FTUAM-05-3
IFT-UAM/CSIC-05-15
NEIP-05/03
ROMA-1385/04
TUM-HEP-578/05

Minimal gauge-Higgs unification with a flavour symmetry

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Abstract

We show that a flavour symmetry à la Froggatt-Nielsen can be naturally incorporated in models with gauge-Higgs unification, by exploiting the heavy fermions that are anyhow needed to realize realistic Yukawa couplings. The case of the minimal five-dimensional model, in which the $SU(2)_L \times U(1)_Y$ electroweak group is enlarged to an $SU(3)_W$ group, and then broken to $U(1)_{\rm em}$ by the combination of an orbifold projection and a Scherk-Schwarz twist, is studied in detail. We show that the minimal way of incorporating a $U(1)_F$ flavour symmetry is to enlarge it to an $SU(2)_F$ group, which is then completely broken by the same orbifold projection and Scherk-Schwarz twist. The general features of this construction, where ordinary fermions live on the branes defined by the orbifold fixed-points and messenger fermions live in the bulk, are compared to those of ordinary four-dimensional flavour models, and some explicit examples are constructed.

1 Introduction

In the last thirty years the central problem in particle physics has been the mechanism for breaking the electroweak gauge symmetry and the consequent generation of masses for gauge bosons and matter fermions. In the Standard Model (SM), the problem manifests itself in two different ways: on the one hand in the instability of the weak interaction scale (the so-called hierarchy problem), on the other in the arbitrariness of the Yukawa couplings, which span at least five orders of magnitude, and the related problem of the strength of the CKM [1] (and PMNS [2]) matrix elements. In the quest for a solution of these fundamental problems, a plethora of extensions of the SM have been proposed, like technicolor, softly broken global supersymmetry, supergravity or string theory. None of the proposed solutions, however, is satisfactory, and this motivates further investigation.

More recently the idea came on the stage that it might be possible to overcome the hierarchy problem by implementing the breaking of gauge symmetries via alternative mechanisms relying on the presence of one or more extra dimensions. In particular, it has been known for a long time that realizing the Higgs field as the zero-mode of the internal component of a higher-dimensional gauge field leads to an effective potential with improved stability [3]. This idea has been recently reinvestigated from various points of view [4], and exploited to construct concrete higher-dimensional orbifold models with this type of gauge-Higgs unification [5–9] (see also [10–12] for supersymmetric models). A simple prototype of this kind of models is the minimal five-dimensional (5D) scenario described in [6], where the gauge symmetry is broken by the combination of a $\mathbf{Z}_2$ orbifold projection [13] and a continuous Scherk-Schwarz (SS) twist [14] along the extra compact dimension. The electroweak symmetry breaking is spontaneous and occurs through the Hosotani mechanism [15]. The order parameter is the Wilson loop $W = \exp\{ig \oint A_5(y)dy\}$ of the internal component A_5 of the gauge field along the internal circle S^1 (here and in the following we denote by y the coordinate along the internal dimension). The role of the Higgs field is played by the zero-mode of A_5, but the effective potential can depend only on the non-local gauge-invariant W and is therefore finite.

In this paper, we study the possibility of endowing the above-mentioned class of higher-dimensional models with a flavour symmetry of the Froggatt-Nielsen (FN) type [16]. This is done by introducing an extended flavour symmetry, which is then broken, as for the electroweak symmetry, by the combination of an orbifold projection and a SS twist. We focus on the model of ref. [6] and describe its minimal flavour extension. We show that by a wise choice of the flavour quantum numbers for bulk and brane fermion fields, it is possible to reproduce the observed pattern of the quark masses and CKM angles, although the mass obtained for the down quark tends to be too small, and observe that a similar approach is possible also for lepton masses and PMNS angles. The resulting model generates a 4D effective theory with a stabilized electroweak scale and a $U(1)$ FN symmetry.

The paper is organized as follows. After quickly reviewing gauge-Higgs unification in sec. 2, we outline in sec. 3 the basic construction discussing explicitly a prototype model. In sec. 4 we generalize our construction to arbitrary representations of the electroweak and flavour groups. In sec. 5 we present more realistic examples of our construction. Finally, in sec. 6 we draw some conclusions and discuss future developments.

2 Gauge-Higgs unification in 5D

Our starting point is the model of gauge-Higgs unification described in ref. [6]. The basic physical idea is to break the electroweak symmetry in a non-local way, so that the Higgs mass is protected by the gauge invariance of the 5D theory. Indeed, in a 5D theory compactified on a circle with SS symmetry breaking, all ultraviolet (UV) divergent quantities at all orders in perturbation theory must be invariant under the full symmetries of the 5D theory [17]. This means that all symmetry-breaking quantities are finite, calculable and insensitive to the unknown UV dynamics. If one could find a 5D symmetry that forbid the Higgs mass term, a non-local breaking of this symmetry would protect the Higgs mass from any divergent radiative correction. Gauge-Higgs unification implements this idea, by identifying the Higgs boson with the internal component of a 5D gauge field, so that the 5D gauge symmetry protects the Higgs mass.

To construct a model of gauge-Higgs unification one must consider a gauge group large enough as to include 4D states corresponding to the $SU(2)_L \otimes U(1)_Y$ gauge bosons plus the Higgs doublet. The minimal possibility corresponds to an $SU(3)_W$ gauge group, broken first to $SU(2)_L \otimes U(1)_Y$ via a Z_2 orbifold projection, and then to $U(1)_{\rm em}$ with a SS twist.[1] The orbifold projection acting on the 5D gauge group leaves as 4D zero modes the SM gauge bosons plus a scalar doublet with the quantum numbers of the SM Higgs: the SS twist corresponds to a Vacuum Expectation Value (VEV) for the Higgs via the Hosotani mechanism. From the 4D point of view, this corresponds to the SM Higgs mechanism: however, higher-dimensional gauge invariance protects the Higgs mass. This remains true even though at the orbifold fixed points only the SM gauge group is present: indeed, the zero-modes of A_5 transform non-homogeneously under gauge transformations belonging to $SU(3)_W/(SU(2)_L \otimes U(1)_Y)$, so that the only possible counterterms are $SU(3)_W$-invariant ones [18] (see also [19]).

The price one has to pay for this UV insensitivity is the absence of a tree-level

[1] In this way, one obtains $\sin^2\theta_W = 3/4$. An acceptable value of the weak mixing angle can be achieved by adding an extra $U(1)'$ factor and tuning its coupling relatively to the weak coupling, as done in ref. [6]. The additional $U(1)_X$ symmetry introduced in this way in the 4D effective theory is anomalous, and must therefore be spontaneously broken and decoupled.

potential for the Higgs. This implies that the Higgs mass generated at one loop is generically too small (see ref. [6] for a detailed discussion of this problem). A related issue is the value of the SS twist that is dynamically generated: unless bulk fermions belonging to very high-rank representations of $SU(3)$ are present, one obtains twist parameters of order 10^{-1}, corresponding to an extra dimension of inverse radius $1/R \sim 10\, m_W \sim 1$ TeV, far below the LEP indirect bounds. Since these problems are unrelated to the issue of flavour symmetry breaking that will be discussed in this work, from now on we will assume that the value of the SS twist α is of order 10^{-2} thanks to some unspecified mechanism, so that $1/R \sim 10$ TeV and Kaluza-Klein (KK) excitations of electroweak gauge bosons do not pose any phenomenological problem.

As in the standard electroweak theory, the VEV of the Higgs field can induce a mass for the matter fermions. The relevant Yukawa couplings, however, originate in this case from the 5D gauge coupling. For bulk fermions, this implies that the Yukawa couplings are universal and their magnitude is simply the gauge coupling times a group-theoretical factor, depending only on the representation. Furthermore, no flavour symmetry breaking can arise from electroweak gauge couplings, so that one is left with a universal fermion mass and no flavour mixing. For brane fields localized at the orbifold fixed-points, on the contrary, the $SU(2)_L \times U(1)_Y$ symmetry would allow Yukawa couplings to be arbitrary and non-universal, but the non-linearly realized $SU(3)_W/(SU(2)_L \times U(1)_Y)$ symmetry implies that they all vanish. In order to achieve realistic Yukawa couplings, one is therefore led to consider the more general case of fermions that are a mixture of bulk and brane fields with wave functions depending non-trivially on the internal dimension [11]. This situation is most easily realized by considering bulk and brane fields that mix through non-universal bilinear couplings localized at the fixed-points [6]. The new eigenstates, resulting from the diagonalization of the quadratic Lagrangian for these fields, will then inherit non-vanishing and non-universal Yukawa couplings to the Higgs field. The structure of the mass couplings is pretty general, but their size is always at most of the order of the gauge coupling. This implies that the natural value of all the fermion masses induced in this way is of the order of m_W.

In the case where the above construction is realized with bulk fields that are much heavier than the brane fields, the lightest eigenstates are sharply localized fields whose dynamics is well approximated by an effective Lagrangian for the original brane fermions, obtained by integrating out the heavy bulk modes. From this perspective, the non-vanishing and non-universal Yukawa couplings for the light localized modes emerge as effective interactions induced through the exchange of the heavy bulk fermions, which have a non-vanishing but universal fundamental Yukawa coupling. This framework is very similar to the one occurring in models with flavour symmetries, the breaking of which is transmitted to the effective Yukawa couplings through a heavy fermion, and suggests that it should be possible to naturally generalize the model of ref. [6] to include a flavour symmetry.

