$

The Δ function is the unique solution of the third order Cauchy problem (63) - (71). To determine Δ explicitly we can take again advantage of the Lorentz covariance and the symmetry properties of the Dirac brackets to write the general form

$$\begin{aligned}
\Delta^{\mu\nu\alpha\beta}(z) = {} & \eta^{\mu\nu}\eta^{\alpha\beta}f_1(z) + \left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}\right)f_2(z) \\
& + \left[\eta^{\mu\alpha}\mathcal{E}^{\nu\beta\sigma} + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta)\right]\partial_\sigma f_3(z) \\
& + \left(\eta^{\mu\nu}\partial^\alpha\partial^\beta + \eta^{\alpha\beta}\partial^\mu\partial^\nu\right)f_4(z) \\
& + \left[\eta^{\mu\alpha}\partial^\nu\partial^\beta + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta)\right]f_5(z) \\
& + \left[\mathcal{E}^{\mu\alpha\sigma}\partial^\nu\partial^\beta + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta)\right]\partial_\sigma f_6(z) \\
& + \partial^\mu\partial^\nu\partial^\alpha\partial^\beta f_7(z) \ ,
\end{aligned} \tag{72}$$

where f_1 - f_7 are scalar functions. Inserting Eq. (72) in the equations defining the Cauchy problem for Δ, we get, after a hard algebraic work, the Wightman function

$$\tilde{W}^{\mu\nu\alpha\beta}(p) = 2\pi\theta(p_0)\left[\delta\left(p^2\right)M^{(0)\mu\nu\alpha\beta}(p) + \delta\left(p^2 - \mu^2\right)M^{(\mu)\mu\nu\alpha\beta}(p)\right] \ , \tag{73}$$

with

$$\begin{aligned}
M^{(0)\mu\nu\alpha\beta}(p) = {} & 2\eta^{\mu\nu}\eta^{\alpha\beta} - \left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}\right) \\
& - \frac{i}{2\mu}\left[\eta^{\mu\alpha}\mathcal{E}^{\nu\beta\sigma} + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta)\right]p_\sigma \\
& - \left(\frac{1}{\mu^2} + \frac{2}{p^2}\right)\left(\eta^{\mu\nu}p^\alpha p^\beta + \eta^{\alpha\beta}p^\mu p^\nu\right) \\
& + \left(\frac{1}{\mu^2} + \frac{\xi - 1}{\xi}\frac{1}{p^2}\right)\left[\eta^{\mu\alpha}p^\nu p^\beta + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta)\right] \\
& + \frac{i}{2\mu}\left(\frac{1}{\mu^2} + \frac{1}{p^2}\right)\left[\mathcal{E}^{\mu\alpha\sigma}p^\nu p^\beta + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta)\right]p_\sigma \\
& - \left(\frac{1}{\mu^4} + \frac{1}{\mu^2 p^2} - \frac{3}{\xi}\frac{1}{p^4}\right)p^\mu p^\nu p^\alpha p^\beta \ ,
\end{aligned} \tag{74}$$

$$M^{(\mu)\mu\nu\alpha\beta}(p) = -\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{\mu^2}\right)\left(\eta^{\alpha\beta} - \frac{p^\alpha p^\beta}{\mu^2}\right)$$
$$+\left[\left(\eta^{\mu\alpha} - \frac{p^\mu p^\alpha}{\mu^2}\right)\left(\eta^{\nu\beta} - \frac{p^\nu p^\beta}{\mu^2}\right) + (\alpha \leftrightarrow \beta)\right] \tag{75}$$
$$+\frac{i}{2\mu}\left[\mathcal{E}^{\mu\alpha\sigma}\left(\eta^{\nu\beta} - \frac{p^\nu p^\beta}{\mu^2}\right) + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta)\right] p_\sigma \ .$$

The propagator follows using Eq. (51)

$$\tilde{S}^{\mu\nu\alpha\beta}(p) = -\frac{i}{p^2+i\epsilon}\left\{\left(P^{\mu\alpha}P^{\nu\beta} + P^{\mu\beta}P^{\nu\alpha}\right) - 2P^{\mu\nu}P^{\alpha\beta}\right.$$
$$\left. + \frac{1}{\xi}\left[\eta^{\mu\alpha}\frac{p^\nu p^\beta}{p^2+i\epsilon} + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta)\right] + \frac{3}{\xi}\frac{p^\mu p^\nu p^\alpha p^\beta}{(p^2+i\epsilon)^2}\right\}$$
$$-\frac{i}{p^2-\mu^2+i\epsilon}\left[P^{\mu\nu}P^{\alpha\beta} - \left(P^{\mu\alpha}P^{\nu\beta} + P^{\mu\alpha}P^{\nu\beta}\right)\right.$$
$$\left. -i\frac{\mu}{2}\frac{p_\sigma}{p^2+i\epsilon}\left(\mathcal{E}^{\mu\alpha\sigma}P^{\beta\nu} + \mathcal{E}^{\nu\alpha\sigma}P^{\beta\mu} + \mathcal{E}^{\mu\beta\sigma}P^{\alpha\nu} + \mathcal{E}^{\nu\beta\sigma}P^{\alpha\mu}\right)\right] \ , \tag{76}$$

with, as before,

$$P^{\mu\nu} = \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2+i\epsilon} \ . \tag{77}$$

4 Quantum states

We are now able to construct the physical Hilbert space which explicitly realizes the operator algebras defined by Eqs.(6) and (36) - (47). This is done by defining the action of the field operators on a suitable space of states [3].