The usual implementation of a FN $U(1)_F$ flavour symmetry goes as follows. One assigns a $U(1)_F$ charge to each of the SM fermions, and introduces some heavy vector-like fermions in order to construct gauge- and flavour-invariant Yukawa couplings. The flavour symmetry is then spontaneously broken by some VEV at a scale smaller than the mass of the heavy fermions, so that the effective Yukawa couplings for SM fermions generated at low energies are $Y_{IJ} \propto (\langle\phi\rangle/M)^{q_I - q_J}$, where $\langle\phi\rangle$ is the $U(1)_F$-breaking VEV, M is the mass of the heavy fermions, and q_I are the SM fermion flavour charges. Wave-function corrections and the potentially dangerous tree-level FCNC generated by the heavy fermions are power suppressed and negligible if the new particles live at a high scale.

It is then natural for us to consider the case of a $U(1)_F$ symmetry broken à la SS. Since rank lowering can only be achieved by combining an orbifold projection with a SS twist, we have to start from an $SU(2)_F$ symmetry in the bulk, broken to $U(1)_F$ by the orbifold and then to nothing via a SS twist. Clearly, since the mass scale of the heavy bulk fermions is around 10 TeV in the case of ref. [6], we should make sure that wave function corrections and tree-level FCNC couplings are under control. We have performed a preliminary analysis of this issue, which indicates that unwanted effects might indeed be kept sufficiently small with reasonable choices of parameters. A detailed analysis, together with a study of loop-induced FCNC's, is currently under way and will be presented elsewhere.

3 A prototype model

A minimal prototype of the models discussed in the previous section can be constructed as follows. The standard fermions are taken to live at the orbifold fixed-points, whereas the messenger fermions that activate the mechanism of symmetry breaking live in the bulk. A spontaneously broken Abelian flavour symmetry is then incorporated much in the same way as for the spontaneously broken electroweak symmetry, and both symmetry breakings are implemented at once by letting the orbifold projection and the SS twist act on both the electroweak and the flavour groups. The minimal choice of 5D flavour group allowing an Abelian group in the intermediate step and a full breaking in the final step is an $SU(2)_F$ group. For simplicity, we assume this to be a global symmetry, but the case of a local symmetry is similar. This flavour group is broken to a $U(1)_F$ subgroup through the orbifold projection, and finally to nothing through the SS twist.

The above construction is very general, and exploits for both the electroweak and the flavour symmetries the same minimal pattern of symmetry breaking discussed in ref. [20], which consists in first promoting the 4D group to a larger 5D group and then performing two non-commuting projections that enable to lower the rank. The standard fermions at the fixed-points form representations of $SU(2)_L \times U(1)_Y \times U(1)_F$, whereas the messenger fermions in the bulk form representations of $SU(3)_W \times SU(2)_F$. The construction can be applied in a perfectly similar way both to the quark and the lepton sectors. Here we shall focus on the quark sector. For the sake of clarity of presentation, we will first illustrate the general qualitative features of the construction with an explicit example, then generalize to arbitrary flavour charges and $SU(3)_W \otimes SU(2)_F$ representations, and finally discuss some realistic models.

3.1 Orbifold projection and SS twist

The projections defining the model are chosen as follows. The orbifold projection on a bulk field $\Phi_{\mathcal{R},\mathcal{R}'}$ in a generic representation $(\mathcal{R},\mathcal{R}')$ of $SU(3)_W \times SU(2)_F$ is taken to be

$$\Phi_{\mathcal{R},\mathcal{R}'}(x,-y) = \pm[P_L \otimes P_W^{\mathcal{R}} \otimes P_F^{\mathcal{R}'}]\, \Phi_{\mathcal{R},\mathcal{R}'}(x,y)\,, \tag{1}$$

where P_L depends on which Lorentz representation the field corresponds to ($P_L = 1$ for a scalar, $P_L = \gamma_5$ for a spinor, etc.) and P_W and P_F define the embedding of the projection into the weak and flavour groups. In order to achieve the desired symmetry breaking down to $SU(2)_W \times U(1)_Y \times U(1)_F$, we use the T_W^8 and T_F^3 generators[2] of $SU(3)_W$ and $SU(2)_F$ respectively, and choose:

$$P_W = e^{2i\pi\sqrt{3}T_W^8}\,, \qquad P_F = e^{-i\pi\,(d(T_F^3)-1)/2}\, e^{i\pi T_F^3}\,, \tag{2}$$

where $d(T)$ is the dimension of the matrix T acting on the representation $\mathcal{R}'$. The residual $SU(2)_F \times U(1)_Y$ electroweak gauge symmetries are associated with the generators T_W^a with $a = 1,2,3,8$ that commute with the projection: $[T_W^a, P_W] = 0$. Similarly, the surviving $U(1)_F$ flavour symmetry is associated to the only generator T_F^3 commuting with the projection: $[T_F^3, P_F] = 0$.

The Scherk-Schwarz twist on the generic representation $(\mathcal{R},\mathcal{R}')$ of $SU(3)_W \times SU(2)_F$ is similarly of the form:

$$\Phi_{\mathcal{R},\mathcal{R}'}(x,y+2\pi R) = \left[T_W^{\mathcal{R}}(\alpha) \otimes T_F^{\mathcal{R}'}(\beta)\right]\, \Phi_{\mathcal{R},\mathcal{R}'}(x,y)\,, \tag{3}$$

where $T_W(\alpha)$ and $T_F(\beta)$ define the embedding of the twist into the weak and flavour groups and depend on two continuous parameters α and β. These must satisfy the usual consistency constraints $(T_W P_W)^2 = (T_F P_F)^2 = 1$ [13, 21]. In order to further break by the twist the electroweak and flavour symmetries $SU(2)_F \times U(1)_Y \times U(1)_F$ preserved by the orbifold projection down to $U(1)_{\rm em}$, we use the T_W^6 and T_F^1 generators of $SU(3)_W$ and $SU(2)_F$ respectively, and choose:

$$T_W(\alpha) = e^{4\pi i\alpha T_W^6}\,, \qquad T_F(\beta) = e^{4\pi i\beta T_F^1}\,. \tag{4}$$

[2]We define the $SU(3)_W$ generators as $T^a = \lambda^a/2$, where λ^a are the standard Gell-Mann matrices with the normalization ${\rm Tr}\lambda^a\lambda^b = 2\delta^{ab}$. Similarly, we define the $SU(2)_F$ as $T^a = \sigma^a/2$, where σ^a are the standard Pauli matrices with the normalization ${\rm Tr}\sigma^a\sigma^b = 2\delta^{ab}$.

The residual $U(1)_{\rm em}$ electromagnetic gauge symmetry is associated with the only generator $T_W^3 + T_W^8/\sqrt{3}$ that commutes also with the twist: $[T_W^3 + T_W^8/\sqrt{3}, T_W] = 0$. Notice that this fixes the hypercharge to be $Y = T_W^8/\sqrt{3}$. The flavour symmetry is instead completely broken since there is no generator commuting also with the twist.

The dimensionless quantities α and β are the order parameters for the rank-reducing breaking of the electroweak and flavour symmetries. Indeed, it is evident from eqs. (2) and (4) that the orbifold projection and the twist do not commute, that is $[P_W, T_W] \neq 0$ in the gauge sector and $[P_F, T_F] \neq 0$ in the flavour sector, unless $\alpha = n/2$ and $\beta = n/2$, with n integer. For the gauge symmetry, it is possible to relate the order parameter to the VEV of the Higgs field A_5 by performing a non-periodic gauge transformation that reabsorbs the twist [15]: $\alpha = g_5 R\langle A_5\rangle/2$. For the flavour symmetry, a similar relation would hold if it were local; the case where it is taken to be global can however be understood in a similar way by taking a suitable decoupling limit [17]. Notice finally that the electroweak and flavour symmetry breaking scales are naturally defined by $m_W = \alpha/R$ and $m_F = \beta/R$.

The effect of the SS twist on the orbifold-projected spectrum of KK modes of bulk fields will as usual amount to shifting the standard integer-moded masses $m_n = n/R$ obtained for fields that are periodic along the internal circle S^1 with radius R through a quantity that depends on the symmetry breaking parameters α and β. To be more precise, notice that the generators appearing in the exponents of the orbifold projection and SS twist do not commute. Starting from the standard basis in which the Cartan generators T_W^8 and T_F^3 appearing in the orbifold projection are diagonal, the generators T_W^6 and T_F^1 appearing in the twist can be brought into diagonal forms, which we denote by t_W and t_F, through some suitable unitary transformations U_W and V_F:

$$t_W = U_W T_W^6 U_W^\dagger\,, \qquad t_F = V_F T_F^1 V_F^\dagger\,. \tag{5}$$

In the transformed basis where the SS twist is diagonal (but the orbifold projection is not diagonal), the mass spectrum can be written in terms of the entries of the diagonalized twist generator simply as $m_n(\alpha,\beta) = (n + 2t_W\alpha + 2t_F\beta)/R$ (see sec. 3.5.1).

3.2 Field content

The field content of the model is a generalization of the one considered in ref. [6], where now all the brane fields must not only belong to $SU(2)_L \times U(1)_Y$ representations but also have definite charges under the $U(1)_F$ subgroup, and similarly all the bulk fields must also belong not only to $SU(3)_W$ but also to $SU(2)_F$ representations. Notice that the charge under the $U(1)_F$ flavour group preserved by the orbifold projection is quantized, as a consequence of the fact that the original flavour group is non-Abelian, and represented by $q_F = T_F^3$. This constrains in an interesting way the allowed charge assignments for the brane fields. The minimal content of brane and bulk fields that is required in order to construct the flavour extension of the model of ref. [6] is then quite rigidly fixed.

The SM fermions are introduced as brane fields at the fixed-points of the orbifold projection. Denoting by y the periodic coordinate of the extra dimension, the two fixed-points are located at $y = 0$ and $y = \pi R$ and represent the two boundaries of the physical space, the segment $[0, \pi R]$ in the extra dimension. Each of the left- and right-handed fields can be located at any of the two fixed-points. The precise distribution that is chosen is qualitatively not too important as far as the low-energy effective theory is concerned, but it is quite relevant for the consistency of the theory, and in particular for the issue of anomalies. Indeed, it is known that globally vanishing localized anomalies occur in theories with a generic content of bulk and brane fields and that requiring their cancellation may have non-trivial implications on the theory [22, 23] (see [24] for a general review). The issue of localized anomalies has already been discussed in ref. [6], and since the flavor extension examined here does not involve any novelty in this respect, we shall not discuss it any further here. For simplicity, we assume that all the left-handed fields are located at $y = 0$ and the right-handed ones at $y = \pi R$, and their interactions are constrained to be invariant under the residual symmetries described above. We introduce the following representations of $SU(2)_L \times U(1)_Y \times U(1)_F$:

- Left-handed fields localized at $y = 0$:

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} : \mathbf{2}_{\frac{1}{6},q} \quad \text{or equivalently} \quad Q_R^c = \begin{pmatrix} d_R^c \\ \text{-}u_R^c \end{pmatrix} = \mathbf{2}_{-\frac{1}{6},-q} \,. \tag{6}$$

- Right-handed fields localized at $y = \pi R$:

$$\begin{aligned} u_R &= \mathbf{1}_{\frac{2}{3},u}\,, \quad \text{or equivalently } \text{-}u_L^c = \mathbf{1}_{-\frac{2}{3},-u}\,, \\ d_R &= \mathbf{1}_{-\frac{1}{3},d}\,, \quad \text{or equivalently } \ d_L^c = \mathbf{1}_{\frac{1}{3},-d}\,, \end{aligned} \tag{7}$$

with the notation $\mathbf{R}_{q_Y,q_F}$, where $\mathbf{R}$ is the $SU(2)_L$ representation and q_Y and q_F are the $U(1)_Y$ and $U(1)_F$ charges respectively. As a first example, we choose the charge assignment reported in Table 1.

The bulk fields consist of the 5D gauge fields and of the heavy fermions that are needed to induce the effective Yukawa couplings as in ref. [6]. The rôle played by the gauge fields has been extensively explained in ref. [6] and will not be discussed again here. The only novelty concerns the heavy messenger fermions, which will now carry flavour quantum numbers. We introduce two pairs $l = u, d$ of fermion fields $(\psi^l, \tilde{\psi}^l)$ with opposite orbifold parities, with a bulk mass term that makes all their

Field	q_F	Field	q_F	Field	q_F
Q_{1L}	4	d_{1R}	-1	u_{1R}	-4
Q_{2L}	3	d_{2R}	0	u_{2R}	-1
Q_{3L}	1	d_{3R}	1	u_{3R}	1

Table 1: *Flavour charges of SM fermions.*

modes heavy. Following ref. [6], we take these two pairs to be weak triplets to generate masses for down-type quarks, and weak sixplets to generate masses for up-type quarks. Concerning the representation under $SU(2)_F$, from Table 1 we see that Q_L and u^c_L have flavour charges with absolute value up to four: the minimal choice is therefore a nineplet, which contains fields with $U(1)_F$ charges from -4 to 4. Summarizing, we have bulk fields in the following representations of $SU(3)_W \times SU(2)_F$:

- Bulk fields with negative overall parity:

$$\psi^d : \Big(\mathbf{3}, \mathbf{9}\Big)\,, \quad \tilde{\psi}^u : \Big(\mathbf{6}, \mathbf{9}\Big)\,, \tag{8}$$

- Bulk fields with positive overall parity:

$$\tilde{\psi}^d : \Big(\mathbf{3}, \mathbf{9}\Big)\,, \quad \psi^u : \Big(\mathbf{6}, \mathbf{9}\Big)\,. \tag{9}$$

The decomposition of the above representations of the $SU(3)_W \times SU(2)_F$ group under its $SU(2)_Y \times U(1)_Y \times U(1)_F$ subgroup, which we will need to determine the coupling of the bulk fields to the brane fields, has the following form:

$$\begin{aligned} \Big(\mathbf{3}, \mathbf{9}\Big) &\to \mathbf{2}_{\frac{1}{6},q} \oplus \mathbf{1}_{-\frac{1}{3},d}\,, \\ \Big(\mathbf{6}, \mathbf{9}\Big) &\to \mathbf{3}_{\frac{1}{3},Q} \oplus \mathbf{2}_{-\frac{1}{6},-q} \oplus \mathbf{1}_{-\frac{2}{3},-u}\,, \end{aligned} \tag{10}$$

with Q, q, u, d ranging from -4 to 4. The only components that have the right quantum numbers to couple to the brane fermions are the $SU(2)_W$ doublets and singlets, with $U(1)_F$ charges matching the SM ones given in Table 1.

The action of the orbifold projection on the bulk fermion fields is given by

$$\begin{aligned} P^{\mathbf{3}}_W &= \mathrm{diag}(-1,-1,1)\,, \quad P^{\mathbf{6}}_W = \mathrm{diag}(1,1,-1,1,-1,1)\,, \\ P^{\mathbf{9}}_F &= \mathrm{diag}(1,-1,1,-1,1,-1,1,-1,1)\,. \end{aligned} \tag{11}$$

This implies that after the projection the particle content is given by an electroweak doublet and an electroweak singlet with flavour charges ranging from -4 to 4, belonging to ψ^l if the flavour charge is even and to $\tilde{\psi}^l$ if it is odd. In other words, one and only one of the two bulk fields ψ^l and $\tilde{\psi}^l$ always has a component with the right quantum numbers to couple to the SM brane fermions.

The choice of the $SU(3)_W$ representation for the messenger fermions in the bulk influences only the overall magnitude of the induced Yukawa couplings, whereas the choice of the $SU(2)_F$ representation for these bulk fermions, together with the $U(1)_F$ charges for the matter brane fermions, determines the flavour structure.

3.3 Lagrangian

The structure of the Lagrangian is the same as in ref. [6]: in addition to the kinetic terms for the bulk and brane fields, we introduce an arbitrary bilinear mixing between them. The couplings of the three generations of left- and right-handed brane fields Q_L, u_R, d_R and their conjugates to the bulk fields ψ^l or $\tilde{\psi}^l$ are parametrized by couplings e_L^l and e_R^l with mass-dimension 1/2, in each sector $l = u, d$. Each brane field can couple either to ψ^l or $\tilde{\psi}^l$, and has therefore only one relevant coupling. To write these couplings more explicitly, it is convenient to embed the brane fields into new fields $\chi_{L,R}^{u,d}$, $\tilde{\chi}_{L,R}^{u,d}$ which have the same matrix structure as the representations of $SU(3)_W \times SU(2)_F$ to which the bulk fields belong, the extra entries being filled with zeroes, and then further combine left and right components into Dirac fields: $\chi^{u,d} = \chi_L^{u,d} + \chi_R^{u,d}$ and $\tilde{\chi}^{u,d} = \tilde{\chi}_L^{u,d} + \tilde{\chi}_R^{u,d}$. Correspondingly, it is convenient to embed the diagonal matrices of couplings e_1^l and e_2^l in family space into new diagonal matrices of couplings $\hat{e}_1^l$ and $\hat{e}_2^l$ in flavour space. In our example, we have:

$$\chi^d = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_W \otimes \begin{pmatrix} u_L^1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_W \otimes \begin{pmatrix} d_L^1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d_R^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F , \tag{12}$$

$$\tilde{\chi}^d = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ u_L^2 \\ 0 \\ u_L^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ d_L^2 \\ 0 \\ d_L^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ d_R^3 \\ 0 \\ d_R^1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F , \tag{13}$$

$$
\chi^u = \begin{pmatrix} 0\\0\\0\\0\\1\\0 \end{pmatrix}_W \otimes \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\\text{-}u_R^{c1} \end{pmatrix}_F + \begin{pmatrix} 0\\0\\1\\0\\0\\0 \end{pmatrix}_W \otimes \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\d_R^{c1} \end{pmatrix}_F + \begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix}_W \otimes \begin{pmatrix} u_L^{c1}\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}_F , \qquad (14)
$$