Consider the vectorial space $\mathcal{H} = \{|\Phi\rangle\}$ of the sequences

$$|\Phi\rangle = \left\{\phi_0, \phi_1^{(\mu_1)}(p_1), \phi_2^{(\mu_1)(\mu_2)}(p_1,p_2), ..., \phi_n^{(\mu_1)...(\mu_n)}(p_1,...,p_n), ...\right\} \ , \tag{78}$$

where $\phi_n^{(\mu_1)...(\mu_n)}(p_1,...,p_n) \in S\left(R^{3n}\right)$ is a completely symmetric tensor with respect to the exchange $(\mu_i, p_i) \leftrightarrow (\mu_j, p_j)$.

The space $\mathcal{H}$ so defined is the Fock space, and the function $\phi_n^{(\mu_1)...(\mu_n)}(p_1,...,p_n)$ is the n-particles component of the generic state; the scalar product is defined in terms of the

[3] The set of all regular functions defined in O is indicated with $S(O)$. The μ-mass-shell is called V_μ^+ and the corresponding integration measure is

$$\widetilde{dp}^{\mu} = \left.\frac{d^2\vec{p}}{2\pi\sqrt{2p_0}}\right|_{p\in V_\mu^+} = \frac{d^3p}{2\pi}\sqrt{2p_0}\,\theta(p_0)\,\delta\left(p^2-\mu^2\right) \ .$$

Wightman function as [7]

$$\langle\Phi|\ \Psi\rangle = \sum_{n=0}^{\infty}\int\frac{d^3p_1}{(2\pi)^3}...\int\frac{d^3p_n}{(2\pi)^3}$$
$$\phi^{*n}_{(\mu_1)...(\mu_n)}(p_1,..,p_n)\,\tilde{W}^{(\mu_1)}{}_{(\nu_1)}(p_1)...\tilde{W}^{(\mu_n)}{}_{(\nu_n)}(p_n)\,\psi_n^{(\nu_1)...(\nu_n)}(p_1,..,p_n)\quad, \tag{79}$$

$\forall\,|\Phi\rangle,\ |\Psi\rangle\ \in\mathcal{H}$. Because of the hermiticity of the Fourier transform of the Wightman function, the scalar product has the essential property $\langle\Phi|\ \Psi\rangle = \langle\Psi|\ \Phi\rangle^*$, $\forall\,|\Phi\rangle,\ |\Psi\rangle\in\mathcal{H}$, but, as we shall see, is in general not positive-defined.

The action of the field operators in the space of the states can be defined using test functions.

From the positive and negative frequency parts of the field operators $\phi^{(i)}(x) = \phi^{+(i)}(x) + \phi^{-(i)}(x)$, one can define the smeared operators

$$\phi^{\pm(i)}[\varphi] = \int d^3x\phi^{\pm(i)}(x)\,\varphi(x) = \left(\phi^{\pm(i)},\varphi\right)\quad. \tag{80}$$

These operators have the following action on the vectors of $\mathcal{H}$ [7]

$$\begin{aligned}&\left(\phi^{+(i)}[\varphi]\,|\Phi\rangle\right)_n^{(\mu_1)...(\mu_n)}(p_1,...,p_n) =\\ &= \frac{1}{\sqrt{n}}\sum_{m=1}^{n}\phi_{n-1}^{(\mu_1)...(\mu_{m-1})(\mu_{m+1})...(\mu_n)}(p_1,...,p_{m-1},p_{m+1},...,p_n)\,\eta^{(i)(\mu_m)}\tilde{\varphi}(p_m)\quad,\end{aligned} \tag{81}$$

$$\begin{aligned}&\left(\phi^{-(i)}[\varphi]\,|\Phi\rangle\right)_n^{(\mu_1)...(\mu_n)}(p_1,...,p_n) =\\ &= \sqrt{n+1}\int\frac{d^3p}{(2\pi)^3}\tilde{W}^{(i)}{}_{(j)}(p)\,\phi_{n+1}^{(j)(\mu_1)...(\mu_n)}(p,p_1,...,p_n)\,\tilde{\varphi}(-p)\quad.\end{aligned} \tag{82}$$

By continuity, from Eqs. (81), (82), one can obtain the action of the unsmeared operators $\phi^{\pm(i)}(x)$; this is usually achieved by replacing the test functions with Dirac δ-functions, obtaining

$$\begin{aligned}&\left(\phi^{+(i)}(x)\,|\Phi\rangle\right)_n^{(\mu_1)...(\mu_n)}(p_1,...,p_n) =\\ &= \frac{1}{\sqrt{n}}\sum_{m=1}^{n}\phi_{n-1}^{(\mu_1)...(\mu_{m-1})(\mu_{m+1})...(\mu_n)}(p_1,...,p_{m-1},p_{m+1},...,p_n)\,\eta^{(i)(\mu_m)}e^{ip_mx}\quad,\end{aligned} \tag{83}$$

$$\begin{aligned}&\left(\phi^{-(i)}(x)\,|\Phi\rangle\right)_n^{(\mu_1)...(\mu_n)}(p_1,...,p_n) =\\ &= \sqrt{n+1}\int\frac{d^3p}{(2\pi)^3}\tilde{W}^{(i)}{}_{(j)}(p)\,\phi_{n+1}^{(j)(\mu_1)...(\mu_n)}(p,p_1,...,p_n)\,e^{-ipx}\quad.\end{aligned} \tag{84}$$

With the help of these definitions, one can easily see that the field operators satisfy the equations of motion and the covariant commutation relations

$$\left[\phi^{(i)}[\varphi],\phi^{(j)}[\chi]\right] = i\int d^3x\int d^3y\varphi(x)\,\Delta^{(i)(j)}(x-y)\,\chi(y)\quad. \tag{85}$$

For the unsmeared operators, Eq. (85) implies

$$\left[\phi^{(i)}(x),\phi^{(j)}(y)\right] = i\Delta^{(i)(j)}(x-y)\quad. \tag{86}$$

Among the states contained in $\mathcal{H}$ the physical states are those having non-vanishing norm, $\langle \Phi_f | \, \Phi_f \rangle \neq 0$, and satisfying the condition $G^- \, |\Phi_f\rangle = 0$ [4]. From the definition of the scalar product (79) one can conclude that the physical degrees of freedom of the theory can be associated with those eigenvectors of the Wightman function giving a non-vanishing and positive contribution to the norm and satisfying the gauge condition.