$$
\tilde{\chi}^u = \begin{pmatrix} 0\\0\\0\\0\\1\\0 \end{pmatrix}_W \otimes \begin{pmatrix} 0\\0\\0\\0\\0\\\text{-}u_R^{c3}\\0\\\text{-}u_R^{c2}\\0 \end{pmatrix}_F + \begin{pmatrix} 0\\0\\1\\0\\0\\0 \end{pmatrix}_W \otimes \begin{pmatrix} 0\\0\\0\\0\\0\\d_R^{c3}\\0\\d_R^{c2}\\0 \end{pmatrix}_F + \begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix}_W \otimes \begin{pmatrix} 0\\0\\0\\u_L^{c2}\\0\\u_L^{c3}\\0\\0\\0 \end{pmatrix}_F , \qquad (15)
$$

with $u^{1,2,3}$ and $d^{1,2,3}$ denoting the three generation quarks, and

$$
\begin{aligned}
\hat{e}_1^d &= \mathrm{diag}(e^d_{1,1}, e^d_{1,2}, 0, e^d_{1,3}, 0, 0, 0, 0, 0)\,, \\
\hat{e}_2^d &= \mathrm{diag}(0, 0, 0, e^d_{2,3}, e^d_{2,2}, e^d_{2,1}, 0, 0, 0)\,, \\
\hat{e}_1^u &= \mathrm{diag}(0, 0, 0, 0, 0, e^u_{1,3}, 0, e^u_{1,2}, e^u_{1,1})\,, \\
\hat{e}_2^u &= \mathrm{diag}(e^u_{2,1}, 0, 0, e^u_{2,2}, 0, e^u_{2,3}, 0, 0, 0)\,.
\end{aligned} \qquad (16)
$$

With this notation, and discarding irrelevant operators, which give negligible physical effects at low energies, the most general local Lagrangian for the light SM fields and the heavy flavour messengers that is compatible with the symmetries of the theory has the structure

$$
\mathcal{L} = \mathcal{L}^{\text{bulk}} + \delta(y)\mathcal{L}^0 + \delta(y-\pi R)\mathcal{L}^{\pi R}\,, \qquad (17)
$$

with

$$
\mathcal{L}^{\text{bulk}} = \sum_{l=u,d} \left[i\bar{\psi}^l\gamma^M D_M\psi^l + i\bar{\tilde{\psi}}^l\gamma^M D_M\tilde{\psi}^l - M_l(\bar{\psi}^l\tilde{\psi}^l + \bar{\tilde{\psi}}^l\psi^l) \right], \qquad (18)
$$

$$
\begin{aligned}
\mathcal{L}^0 &= i\bar{\chi}^d_L\gamma^\mu D_\mu\chi^d_L + i\bar{\tilde{\chi}}^d_L\gamma^\mu D_\mu\tilde{\chi}^d_L + i\bar{\chi}^u_R\gamma^\mu D_\mu\chi^u_R + i\bar{\tilde{\chi}}^u_R\gamma^\mu D_\mu\tilde{\chi}^u_R \\
&\quad + \left[\bar{\chi}^d_L\,\hat{e}_1^{d\dagger}\,\psi^d + \bar{\tilde{\chi}}^d_L\,\hat{e}_1^{d\dagger}\,\tilde{\psi}^d + \bar{\chi}^u_R\,\hat{e}_1^{u\dagger}\,\psi^u + \bar{\tilde{\chi}}^u_R\,\hat{e}_1^{u\dagger}\,\tilde{\psi}^u + \text{h.c.} \right], \qquad (19) \\
\mathcal{L}^{\pi R} &= i\bar{\chi}^u_L\gamma^\mu D_\mu\chi^u_L + i\bar{\tilde{\chi}}^u_L\gamma^\mu D_\mu\tilde{\chi}^u_L + i\bar{\tilde{\chi}}^d_R\gamma^\mu D_\mu\tilde{\chi}^d_R + i\bar{\chi}^d_R\gamma^\mu D_\mu\chi^d_R \\
&\quad + \left[\bar{\chi}^d_R\,\hat{e}_2^{d\dagger}\,\psi^d + \bar{\tilde{\chi}}^d_R\,\hat{e}_2^{d\dagger}\,\tilde{\psi}^d + \bar{\chi}^u_L\,\hat{e}_2^{u\dagger}\,\psi^u + \bar{\tilde{\chi}}^u_L\,\hat{e}_2^{u\dagger}\,\tilde{\psi}^u + \text{h.c.} \right]. \qquad (20)
\end{aligned}
$$

We have here tacitly excluded the possibility that odd operators might appear in the Lagrangian with coefficients that are themselves odd functions of the coordinates, behaving as constants in the bulk and jumping discontinuously at the branes. This is reasonable, since this kind of odd operators can be distinguished from ordinary even operators by a local parity symmetry [23]. It should be noticed, however, that imposing the latter symmetry significantly restricts the possibilities for canceling potential localized anomalies, since it forbids bulk Chern-Simons counterterms.

3.4 Structure of the induced couplings

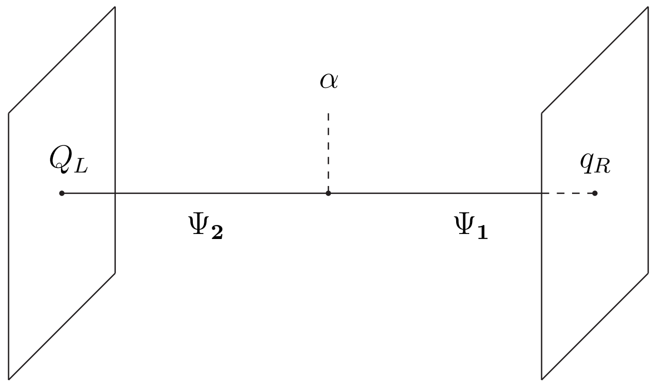

Figure 1: *Diagram inducing the effective mass in the presence of $SU(3)_W$ gauge symmetry breaking only: all the fields carry the same flavour charge. The insertion of α switches from the doublet to the singlet components of the bulk field. Here Q_L and q_R can be any left- and right-handed brane fermion and Ψ represents the pair of bulk fermions.*

As in ref. [6], effective Yukawa couplings, wave-function and vertex corrections for the standard matter fermions are generated in the low-energy effective theory defined by integrating out the heavy messenger fermions. For example, mass terms are obtained from the diagrams in Fig. 1. In this case, however, a given brane fermion can couple only to the flavour component of the bulk fermions that has the same $U(1)_F$ charge. This implies that a non-vanishing Yukawa coupling, wave-function or vertex correction is generated only if the involved brane fields have equal $U(1)_F$ charge, as long as the $U(1)_F$ symmetry stays unbroken, that is for $\beta = 0$. The other Yukawa couplings, wave-function and vertex corrections, involving brane fields with different $U(1)_F$ charges, can be generated only if the $U(1)_F$ symmetry is broken, that is $\beta \neq 0$. In this case, we have the diagrams in Fig. 2. Since $T_F^1 = (T_F^+ + T_F^-)/2$ can change the $U(1)_F$ charge by 1 unit, in order to connect two brane fields with charges differing by some integer k, we need $|k|$ insertions of βT_F^1. The effect will thus be of order $\beta^{|k|}$.

Actually, a further restriction turns out to be present, depending on whether k is

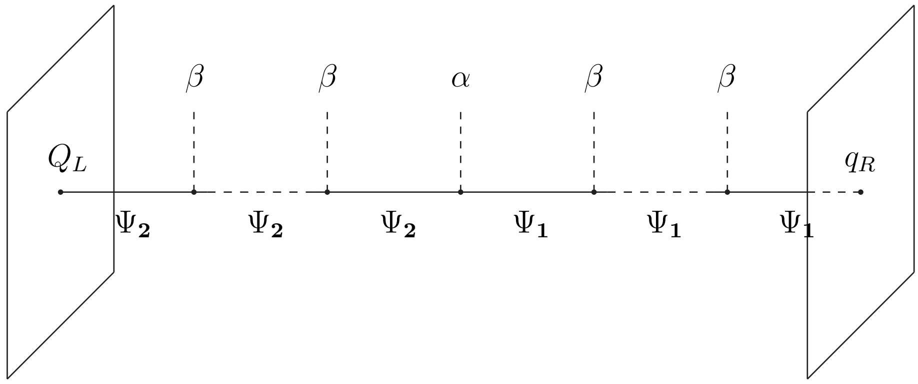

Figure 2: *Diagram inducing the effective mass in the presence of both gauge and flavour symmetry breaking. Each insertion of β switches between two components of bulk fields with flavour charges differing by one unit. The minimal number of such insertions that is needed to get a non-vanishing result is equal to the difference between the flavour charges of the left- and right-handed brane fields. Moreover, if this number is even, there is no mass insertion for the bulk fields, whereas when it is odd, there must be one mass insertion. Here Q_L and q_R can be any left- and right-handed brane fermion and Ψ represents the pair of bulk fermions.*

even or odd, as a consequence of the fact that the two types of bulk fermions ψ^l and $\tilde{\psi}^l$ can couple only to SM fermions with even and odd flavour charges respectively (in the example under consideration). The Yukawa couplings can be generated through the exchange of bulk fermions with an even or odd number of bulk mass insertions, *i.e.* with or without a $\psi^l \Leftrightarrow \tilde{\psi}^l$ transition. Non-vanishing entries can therefore be generated only with even or odd k, depending on whether the involved fields couple to the same or to a different kind of bulk fields ψ or $\tilde{\psi}$. Wave-function and vertex corrections can instead be generated only with an even number of bulk mass insertions, *i.e.* without an overall $\psi^l \Leftrightarrow \tilde{\psi}^l$ transition, and a non-vanishing correction is therefore generated only for k even.

The above reasoning shows that with a suitable assignment of the $SU(2)_F$ quantum numbers for brane and bulk fermions, it is possible to induce effective mass matrices with a pattern of matrix elements that can naturally explain the hierarchies among observed masses and mixing angles for matter fermions. Just as with the FN mechanism, the entries of the Yukawa couplings $Y_{IJ}^{u,d}$ (from now on we denote by I and J the family index) and the wave-function factors Z_{IJ}^Q and $Z_{IJ}^{u,d}$ for doublets and singlets respectively, can be expressed as powers of the order parameter $\lambda \equiv \pi\beta$ for the breaking of the Abelian flavour symmetry, modulo numerical factors of order one. The results can be written in terms of the charges q_I of the left-handed doublets Q

and the charges l_I of the right-handed singlets $l = u, d$ as:

$$Y^l_{IJ} \sim \lambda^{|q_I - l_J|}\,, \tag{21}$$

$$Z^Q_{IJ} \sim \begin{cases} \delta_{IJ} + \lambda^{|q_I - q_J|} & \text{for } |q_I - q_J| \text{ even} \\ \delta_{IJ} & \text{for } |q_I - q_J| \text{ odd} \end{cases}\,, \tag{22}$$

$$Z^l_{IJ} \sim \begin{cases} \delta_{IJ} + \lambda^{|l_I - l_J|} & \text{for } |l_I - l_J| \text{ even} \\ \delta_{IJ} & \text{for } |l_I - l_J| \text{ odd} \end{cases}\,. \tag{23}$$

The physical Yukawa couplings are obtained after performing a transformation on matter fermions that brings their kinetic terms to a canonical form. To do so (see for example [25]), we first diagonalize the wave functions as $Z^Q = U^{Q\dagger} D^Q U^Q$ and $Z^l = U^{l\dagger} D^l U^l$ in terms of some unitary matrices U^Q and U^l. In general, the diagonal matrices have entries of order one, $D^Q_{II} \sim 1$ and $D^l_{II} \sim 1$, but differ from the identity, while the U matrices have the same form as the wave-function corrections themselves, *i.e.* $U^Q_{IJ} \sim \delta_{IJ} + \lambda^{|q_I - q_J|}$ and $U^l_{IJ} \sim \delta_{IJ} + \lambda^{|l_I - l_J|}$. We then redefine the matter fields to be $\hat{Q} = \sqrt{D^Q} U^Q Q$ and $\hat{l} = \sqrt{D^l} U^l l$. In this way, the new wave-function factors are $\hat{Z}^q = 1$ and $\hat{Z}^l = 1$, whereas the new Yukawa coupling is given by $\hat{Y}^l = (D^Q)^{-\frac{1}{2}} U^Q Y^l U^{l\dagger} (D^l)^{-\frac{1}{2}}$. In terms of λ this means

$$\hat{Y}^l_{IJ} \sim \sum_{KL} \lambda^{|q_I - q_K| + |q_K - l_L| + |l_L - l_J|} \sim \lambda^{|q_I - l_J|}\,. \tag{24}$$

The last step, which follows from the inequality $|x| + |y| \geq |x + y|$, shows that as in standard 4D flavour models the wave function corrections do not mess up the structure of the Yukawa couplings.

Equations (24) realize the starting point for building interesting flavour models. However, a more careful analysis shows that our 5D construction presents a number of peculiarities that make it much more constrained than a generic 4D flavour model of the FN type, mostly due to the embedding in a non-Abelian group and to the structure of the mediator sector:

- The flavour charge is quantized and charge differences are integer.
- The precise numerical factors appearing in the induced Yukawa couplings are correlated, and contain potentially large group-theoretical coefficients.

3.5 Effective Lagrangian and induced couplings

We now present the explicit computation of the 4D effective Lagrangian, and in particular the corrections to the kinetic and mass terms for the SM fields. The leading effects are obtained by integrating out the heavy bulk fermions at the classical level.

The computation can be done along the lines of ref. [6]. In order to illustrate the procedure, we start by discussing in detail the d-quark sector. The up quark sector will then be easily explained.

3.5.1 Mode decomposition

In general, matter fields obey the compactification conditions in eqs. (1) and (3). In the following, $SU(3)_W$ and $SU(2)_F$ indices will be denoted by $i, j, \ldots$ and $a, b, \ldots$ respectively, and we work in a basis where the orbifold projection is diagonal, whereas, in general, the twist is non diagonal. For fixed electroweak and flavour indices, the γ_5 matrix acting in the orbifold projection causes the right- and left-handed matter field components to have different parity. Hence we can write the matter field components as follows:

$$\psi_{i,a}(x,y) = \psi_{i,a}^{+}(x,y) + \psi_{i,a}^{-}(x,y) \,, \tag{25}$$

where $\psi_{i,a}^{+}(x,y)$ and $\psi_{i,a}^{-}(x,y)$ are fields with positive and negative orbifold parity respectively and a given chirality which depends on i and a. Thus the superscript $\pm$ refers to the orbifold parity. The fields satisfying the condition (1) can be expanded in KK modes as

$$\begin{aligned}
\psi_{i,a}^{+}(x,y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{+\infty} (\frac{1}{\sqrt{2}})^{\delta_{n,0}} (\psi_{i,a}^{+})_n(x) \; \cos(\frac{ny}{R}) \,, \\
\psi_{i,a}^{-}(x,y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{+\infty} (\psi_{i,a}^{-})_n(x) \; \sin(\frac{ny}{R}) \,.
\end{aligned} \tag{26}$$

It is convenient to express $\psi_{i,a}^{+}(x,y)$, $\psi_{i,a}^{-}(x,y)$ as sums over all integer modes, both positive and negative; this is done by defining the negative modes of a given component as reflection of the positive modes: $(\psi_{i,a}^{\pm})_{-n}^{\dagger}(x) = \pm(\psi_{i,a}^{\pm})_n(x)$. The new mode expansion for untwisted fields is then

$$\begin{aligned}
\psi_{i,a}^{+}(x,y) &= \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{+\infty} \eta_n (\psi_{i,a}^{+})_n(x) \cos(\frac{ny}{R}) \,, \\
\psi_{i,a}^{-}(x,y) &= \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{+\infty} \eta_n (\psi_{i,a}^{-})_n(x) \sin(\frac{ny}{R}) \,,
\end{aligned} \tag{27}$$

where

$$\eta_n = \begin{cases} 1/\sqrt{2} & \text{if} \quad n \neq 0 \\ 1 & \text{if} \quad n = 0 \end{cases} \,. \tag{28}$$

We now switch to the basis in which the SS twist is diagonal. The eigenvectors $\Psi_{i,a}^{\pm}$ of the twist $T_{\mathcal{R},\mathcal{R}'}$ can be written as follows:

$$\Psi_{i,a}^{+} = (U_W)_{ij}^{\mathcal{R}} (V_F)_{ab}^{\mathcal{R}'} \psi_{j,b}^{+} \,, \qquad \Psi_{i,a}^{-} = (U_W)_{ij}^{\mathcal{R}} (V_F)_{ab}^{\mathcal{R}'} \psi_{j,b}^{-} \,, \tag{29}$$

where $(U_W)^{\mathcal{R}}$ and $(V_F)^{\mathcal{R}'}$ are two unitary matrices in the gauge and flavour space and the labels $\mathcal{R}$ and $\mathcal{R}'$ denote the representation to which the matter fields belong. Since the rotation mixes different indices i and a, corresponding to different chiralities, the fields $\Psi^+_{i,a}$ and $\Psi^-_{i,a}$ do not have a definite chirality, when expanded in KK modes as in eqs. (27). On the other hand, since the twist mixes fields with the same orbifold parity, it is possible to diagonalize it with transformations acting separately on ψ^+ and ψ^-.