4.1 Electrodynamics

Consider a one-particle state. Since the Wightman function has domain $V_0^+ \cup V_\mu^+$, such a state can be written as

$$|\Phi_1\rangle = \left\{0, \phi^\alpha_{(0)}\left(p\right) + \phi^\alpha_{(\mu)}\left(p\right), 0, 0, 0, ...\right\} \quad ,$$

with

$$\phi^\alpha_{(0)}\left(p\right) \in S\left(V_0^+\right) \; , \; \phi^\alpha_{(\mu)}\left(p\right) \in S\left(V_\mu^+\right) \quad . \tag{87}$$

Its norm is given by

$$\begin{aligned} \langle \Phi_1 | \, \Phi_1 \rangle = & \int \widetilde{dp}^0 \frac{1}{2\pi\sqrt{2p_0}} \phi^{*\alpha}_{(0)}\left(p\right) M^{(0)}_{\alpha\beta}\left(p\right) \phi^\beta_{(0)}\left(p\right) \\ & + \int \widetilde{dp}^\mu \frac{1}{2\pi\sqrt{2p_0}} \phi^{*\alpha}_{(\mu)}\left(p\right) M^{(\mu)}_{\alpha\beta}\left(p\right) \phi^\beta_{(\mu)}\left(p\right) \quad . \end{aligned} \tag{88}$$

From the gauge condition $\left(\partial_\mu A^\mu\left(x\right)\right)^- |\Phi_f\rangle = 0$, we have

$$p_\alpha \phi^\alpha_{(0)}\left(p\right) = 0 \quad . \tag{89}$$

From (89) [5], one can see that the massless part $\phi^\alpha_{(0)}\left(p\right)$ of the state does not contribute to the norm (88), and, consequently, it is unphysical.

The massive part of the state appears in (88) only through a projector $P^{(\mu)}_{\alpha\beta}$ according to

$$\langle \Phi_1 | \, \Phi_1 \rangle = -2 \int \widetilde{dp}^\mu \frac{1}{2\pi\sqrt{2p_0}} \phi^{*\alpha}_{(\mu)}\left(p\right) P^{(\mu)}_{\alpha\beta}\left(p\right) \phi^\beta_{(\mu)}\left(p\right) \quad , \tag{90}$$

with

$$P^{(\mu)\alpha\beta}\left(p\right) = \frac{1}{2}\left(P^{\alpha\beta} + \frac{i}{\mu}\mathcal{E}^{\alpha\beta\rho} p_\rho\right) \quad , \tag{91}$$

$$P^{(\mu)}_{\alpha\rho}\left(p\right) P^{(\mu)\rho}{}_\beta\left(p\right) = P^{(\mu)}_{\alpha\beta}\left(p\right) \quad , \quad P^{(\mu)}_{\alpha\beta}\left(p\right) = P^{*(\mu)}_{\beta\alpha}\left(p\right) \quad . \tag{92}$$

We can then conclude that the physical states have mass $|\mu|$ and polarization given by the eigenvector of the projector $P^{(\mu)}_{\alpha\beta}\left(p\right)$, with non vanishing eigenvalue.

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[4]For electrodynamics,

$$G = \partial_\mu A^\mu \quad ,$$

while for gravity,

$$G = \partial_\mu h^{\mu\nu} \quad .$$

[5]and using

$$\phi^{*\alpha}_{(0)}\left(p\right) \mathcal{E}_{\alpha\beta\rho} p^\rho \phi^\beta_{(0)}\left(p\right) = \frac{\left(p\right)^2}{p_0}\left(\phi^{*1}_{(0)}\left(p\right)\phi^2_{(0)}\left(p\right) - \phi^1_{(0)}\left(p\right)\phi^{*2}_{(0)}\left(p\right)\right)$$

Solving explicitly the eigenvalue problem for the projector (91), one obtains

$$f^{\mu}(p) = \frac{1}{\sqrt{2\mu^2\left(p_0^2-\mu^2\right)}} \begin{pmatrix} \mu^2 - p_0^2 \\ i\mu p^2 - p^0 p^1 \\ -i\mu p^1 - p^0 p^2 \end{pmatrix} e^{i\beta(p)} \ , \tag{93}$$

with the normalization

$$f^{*\alpha}(p) f_{\alpha}(p) = -1 \ . \tag{94}$$

The phase factor $\beta(p)$ has an important role and is not completely arbitrary. As a matter of fact the infrared behavior of the eigenvector depends upon the choice of the phase. Taking the infrared limit, we have

$$f^{\mu}(0) = \lim_{|\vec{p}|\to 0} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ -i\frac{\mu}{|\mu|} \end{pmatrix} e^{i\left(\beta(p)-\frac{\mu}{|\mu|}\theta(p)\right)} \ , \tag{95}$$

with

$$\theta(p) = \arctan\left(\frac{p^2}{p^1}\right) \ . \tag{96}$$

Since $\theta(p)$ is not defined in the origin, one must take

$$\beta(p) = \frac{\mu}{|\mu|}\theta(p) + \gamma(p) \ . \tag{97}$$

In (97) $\gamma(p)$ is an arbitrary but infrared regular phase factor, which can be chosen to vanish.