In the new basis, the twist is diagonal. The explicit expressions for the unitary matrices $(U_W)^{\mathbf{3}}$, $(U_W)^{\mathbf{6}}$ and $(V_F)^{\mathbf{9}}$ that diagonalize the twist matrices $(T^6_W)^{\mathbf{3}}$, $(T^6_W)^{\mathbf{6}}$ and $(T^1_W)^{\mathbf{9}}$ to the forms $(t_W)^{\mathbf{3}} = \mathrm{diag}(1/2, 0, -1/2)$, $(t_W)^{\mathbf{6}} = \mathrm{diag}(1, 1/2, 0, 0, 0, -1/2, -1)$ and $(t_F)^{\mathbf{9}} = \mathrm{diag}(-4, 4, -3, 3, -2, 2, -1, 1, 0)$ are given by:

$$(U_W)^{\mathbf{3}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ \sqrt{2} & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad (U_W)^{\mathbf{6}} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 & \sqrt{2} & 1 \\ 0 & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & \sqrt{2} \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\sqrt{2} & 1 \end{pmatrix},$$

$$(V_F)^{\mathbf{9}} = \frac{1}{16} \begin{pmatrix} 1 & -\sqrt{8} & \sqrt{28} & -\sqrt{56} & \sqrt{70} & -\sqrt{56} & \sqrt{28} & -\sqrt{8} & 1 \\ 1 & \sqrt{8} & \sqrt{28} & \sqrt{56} & \sqrt{70} & \sqrt{56} & \sqrt{28} & \sqrt{8} & 1 \\ -\sqrt{8} & 6 & -\sqrt{56} & \sqrt{28} & 0 & -\sqrt{28} & \sqrt{56} & -6 & \sqrt{8} \\ -\sqrt{8} & -6 & -\sqrt{56} & -\sqrt{28} & 0 & \sqrt{28} & \sqrt{56} & 6 & \sqrt{8} \\ \sqrt{28} & -\sqrt{56} & 4 & \sqrt{8} & -\sqrt{40} & \sqrt{8} & 4 & -\sqrt{56} & \sqrt{28} \\ \sqrt{28} & \sqrt{56} & 4 & -\sqrt{8} & -\sqrt{40} & -\sqrt{8} & 4 & \sqrt{56} & \sqrt{28} \\ -\sqrt{56} & \sqrt{28} & \sqrt{8} & -6 & 0 & 6 & -\sqrt{8} & -\sqrt{28} & \sqrt{56} \\ -\sqrt{56} & -\sqrt{28} & \sqrt{8} & 6 & 0 & -6 & -\sqrt{8} & \sqrt{28} & \sqrt{56} \\ \sqrt{70} & 0 & -\sqrt{40} & 0 & 6 & 0 & -\sqrt{40} & 0 & \sqrt{70} \end{pmatrix}. \quad (30)$$

Let us now define

$$\hat{\Psi}_{i,a} = \left(\Psi^+_{i,a}, \Psi^-_{i,a} \right) . \quad (31)$$

In this basis, the effect of the twist amounts to shifting the masses of the KK modes by the quantity $2(t_W)_{ii}\alpha + 2(t_F)_{aa}\beta$. Therefore, suppressing all the indices, the new KK mass spectrum is given by

$$m_n(\alpha, \beta) = \frac{n\sigma_1 + (2t_W\alpha + 2t_F\beta)\mathbb{1}}{R}, \quad (32)$$

where σ_1 and $\mathbb{1}$ act at fixed i, a on the space $(\Psi^+_{i,a}, \Psi^-_{i,a})$ and connect terms with opposite and equal orbifold parity respectively. Finally, a complete diagonalization can be achieved by mixing states with opposite orbifold chirality:

$$(\Psi_{i,a})_n = \eta_n \left[(\Psi^+_{i,a})_n + (\Psi^-_{i,a})_n\right] \tag{33}$$

where now positive and negative n components of $\Psi_{i,a}$ are independent, and their mass is simply given by

$$m_n(\alpha,\beta) = \frac{n + (2t_W\alpha + 2t_F\beta)}{R}\,. \tag{34}$$

3.5.2 Construction of the effective Lagrangian

In order to derive the effective Lagrangian that is induced for the SM fermions by integrating out the bulk fermions at the classical level, we use for the latter the mode decomposition derived in previous subsection, and switch to 4D momentum space. The relevant linear and quadratic parts of the Lagrangian for the modes of the bulk fermions then becomes

$$\mathcal{L} = \sum_{n=-\infty}^{\infty} \left[\mathcal{L}_n^{\rm bulk} + \mathcal{L}_n^0 + (-1)^n \mathcal{L}_n^{\pi R}\right], \tag{35}$$

where

$$\begin{aligned}
\mathcal{L}_n^{\rm bulk} &= \sum_{l=u,d} \left[\bar\Psi^l_n(\not p - m_n)\Psi^l_n + \bar{\tilde\Psi}^l_n(\not p + m_n)\tilde\Psi^l_n - M_l\Big(\bar\Psi^l_n\tilde\Psi^l_n + \bar{\tilde\Psi}^l_n\Psi^l_n\Big)\right], &(36)\\
\mathcal{L}_n^0 &= \frac{1}{\sqrt{2\pi R}}\Big[\bar\chi^d_L(U_W V_F\,\hat e^d_1)^\dagger\Psi^d_n + \bar\chi^d_L(U_W V_F\,\hat e^d_1)^\dagger\tilde\Psi^d_n \\
&\qquad + \bar\chi^u_R(U_W V_F\,\hat e^u_1)^\dagger\Psi^u_n + \bar{\tilde\chi}^u_R(U_W V_F\,\hat e^u_1)^\dagger\tilde\Psi^u_n + \text{h.c.}\Big], &(37)\\
\mathcal{L}_n^{\pi R} &= \frac{1}{\sqrt{2\pi R}}\Big[\bar\chi^d_R(U_W V_F\,\hat e^d_2)^\dagger\Psi^d_n + \bar{\tilde\chi}^d_R(U_W V_F\,\hat e^d_2)^\dagger\tilde\Psi^d_n \\
&\qquad + \bar\chi^u_L(U_W V_F\,\hat e^u_2)^\dagger\Psi^u_n + \bar{\tilde\chi}^u_L(U_W V_F\,\hat e^u_2)^\dagger\tilde\Psi^u_n + \text{h.c.}\Big]. &(38)
\end{aligned}$$

From these expressions it is clear that the physics of the light modes depends on the mass mixings $\hat e_{1,2}/\sqrt{2\pi R}$ encoding the couplings between brane and bulk modes and on the masses M_l for the bulk modes. The relevant dimensionless parameters are then the products of these masses with the length πR of the internal dimension:

$$\epsilon^l_{1,2} = \sqrt{\pi R/2}\,\hat e^l_{1,2}\,, \qquad x_l = \pi R M_l\,. \tag{39}$$

For convenience, we also define the ϵ couplings in the basis of diagonal twist:

$$\varepsilon^l_{1,2} = U_W V_F \epsilon^l_{1,2}\,. \tag{40}$$

The mass and wave-function corrections are generated by diagrams similar to the ones in Figs. 1 and 2. The result depends on the vacuum expectation value of the

Higgs field A_5 through the dimensionless parameter $\alpha = g_5 R\langle A_5\rangle/2$, and on the phase β induced by the twist. If the flavour symmetry were local, β would be related to the fifth component of the corresponding $SU(2)_F$ field. For a global symmetry, β is a free parameter related to the phase accumulated by the twist. The tree-level propagator in momentum space for the KK modes of Ψ^l and $\tilde{\Psi}^l$ is given, in two-by-two matrix notation, by the following expression:

$$S_l^n = \frac{i}{p^2 - m_n(\alpha,\beta)^2 - (M_l)^2}\begin{pmatrix} \not{p} + m_n(\alpha,\beta) & M_l \\ M_l & \not{p} - m_n(\alpha,\beta)\end{pmatrix}. \tag{41}$$

The effective action is then obtained by integrating out the bulk fermions at the classical level, using the above propagator and treating the brane fields as sources, as in ref. [6]. We find a contribution to the effective Lagrangian in momentum space of the form $\mathcal{L}_d^{\rm eff} = \mathcal{L}_d^{\rm kin} + \mathcal{L}_d^{\rm m}$, where $\mathcal{L}_d^{\rm kin}$ contains the kinetic term corrections of SM matter fields and is given by

$$\begin{aligned}\mathcal{L}_d^{\rm kin} = \; & \overline{\chi}_L^d \varepsilon_1^{d\dagger}\; (\not{p})\; F(p, M_d, 2t_W\alpha + 2t_F\beta)\; \varepsilon_1^d \chi_L \\ & + \overline{\chi}_R^d \varepsilon_2^{d\dagger}\; (\not{p})\; F(p, M_d, 2t_W\alpha + 2t_F\beta)\; \varepsilon_2^d \chi_R^d \\ & + \overline{\tilde{\chi}}_L^d \varepsilon_1^{d\dagger}\; (\not{p})\; F(p, M_d, 2t_W\alpha + 2t_F\beta)\; \varepsilon_1^d \tilde{\chi}_L^d \\ & + \overline{\tilde{\chi}}_R^d \varepsilon_2^{d\dagger}\; (\not{p})\; F(p, M_d, 2t_W\alpha + 2t_F\beta)\; \varepsilon_2^d \tilde{\chi}_R^d\,,\end{aligned} \tag{42}$$

whereas $\mathcal{L}_d^{\rm m}$ contains the effective mass terms and is given by

$$\begin{aligned}\mathcal{L}_d^{\rm m} = \; & \frac{1}{\pi R}\Big\{ \Big[\overline{\chi}_L^d \varepsilon_1^{d\dagger}\; G_1(p, M_d, 2t_W\alpha + 2t_F\beta)\; \varepsilon_2^d \chi_R^d + \text{h.c.}\Big] \\ & - \Big[\overline{\tilde{\chi}}_L^d \varepsilon_1^{d\dagger}\; G_1(p, M_d, 2t_W\alpha + 2t_F\beta)\; \varepsilon_2^d \tilde{\chi}_R^d + \text{h.c.}\Big] \\ & + \Big[\overline{\chi}_L^d \varepsilon_1^{d\dagger}\; G_2(p, M_d, 2t_W\alpha + 2t_F\beta)\; \varepsilon_2^d \tilde{\chi}_R^d + \text{h.c.}\Big] \\ & + \Big[\overline{\tilde{\chi}}_L^d \varepsilon_1^{d\dagger}\; G_2(p, M_d, 2t_W\alpha + 2t_F\beta)\; \varepsilon_2^d \chi_R^d + \text{h.c.}\Big]\Big\}\,.\end{aligned} \tag{43}$$