The final result for the polarization vector then is

$$f^{\mu}(p) = \frac{1}{\sqrt{2\mu^2\left(p_0^2-\mu^2\right)}} \begin{pmatrix} \mu^2 - p_0^2 \\ i\mu p^2 - p^0 p^1 \\ -i\mu p^1 - p^0 p^2 \end{pmatrix} e^{i\frac{\mu}{|\mu|}\theta(p)} \ . \tag{98}$$

The polarization vector (98) satisfies

$$P^{(\mu)\mu}{}_{\nu}(p) f^{\nu}(p) = f^{\mu}(p) \ , \tag{99}$$

$$p_{\mu} f^{\mu}(p) = 0 \ , \tag{100}$$

$$f^{\mu}(p) f^{*\nu}(p) = -P^{(\mu)\mu\nu}(p) \ . \tag{101}$$

Eqs. (99) - (101) are very useful in computing the generators of the Poincaré group. One verifies that the contribution of the eigenvector (98) to the norm is positive.

The phase in (98) was first introduced in Ref. [3] as a regularization of the field operators. In our approach it appears directly in the definition of the physical states.

In conclusion, the normalized physical one-particle states of topologically massive electrodynamics are

$$|\Phi_{f1}\rangle = \{0, 2\pi\sqrt{p_0}a(p) f^{\alpha}(p), 0, 0, 0, ...\} \ . \tag{102}$$

Here $a(p)$ is an arbitrary scalar function defined on V_{μ}^{+} and such that $\int d^2\vec{p}\,|a(p)|^2 = 1$. $a(p)$ plays the role of the one-particle wave function.

4.2 Gravity

The construction of the one-particle state for the gravitational case is very similar to that of the electrodynamics. Since the Wightman function has domain given by $V_0^+ \cup V_\mu^+$, this state can be written as

$$|\Phi_1\rangle = \left\{0, \phi^{\alpha\beta}_{(0)}(p) + \phi^{\alpha\beta}_{(\mu)}(p), 0, 0, 0, ...\right\} \ ,$$

with

$$\phi^{\alpha\beta}_{(0)}(p) \in S\left(V_0^+\right) \ , \ \phi^{\alpha\beta}_{(\mu)}(p) \in S\left(V_\mu^+\right) \ . \tag{103}$$

Its norm is given by

$$\begin{aligned} \langle \Phi_1 | \Phi_1 \rangle = & \int \widetilde{dp}^0 \frac{1}{2\pi\sqrt{2p_0}} \phi^{*\alpha\beta}_{(0)}(p) M^{(0)}_{\alpha\beta\mu\nu}(p) \phi^{\mu\nu}_{(0)}(p) \\ & + \int \widetilde{dp}^\mu \frac{1}{2\pi\sqrt{2p_0}} \phi^{*\alpha\beta}_{(\mu)}(p) M^{(\mu)}_{\alpha\beta\mu\nu}(p) \phi^{\mu\nu}_{(\mu)}(p) \ . \end{aligned} \tag{104}$$

The gauge condition $\left(\partial_\mu h^{\mu\nu}(x)\right)^- |\Phi_f\rangle = 0$, reads

$$p_\alpha \phi^{\alpha\beta}_{(0)}(p) = 0 \ . \tag{105}$$

Using (105) [6], one sees that the massless part of the state $\phi^{\alpha\beta}_{(0)}(p)$ does not contribute to the norm, and is indeed unphysical.

Again the norm involves only a projection on the massive part of the state, and reads

$$\langle \Phi_1 | \Phi_1 \rangle = 4 \int \widetilde{dp}^\mu \frac{1}{2\pi\sqrt{2p_0}} \phi^{*\alpha\beta}_{(\mu)}(p) P^{(\mu)}_{\alpha\beta\mu\nu}(p) \phi^{\mu\nu}_{(\mu)}(p) \ , \tag{106}$$

with

$$\begin{aligned} P^{(\mu)\mu\nu\alpha\beta}(p) = & \frac{1}{4}\left\{\left(P^{\mu\alpha}P^{\nu\beta} + P^{\mu\beta}P^{\nu\alpha}\right) - P^{\mu\nu}P^{\alpha\beta}\right. \\ & \left. + \frac{i}{2\mu}\left[\mathcal{E}^{\mu\alpha\sigma}P^{\nu\beta} + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta)\right] p_\sigma \right\} \ , \end{aligned} \tag{107}$$

$$P^{(\mu)}_{\mu\nu\rho\tau}(p) P^{(\mu)\rho\tau}{}_{\alpha\beta}(p) = P^{(\mu)}_{\mu\nu\alpha\beta}(p) \ , \ P^{(\mu)}_{\mu\nu\alpha\beta}(p) = P^{*(\mu)}_{\alpha\beta\mu\nu}(p) \ . \tag{108}$$