For the u quarks, one can proceed exactly in the same way. In the Euclidean space-time, the explicit expressions of the functions F, G_1 and G_2 are given by

$$\begin{aligned} F(p,M,\rho) &= \frac{1}{(\pi R)^2}\sum_{n=-\infty}^{\infty}\frac{1}{p^2 + (\frac{n+\rho}{R})^2 + M^2} \\ &= \frac{1}{\pi R\sqrt{p^2+M^2}}\,\mathrm{Re}\left[\sum_{m=-\infty}^{\infty} e^{-|2m|\pi R\sqrt{p^2+M^2}} e^{-|2m|\pi i\rho}\right] \\ &= \frac{1}{\pi R\sqrt{p^2+M^2}}\,\mathrm{Re}\coth(\pi R\sqrt{p^2+M^2} + i\rho\pi)\,, \\ G_1(p,M,\rho) &= \frac{1}{\pi R}\sum_{n=-\infty}^{\infty}(-1)^n\frac{\frac{n+\rho}{R}}{p^2 + (\frac{n+\rho}{R})^2 + M^2} \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Im}\left[\sum_{m=-\infty}^{\infty} e^{-|2m+1|\pi R\sqrt{p^2+M^2}} e^{-|2m+1|\pi i\rho}\right] \qquad (44)\\
&= -\mathrm{Im}\ \mathrm{csch}\,(\pi R\sqrt{p^2+M^2}+i\rho\pi)\,,\\
G_2(p,M,\rho) &= \frac{1}{\pi R}\sum_{n=-\infty}^{\infty}(-1)^n\frac{M}{p^2+(\frac{n+\rho}{R})^2+M^2}\\
&= \frac{M}{\sqrt{p^2+M^2}}\,\mathrm{Re}\left[\sum_{m=-\infty}^{\infty} e^{-|2m+1|\pi R\sqrt{p^2+M^2}} e^{-|2m+1|\pi i\rho}\right]\\
&= \mathrm{Re}\ \mathrm{csch}\,(\pi R\sqrt{p^2+M^2}+i\rho\pi)\,.
\end{aligned}
$$

3.6 Low energy limit

We now study the effective Lagrangian in the low energy limit $p^2 \ll M_l^2$. In this limit, the non-local p-dependent couplings of eqs. (42)-(43) and the analogous terms for up-type quarks reduce to local kinetic and mass terms. After diagonalization and canonical normalization of the physical fields, these generate the physical fermion masses and mixings.

In the low-energy limit $p^2 \ll M_l^2$, the momentum variable $\pi R\sqrt{p^2+M_l^2}$ reduces to the constant parameter x_l defined in eq. (39). The functions F, G_1 and G_2 become simple trigonometric functions of the three parameters x_l, α and β. Notice moreover that not all the functional dependence on the parameters α and β is relevant. First of all, for various phenomenological reasons that were explained in ref. [6] and in sec. 2 and that we will review below, we must assume that α is small and retain only the leading effects that are at most linear in α. Since α is related to the VEV of the Higgs field, this corresponds to keeping only those effective operators that involve at most one Higgs field. Moreover, it is easy to check that only even powers of β are relevant in F and G_1, and similarly only odd powers of β are relevant in G_2, due to the flavour quantum numbers of the brane fields (see eqs. (12)-(15)). The above functions can therefore be effectively substituted with:

$$
\begin{aligned}
F(p,M_l,2t_W\alpha+2t_F\beta) &\Rightarrow f(x_l,2t_F\beta)\,,\\
G_1(p,M_l,2t_W\alpha+2t_F\beta) &\Rightarrow (2\pi t_W\alpha)\,g_1(x_l,2t_F\beta)\,,\\
G_2(p,M_l,2t_W\alpha+2t_F\beta) &\Rightarrow (2\pi t_W\alpha)\,g_2(x_l,2t_F\beta)\,,
\end{aligned}
\tag{45}
$$

where

$$
\begin{aligned}
f(x_l,2t_F\beta) &= \frac{1}{x_l}\,\mathrm{Re}\ \coth\left[x_l+2\pi i t_F\beta\right],\\
g_1(x_l,2t_F\beta) &= \mathrm{Re}\left(\coth\left[x_l+2\pi i t_F\beta\right]\,\mathrm{csch}\left[x_l+2\pi i t_F\beta\right]\right),\\
g_2(x_l,2t_F\beta) &= \mathrm{Im}\left(\coth\left[x_l+2\pi i t_F\beta\right]\,\mathrm{csch}\left[x_l+2\pi i t_F\beta\right]\right).
\end{aligned}
\tag{46}
$$

Let us be more quantitative on the range of values that the above dimensionless parameters are allowed to take by basic phenomenological constraints. A first important requirement is that $m_W \ll 1/R$, since indirect experimental constraints imply that the compactification scale should be at least a few TeV. A second requirement is that $M_l \gg m_W$, in such a way that even the lightest modes of the extra bulk fermions that we have introduced are heavy enough to satisfy direct experimental constraints. This two conditions imply respectively the following restrictions:

$$\pi\alpha \ll 1\,, \qquad \pi\alpha \ll x_l\,. \tag{47}$$

Notice that the above conditions justify in a more precise way the approximation done to derive eqs. (46). Notice also that they do not fix the size of the parameters x_l related to the masses of the bulk fermions.

The total effective Lagrangian is obtained by adding up the first rows of the brane Lagrangians (19)-(20) and the correction $\mathcal{L}_u^{\rm eff} + \mathcal{L}_d^{\rm eff}$. After simplifying the traces over gauge and flavour indices, which in the approximation leading to eqs. (46) are disentangled, it can be rewritten in terms of the original three generations of fields u_L, u_R, d_L, d_R and couplings $\epsilon^l_{L,R}$, and has the following general form:

$$\begin{aligned}\mathcal{L}^{\rm phen} = \sum_{a,b=1}^{3} \Big\{ & \bar u_L^a \not p\, \mathcal{Z}_{ab}^{u_L} u_L^b + \bar u_R^a \not p\, \mathcal{Z}_{ab}^{u_R} u_R^b + \left(\bar u_L^a \mathcal{M}_{ab}^u u_R^b + \text{h.c.}\right) \\ & + \bar d_L^a \not p\, \mathcal{Z}_{ab}^{d_L} d_L^b + \bar d_R^a \not p\, \mathcal{Z}_{ab}^{d_R} d_R^b + \left(\bar d_L^a \mathcal{M}_{ab}^d d_R^b + \text{h.c.}\right) \Big\}\,. \end{aligned} \tag{48}$$

3.7 Fermion masses and mixings

Let us now specialize to the case at hand and work out in detail the expressions for fermion masses and mixing angles that can be obtained from $\mathcal{L}^{\rm phen}$ in eq. (48). In order to make the physics behind $\mathcal{L}^{\rm phen}$ clear, it is instructive to study the two limits $x_l \ll 1$ and $x_l \gg 1$, where many of the expressions drastically simplify. We start by discussing the case $x_l \gg 1$, since the corrections to the quark field wave functions are simpler in this limit.