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[6] and

$$\begin{aligned} & \phi^{*\mu\nu}_{(0)}(p)\left[2\eta_{\mu\nu}\eta_{\alpha\beta} - (\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha})\right]\phi^{\alpha\beta}_{(0)}(p) = \\ & = 2(p)^2\left[\frac{1}{p^1p^2}\left(\phi^{*01}_{(0)}(p)\phi^{02}_{(0)}(p) + \phi^{01}_{(0)}(p)\phi^{*02}_{(0)}(p)\right)\right. \\ & \quad - \frac{1}{p_0p^1}\left(\phi^{*02}_{(0)}(p)\phi^{12}_{(0)}(p) + \phi^{02}_{(0)}(p)\phi^{*12}_{(0)}(p)\right) \\ & \quad \left. - \frac{1}{p_0p^2}\left(\phi^{*01}_{(0)}(p)\phi^{12}_{(0)}(p) + \phi^{01}_{(0)}(p)\phi^{*12}_{(0)}(p)\right)\right] \ , \\ & \phi^{*\mu\nu}_{(0)}(p)\left[\eta_{\mu\alpha}\mathcal{E}_{\nu\beta\sigma} + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta)\right]p^\sigma \phi^{\alpha\beta}_{(0)}(p) = \\ & = 4\frac{(p)^2}{p_0}\left(\phi^{*1\lambda}_{(0)}(p)\phi^2_{(0)\lambda}(p) - \phi^{1\lambda}_{(0)}(p)\phi^{*2}_{(0)\lambda}(p)\right) \end{aligned}$$

We can therefore conclude that physical states have mass $|\mu|$ and polarization given by the eigentensor of the projector $P^{(\mu)}_{\mu\nu\alpha\beta}(p)$ with non-vanishing eigenvalue.

Solving explicitly the eigenvalue problem for the projector (107), and taking care of the infrared behavior of the eigentensor, one obtains

$$f^{\mu\nu}(p) = \frac{1}{2\mu^2}\begin{pmatrix} p_0^2-\mu^2 & p^0p^1 - i\mu p^2 & p^0p^2 + i\mu p^1 \\ p^0p^1 - i\mu p^2 & \frac{\left(p^0p^1 - i\mu p^2\right)^2}{p_0^2-\mu^2} & \frac{\left(p^0p^1 - i\mu p^2\right)\left(p^0p^2 + i\mu p^1\right)}{p_0^2-\mu^2} \\ p^0p^2 + i\mu p^1 & \frac{\left(p^0p^1 - i\mu p^2\right)\left(p^0p^2 + i\mu p^1\right)}{p_0^2-\mu^2} & \frac{\left(p^0p^2 + i\mu p^1\right)^2}{p_0^2-\mu^2} \end{pmatrix} e^{i2\frac{\mu}{|\mu|}\theta(p)} \ , \tag{109}$$

with the normalization

$$f^{*\alpha\beta}(p)\, f_{\alpha\beta}(p) = 1 \ . \tag{110}$$

This is the infrared well-defined polarization tensor that has to be used in any perturbative computation. This eigentensor has the following important properties

$$P^{(\mu)\mu\nu}{}_{\alpha\beta}(p)\, f^{\alpha\beta}(p) = f^{\mu\nu}(p) \ , \tag{111}$$

$$p_\mu f^{\mu\nu}(p) = 0 \ , \tag{112}$$

$$f^{\mu}{}_{\mu}(p) = 0 \ , \tag{113}$$

$$f^{\mu\nu}(p)\, f^{*\alpha\beta}(p) = P^{(\mu)\mu\nu\alpha\beta}(p) \ . \tag{114}$$

One can verify that the contribution of the eigentensor (109) to the norm is positive.

Summarizing, the normalized physical one-particle states of topologically massive gravity are

$$|\Phi_{f1}\rangle = \left\{0, \pi\sqrt{2p_0}a(p)\, f^{\alpha\beta}(p), 0, 0, 0, ...\right\} \ , \tag{115}$$

where $a(p)$ is an arbitrary scalar function defined on V_μ^+ and such as $\int d^2\vec{p}\,|a(p)|^2 = 1$. This plays the role of one-particle wave function.

Note that the eigentensor of gravity is the tensorial product of two eigenvectors of electrodynamics:

$$f^{\mu\nu}(p) = f^{\mu}(p)\, f^{\nu}(p) \ . \tag{116}$$

Mathematically, this stems from the fact that on V_μ^+ one has

$$P^{(\mu)\mu\nu\alpha\beta}(p) = \frac{1}{2}\left(P^{(\mu)\mu\alpha}(p)\, P^{(\mu)\nu\beta}(p) + P^{(\mu)\mu\beta}(p)\, P^{(\mu)\nu\alpha}(p)\right) \ . \tag{117}$$

Physically, the two theories essentially differ in the spin representation to which they correspond. Since the spin is a scalar quantity in 2+1 dimensions, and since the two theories correspond to representations of the Poincaré group which are related by a tensorial product (as expected [3]), the spin of the graviton will be twice that of the photon. We shall show explicitly in the next section that the states constructed above satisfy this property.

5 Observables

The last step in the analysis of the two theories is to extract the physical properties of the states. This is readily done by expanding the fields in normal modes and defining creation and annihilation operators for the physical states.

We start by observing that a field satisfying the equations of motion can be expanded in normal modes in a completely general way, once the Wightman function of the theory is known. In fact, one can write

$$\phi^{(i)}(x) = \int \frac{d^3p}{(2\pi)^2}\sqrt{2p_0}\left[\tilde{W}^{(i)(j)}(p)\, a_{(j)}(p)\, e^{-ipx} + \tilde{W}^{*(i)(j)}(p)\, a^{+}_{(j)}(p)\, e^{ipx}\right] \quad . \tag{118}$$

According to the spectral theorem, the Wightman function can always be written as

$$\tilde{W}^{(i)(j)}(p) = 2\pi\theta(p_0)\sum_{n=1}^{N}\delta\left(p^2-\mu_n^2\right) M^{\mu_n(i)(j)}(p) \quad , \tag{119}$$

where the μ_n have the dimension of a mass, the $M^{\mu_n(i)(j)}(p)$ are hermitian matrices and N is the number of massive and massless excitation of the theory [7]. The field expansion can then be written as

$$\phi^{(i)}(x) = \sum_{n=1}^{N}\int \widetilde{dp}^{\mu_n}\left[M^{\mu_n(i)(j)}(p)\, a^{\mu_n}_{(j)}(p)\, e^{-ipx} + M^{*\mu_n(i)(j)} a^{+\mu_n}_{(j)}(p)\, e^{ipx}\right] \quad . \tag{120}$$

Using the eigenvectors of the N matrices $M^{\mu_n(i)(j)}(p)$, defined by the eigenvalues problem

$$M^{\mu_n(i)}{}_{(j)}(p)\, f^{\mu_n(j)}_{[k]}(p) = \lambda^{\mu_n}_{[k]}(p)\, f^{\mu_n(i)}_{[k]}(p) \quad , \quad p\in V^{+}_{\mu_n} \quad , \tag{121}$$

and normalized to [8]

$$f^{\mu_n(i)}_{[k]}(p)\, f^{*\mu_n}_{[l](i)}(p) = \pm\eta_{[k][l]} \quad , \quad p\in V^{+}_{\mu_n} \quad , \tag{122}$$

the expansion finally reads

$$\phi^{(i)}(x) = \sum_{n=1}^{N}\sum\nolimits_{[k]}\int \widetilde{dp}^{\mu_n}\sqrt{\left|\lambda^{\mu_n}_{[k]}(p)\right|}\left[a^{\mu_n}_{[k]}(p)\, f^{\mu_n(i)}_{[k]}(p)\, e^{-ipx} + a^{+\mu_n}_{[k]}(p)\, f^{*\mu_n(i)}_{[k]}(p)\, e^{ipx}\right] \quad . \tag{123}$$

The explicit action of the operators $a^{\mu_n}_{[k]}[\varphi]$ and $a^{+\mu_n}_{[k]}[\varphi]$ on the states can be deduced from that of the fields; the smeared operators

$$a^{\mu_n}_{[k]}[\varphi] = \int d^2\vec{p}\, a^{\mu_n}_{[k]}(p)\,\varphi(p) = \left(a^{\mu_n}_{[k]},\varphi\right) \quad , \tag{124}$$

$$a^{+\mu_n}_{[k]}[\varphi] = \int d^2\vec{p}\, a^{+\mu_n}_{[k]}(p)\,\varphi^{*}(p) = \left(a^{+\mu_n}_{[k]},\varphi^{*}\right) \quad , \tag{125}$$

[7] As already seen, in both our cases we have $\mu_1 = 0$ and $\mu_2 = \mu$.

[8] Since the internal field metric $\eta^{(i)(j)}$ is not in general positive defined, one has to consider both signs.

act like

$$\begin{aligned}&\left(a_{[k]}^{+\mu_p}\left[\varphi\right]\left|\Phi\right\rangle\right)_n^{(\mu_1)\ldots(\mu_n)}\left(p_1,...,p_n\right)=\\&=\pm\frac{2\pi\sqrt{2p_0}}{\sqrt{n}}\sum_{m=1}^{n}\phi_{n-1}^{(\mu_1)\ldots(\mu_{m-1})(\mu_{m+1})\ldots(\mu_n)}\left(p_1,...,p_{m-1},p_{m+1},...,p_n\right)\\&\frac{\eta_{[k][k]}}{\sqrt{\left|\lambda_{[k]}^{\mu_p}\left(p_m\right)\right|}}f_{[k]}^{\mu_p(\mu_m)}\left(p_m\right)\varphi^*\left(p_m\right)\quad,\end{aligned}\tag{126}$$

$$\begin{aligned}&\left(a_{[k]}^{\mu_p}\left[\varphi\right]\left|\Phi\right\rangle\right)_n^{(\mu_1)\ldots(\mu_n)}\left(p_1,...,p_n\right)=\\&=\pm\sqrt{n+1}\int\widetilde{dp}^{\mu_p}\eta_{[k][k]}\frac{\lambda_{[k]}^{\mu_p}\left(p\right)}{\sqrt{\left|\lambda_{[k]}^{\mu_p}\left(p\right)\right|}}f_{[k](i)}^{*\mu_p}\left(p\right)\phi_{n+1}^{(i)(\mu_1)\ldots(\mu_n)}\left(p,p_1,...,p_n\right)\varphi\left(p\right)\quad.\end{aligned}\tag{127}$$

It then follows that the unsmeared operators act like

$$\begin{aligned}&\left(a_{[k]}^{+\mu_p}\left(p\right)\left|\Phi\right\rangle\right)_n^{(\mu_1)\ldots(\mu_n)}\left(p_1,...,p_n\right)=\\&=\pm\frac{2\pi\sqrt{2p_0}}{\sqrt{n}}\sum_{m=1}^{n}\phi_{n-1}^{(\mu_1)\ldots(\mu_{m-1})(\mu_{m+1})\ldots(\mu_n)}\left(p_1,...,p_{m-1},p_{m+1},...,p_n\right)\\&\frac{\eta_{[k][k]}}{\sqrt{\left|\lambda_{[k]}^{\mu_p}\left(p_m\right)\right|}}f_{[k]}^{\mu_p(\mu_m)}\left(p_m\right)\delta^2\left(\vec{p}-\vec{p}_m\right)\quad,\end{aligned}\tag{128}$$

$$\begin{aligned}&\left(a_{[k]}^{\mu_p}\left(p\right)\left|\Phi\right\rangle\right)_n^{(\mu_1)\ldots(\mu_n)}\left(p_1,...,p_n\right)=\\&=\pm\frac{\sqrt{n+1}}{2\pi\sqrt{2p_0}}\eta_{[k][k]}\frac{\lambda_{[k]}^{\mu_p}\left(p\right)}{\sqrt{\left|\lambda_{[k]}^{\mu_p}\left(p\right)\right|}}f_{[k](i)}^{*\mu_p}\left(p\right)\phi_{n+1}^{(i)(\mu_1)\ldots(\mu_n)}\left(p,p_1,...,p_n\right)\quad.\end{aligned}\tag{129}$$

The commutation relations of $a_{[k]}^{\mu_n}\left[\varphi\right]$ and $a_{[k]}^{+\mu_n}\left[\varphi\right]$ are

$$\left[a_{[k]}^{\mu_m}\left[\varphi\right],a_{[l]}^{+\mu_n}\left[\chi\right]\right]=\pm\delta^{mn}\eta_{[k][l]}\frac{\left|\lambda_{[k]}^{\mu_m}\left(p\right)\right|}{\lambda_{[k]}^{\mu_m}\left(p\right)}\int d^2\vec{p}\varphi\left(p\right)\chi^*\left(p\right)\quad,\tag{130}$$

$$\left[a_{[k]}^{\mu_m}\left[\varphi\right],a_{[l]}^{\mu_n}\left[\chi\right]\right]=\left[a_{[k]}^{+\mu_m}\left[\varphi\right],a_{[l]}^{+\mu_n}\left[\chi\right]\right]=0\quad,\tag{131}$$

which imply

$$\left[a_{[k]}^{\mu_m}\left(p\right),a_{[l]}^{+\mu_n}\left(q\right)\right]=\pm\delta^{mn}\eta_{[k][l]}\frac{\left|\lambda_{[k]}^{\mu_m}\left(p\right)\right|}{\lambda_{[k]}^{\mu_m}\left(p\right)}\delta^2\left(\vec{p}-\vec{q}\right)\quad,\tag{132}$$

$$\left[a_{[k]}^{\mu_m}\left(p\right),a_{[l]}^{\mu_n}\left(q\right)\right]=\left[a_{[k]}^{+\mu_m}\left(p\right),a_{[l]}^{+\mu_n}\left(q\right)\right]=0\quad.\tag{133}$$

Thus, $a_{[k]}^{+\mu_n}\left(p\right)$ and $a_{[k]}^{\mu_n}\left(p\right)$ are creation and annihilation operators for the states of mass $\left|\mu_n\right|$ and polarization $f_{[k]}^{\mu_n(i)}\left(p\right)$. The physical Fock space can be constructed by the cyclic action of $a_{[k]}^{+\mu_n}\left(p\right)$ on the vacuum.

In the next subsections we shall compute, on the physical states, the mean-values of the Poincaré charges

$$P^{\mu}=\int d^{2}\vec{x}T^{0\mu}\ ,\ \ M^{\mu\nu}=\int d^{2}\vec{x}M^{0\mu\nu}\ , \tag{134}$$

and of the Pauli-Lubanski scalar, which defines the spin through

$$S=\frac{1}{2\left|\mu\right|}\mathcal{E}_{\alpha\mu\nu}P^{\alpha}M^{\mu\nu}\ . \tag{135}$$

Then, by means of the algebra (132), (133), we shall verify that the Poincaré is satisfied on the space of the physical states.

5.1 Electrodynamics

The physical field is

$$A_{f}^{\mu}\left(x\right)=\sqrt{2}\int\widetilde{dp}^{\mu}\left[a\left(p\right)f^{\mu}\left(p\right)e^{-ipx}+a^{+}\left(p\right)f^{*\mu}\left(p\right)e^{ipx}\right]\ . \tag{136}$$

From the properties of $f^{\mu}\left(p\right)$, one obtains for the physical field

$$A_{f}^{\mu}\left(x\right)-\frac{1}{\mu}\mathcal{E}^{\mu\alpha\beta}\partial_{\alpha}A_{f\beta}\left(x\right)=0\ , \tag{137}$$

$$\partial_{\mu}A_{f}^{\mu}\left(x\right)=0\ , \tag{138}$$

so that the physical excitation has Klein-Gordon dynamics

$$\left(\Box+\mu^{2}\right)A_{f}^{\mu}\left(x\right)=0\ . \tag{139}$$

The complete Poincaré generators contain terms depending also on the unphysical components (the massless and the massive with vanishing norm) of the field. Our analysis enables to select only the physical part of the Poincaré generators, which are given by

$$P_{f}^{\mu}=\int d^{2}\vec{p}p^{\mu}a^{+}\left(p\right)a\left(p\right)\ , \tag{140}$$

$$M_{f}^{ij}=\mathcal{E}^{ij}\int d^{2}\vec{p}\left[a^{+}\left(p\right)\left(-\frac{i}{2}\overleftrightarrow{\frac{\partial}{\partial\theta}}\right)a\left(p\right)+\frac{\mu}{\left|\mu\right|}a^{+}\left(p\right)a\left(p\right)\right]\ , \tag{141}$$

$$M_{f}^{0i}=tP_{f}^{i}+\int d^{2}\vec{p}\left[a^{+}\left(p\right)\left(\frac{i}{2}p^{0}\overleftrightarrow{\partial^{i}}\right)a\left(p\right)+\frac{\mu}{\left|\mu\right|}\frac{1}{\left|\mu\right|+p^{0}}\mathcal{E}^{i}{}_{k}p^{k}a^{+}\left(p\right)a\left(p\right)\right]\ , \tag{142}$$

and satisfy the Poincaré algebra.

On a physical state

$$\left|\Phi_{f1}\left(k\right)\right\rangle=a^{+}\left(k\right)\left|\Phi_{f0}\right\rangle\ , \tag{143}$$

one has

$$\left\langle\Phi_{f1}\left(k\right)\right|P_{f}^{\mu}\left|\Phi_{f1}\left(k\right)\right\rangle=k^{\mu}\ , \tag{144}$$

$$\left\langle\Phi_{f1}\left(k\right)\right|S_{f}\left|\Phi_{f1}\left(k\right)\right\rangle=\frac{\mu}{\left|\mu\right|}\ . \tag{145}$$

Therefore, $a^{+}(k)$ creates a photon with mass $|\mu|$, four-momentum k^{μ}, spin $\frac{\mu}{|\mu|}$ and polarization $f^{\mu}(k)$. This is the physical excitation of the theory already found in Ref. [3].

It is important to stress that the algebra realization exhibited in this paper, is the only one compatible with the closure of the Poincaré algebra. The phase choice performed in Section 4, not only makes the polarization vector infrared well-behaved, but it is also the phase choice that allows the Poincaré algebra to close. If we did not include the regularization phase in the definition of the $f^{\mu}(p)$, the Poincaré algebra would present the well-known anomaly obtained in [3]. In fact the generators become

$$P_f^{\mu} = \int d^2\vec{p}p^{\mu}a^{+}(p)\,a(p)\ \ , \tag{146}$$

$$M_f^{ij} = \mathcal{E}^{ij}\int d^2\vec{p}a^{+}(p)\left(-\frac{i}{2}\overleftrightarrow{\frac{\partial}{\partial\theta}}\right)a(p)\ \ , \tag{147}$$

$$M_f^{0i} = tP_f^i + \int d^2\vec{p}\left[a^{+}(p)\left(\frac{i}{2}p^0\overleftrightarrow{\partial^i}\right)a(p) + \frac{\mathcal{E}^i{}_k p^k}{p^m p_m}a^{+}(p)\,a(p)\right]\ \ , \tag{148}$$

and the Poincaré algebra acquires the anomaly

$$\Delta = 2\pi\mu^2 S a^{+}(0)a(0) \tag{149}$$

in the commutator of two boosts

$$\left[M_f^{0i}, M_f^{0j}\right] = -i\left(M_f^{ij} - \mathcal{E}^{ij}\Delta\right)\ \ . \tag{150}$$

Consequently, in any phenomenological computation, if we want a true representation of the Poincaré algebra, we need to use the polarization vector (98).

5.2 Gravity

The physical field is

$$h_f^{\mu\nu}(x) = 2\int \widetilde{dp}^{\mu}\left[a(p)\,f^{\mu\nu}(p)\,e^{-ipx} + a^{+}(p)\,f^{*\mu\nu}(p)\,e^{ipx}\right]\ \ . \tag{151}$$

From the properties of $f^{\mu\nu}(p)$ follow the properties of the physical field

$$h_f^{\mu\nu}(p) - \frac{1}{2\mu}\left[\mathcal{E}^{\mu\alpha\beta}\partial_\alpha h_{f\beta}{}^{\nu}(x) + \mathcal{E}^{\nu\alpha\beta}\partial_\alpha h_{f\beta}{}^{\mu}(x)\right] = 0\ \ , \tag{152}$$

$$\partial_\mu h_f^{\mu\nu}(x) = 0\ \ , \tag{153}$$

$$h^{\mu}_{f\mu}(x) = 0\ \ , \tag{154}$$

and the physical excitation has Klein-Gordon dynamics

$$\left(\Box + \mu^2\right)h_f^{\mu\nu}(x) = 0\ \ . \tag{155}$$

The physical Poincaré generators are those of the electrodynamic case, except for a factor two in the spin terms. The Poincaré algebra is satisfied and $a^{+}(k)$ creates a graviton with mass $|\mu|$, four-momentum k^{μ}, spin $2\frac{\mu}{|\mu|}$ and polarization $f^{\mu\nu}(k)$ [3]. As before, the spin dependent phase choice performed in Section 4, to regularize the infrared behavior of the polarization tensor, allows the closure of the Poincaré algebra.

6 Conclusion

The detailed analysis of the free quantum states of the topologically massive theories performed in this paper, has provided us with the explicit form of the one-particle states for these theories. We have shown that the states are infrared well-behaved representations of the Poincaré group with the well known values for the physical charges (mass and spin). These states define the polarization vector and tensor needed in phenomenological computations. The phase, regularizing the infrared behavior of the physical states, allows also for the closure of the Poincaré algebra on the physical states.

The derivation of the electrodynamical physical states might have relevant consequences on the knowledge of the bound-state spectrum of the theory. In particular the existence of photon-fermion bound-states suggested in Ref. [5], might be proved by the analysis of the effective non-relativistic fermion-photon potential. In fact, the inclusion of the regularizing phase in the photon polarization vector, simplifies dramatically the angular dependence of such a potential. We shall discuss how the regularization phase affects the bound-state spectrum of the theory in a forthcoming paper [17].

For gravity it would be interesting to compare our results with an analogous treatment of the gauge theoretical first order formulation [6]. The comparison of the free quantum states of the two theories might provide important informations on the relation between the first order and metric gravity at the quantum level.

7 Acknowledgments

We are grateful to M. Mintchev and G. Nardelli for useful discussions.

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