$\mathbf{x_l \gg 1}$

In the limit of $x_l \gg 1$, the functions in eqs. (46) take the form

$$\begin{aligned} f(x_l, 2t_F\beta) &\sim \frac{1}{x_l}\Big(1 + 2\,e^{-2x_l}\cos 4\pi t_F\beta\Big) \sim \frac{1}{x_l}\,, \\ g_1(x_l, 2t_F\beta) &\sim 2e^{-x_l}\cos(2\pi t_F\beta)\,, \\ g_2(x_l, 2t_F\beta) &\sim -2e^{-x_l}\sin(2\pi t_F\beta)\,. \end{aligned} \tag{49}$$

Under the hypothesis that $\lambda = \pi\beta$ is of the order of the Cabibbo angle, we expand up to the appropriate order equations (49). For the case at hand, this order is λ^8. Our expansion gives the following effective mass matrices:

$$\mathcal{M}^d_{ad} = -\, m_W\, e^{-x_d}\, (\mathcal{E}^d_1)^\dagger_{ab}\, \widetilde{Y}^d_{bc}\, (\mathcal{E}^d_2)_{cd} \tag{50}$$

$$\mathcal{M}^u_{ad} = -\sqrt{2}\, m_W\, e^{-x_u}\, (\mathcal{E}^u_1)^\dagger_{ab}\, \widetilde{Y}^u_{bc}\, (\mathcal{E}^u_2)_{cd}\,, \tag{51}$$

where, keeping only the leading terms for each entry,

$$\widetilde{Y}^d = \begin{pmatrix} -2\sqrt{14}\,\lambda^5 & \sqrt{70}\,\lambda^4 & 2\sqrt{14}\,\lambda^3 \\ -5\sqrt{7}\,\lambda^4 & 2\sqrt{35}\,\lambda^3 & 3\sqrt{7}\,\lambda^2 \\ 10\,\lambda^2 & -2\sqrt{5}\,\lambda & -1 \end{pmatrix}, \tag{52}$$

$$\widetilde{Y}^u = \begin{pmatrix} \lambda^8 & -2\sqrt{14}\,\lambda^5 & 2\sqrt{14}\,\lambda^3 \\ 2\sqrt{2}\,\lambda^7 & -5\sqrt{7}\,\lambda^4 & 3\sqrt{7}\,\lambda^2 \\ -2\sqrt{14}\,\lambda^5 & 10\,\lambda^2 & -1 \end{pmatrix}, \tag{53}$$

and for convenience we have defined

$$\mathcal{E}^d_k = \mathrm{diag}(\epsilon^d_{k,1}, \epsilon^d_{k,2}, \epsilon^d_{k,3})\,, \qquad \mathcal{E}^u_k = \mathrm{diag}(\epsilon^u_{k,1}, \epsilon^u_{k,2}, \epsilon^u_{k,3})\,. \tag{54}$$

We see from eqs. (52) and (53) that we have obtained the desired structure in powers of λ, but the group-theoretical coefficients are large and modify substantially masses and mixing angles. However, since these coefficients are entirely fixed by the flavour symmetry, one can tolerate their presence and design the texture in such a way to obtain suitable additional powers of λ to compensate for the fact that they are not of order 1. In other words, we can still obtain a good description of masses and mixings in terms of a single parameter λ, but with non-conventional textures, which take into account the fact that the numerical coefficients can become of order λ^{-1} or larger. We will discuss in sec. 5.1 an explicit realization of this idea. The wave-function corrections are instead given by

$$\begin{aligned} \mathcal{Z}^{u_L} &= \mathcal{Z}^{d_L} = \mathbf{1} + \frac{1}{x_d}\mathcal{E}^{d\dagger}_1\mathcal{E}^d_1 + \frac{1}{x_u}\mathcal{E}^{u\dagger}_1\mathcal{E}^u_1\,, \\ \mathcal{Z}^{d_R} &= \mathbf{1} + \frac{1}{x_d}\mathcal{E}^{d\dagger}_2\mathcal{E}^d_2\,, \\ \mathcal{Z}^{u_R} &= \mathbf{1} + \frac{1}{x_u}\mathcal{E}^{u\dagger}_2\mathcal{E}^u_2\,. \end{aligned} \tag{55}$$

The physical quark Yukawa couplings are obtained by redefining the quark fields to reabsorb the wave-function corrections $\mathcal{Z}$. The structure of the latter is such that the physical mass matrix cannot grow indefinitely when the $\epsilon^{u,d}_a$ are increased. The reason is that the ϵ-parameters encode the mixing between bulk and brane fermions. The resulting mass of the hybrid fields must therefore interpolate between the value that one would get for a bulk field ($\epsilon_a^{u,d} \to \infty$) and the vanishing value that one would get for a brane field ($\epsilon_a^{u,d} \to 0$).

In the simple case where the ϵ_a^u and ϵ_a^d are real, it is useful to introduce the following bulk-brane mixing angles:

$$
\begin{aligned}
\alpha_{1,a}^u &= \mathrm{Arctan}\left(\frac{\sqrt{1/x_u}\,\epsilon_{1,a}^u}{\sqrt{1+1/x_d\,\left(\epsilon_{1,a}^d\right)^2}}\right) , \qquad & \alpha_{1,a}^d &= \mathrm{Arctan}\left(\frac{\sqrt{1/x_d}\,\epsilon_{1,a}^d}{\sqrt{1+1/x_u\,\left(\epsilon_{1,a}^u\right)^2}}\right) , \\
\alpha_{2,a}^u &= \mathrm{Arctan}\left(\sqrt{1/x_u}\,\epsilon_{2,a}^u\right) , & \alpha_{2,a}^d &= \mathrm{Arctan}\left(\sqrt{1/x_d}\,\epsilon_{2,a}^d\right) .
\end{aligned}
\tag{56}
$$

The physical masses, obtained by rescaling the quarks fields in order to have a canonically normalized kinetic term, e.g. $\bar{u}_L \not{p}\, u_L$, are then found to be:

$$
\begin{aligned}
\mathcal{M}^u &= -\sqrt{2}\,x_u\,e^{-x_u}\,m_W\,S_1^u\,\tilde{Y}^u\,S_2^u , \\
\mathcal{M}^d &= -\,x_d\,e^{-x_d}\,m_W\,S_1^d\,\tilde{Y}^d\,S_2^d ,
\end{aligned}
\tag{57}
$$

where

$$
\begin{aligned}
S_1^l &= \mathrm{diag}(\sin\alpha_{1,1}^l, \sin\alpha_{1,2}^l, \sin\alpha_{1,3}^l) , \\
S_2^l &= \mathrm{diag}(\sin\alpha_{2,1}^l, \sin\alpha_{2,2}^l, \sin\alpha_{2,3}^l) .
\end{aligned}
\tag{58}
$$

At this point, we proceed exactly as in the SM, by diagonalizing the mass matrices via a bi-unitary transformation

$$
u_{L,R}^\alpha \to \mathcal{U}_{L,R}^{\alpha\beta} u_{L,R}^\beta , \quad d_{L,R}^\alpha \to \mathcal{D}_{L,R}^{\alpha\beta} d_{L,R}^\beta \;\;\Rightarrow\;\; V_{CKM} = \mathcal{U}_L^\dagger \mathcal{D}_L . \tag{59}
$$

The masses in eq. (57) are suppressed with respect to $m_W = \alpha/R$ by the factor $x_l e^{-x_l}$, which is a small parameter since we are now considering the limit $x_l \gg 1$, and by a trigonometric factor parametrizing the bulk-brane mixing. In this situation we therefore obtain mass matrices with an absolute scale much smaller than the W mass:

$$
\mathcal{M}^{u,d} \sim x_{u,d} e^{-x_{u,d}} m_W \ll m_W . \tag{60}
$$

This is phenomenologically not acceptable for the top quark mass. Notice, nevertheless, that the exponential dependence on x_u and x_d of the overall scale for the masses in the up and down sectors could allow to account for the significant hierarchy observed between the latter through a modest hierarchy between the two parameters x_u and x_d. The physical origin of the above exponential suppression is related to the higher-dimensional gauge symmetry constraining the Higgs interactions. More precisely, the only invariant Yukawa-type effective operators turn out to involve the Higgs field in the form of a Wilson line, which connects the two branes where the relevant left- and right-handed fermions are located and winds at least once around the internal interval [6]. The exchanged bulk fermion of mass M_l must therefore propagate at least over a distance πR and this implies a suppression factor proportional to e^{-x_l} in the limit $x_l \gg 1$.

$\mathbf{x_l \ll 1}$

In the limit of $x_l \ll 1$, the functions in eqs. (46) also simplify. Actually, to have a significant simplification we really need $x_l \ll \pi\beta$, but deriving an asymptotic expression in this limit would contrast with the philosophy of flavour models, which always assumes a power expansion in the order parameter $\pi\beta \ll 1$. For this reason, we will consider this situation only for the case of flavour-singlet bulk fermions, which are blind to the flavour symmetry. We will see in sec. 5.2 that it is possible to take advantage of the possibility of adding such a flavour-neutral fermion, in addition to flavour-charged ones, to improve the magnitude of the masses of the third familiy of quarks. We therefore set $\pi t_F \beta$ to 0. Under these assumptions, the functions of eqs. (46) reduce to

$$f(x_l, 0) \simeq \frac{1}{x_l^2}\,, \qquad g_1(x_l, 0) \simeq \frac{1}{x_l^2}\,, \qquad g_2(x_l, 0) \simeq 0\,. \tag{61}$$

The induced wave functions are then given by (there is no matrix structure here since we are considering flavour singlets):

$$\mathcal{Z}_L^l \simeq 1 + \frac{1}{x_d^2}\epsilon_L^{d\dagger}\epsilon_L^d + \frac{1}{x_u^2}\epsilon_L^{u\dagger}\epsilon_L^u\,, \qquad \mathcal{Z}_R^l \simeq 1 + \frac{1}{x_l^2}\epsilon_R^{l\dagger}\epsilon_R^l\,. \tag{62}$$

Similarly, the induced masses are found to be

$$\mathcal{M}^u \simeq \sqrt{2}\frac{1}{x_u^2}\epsilon_L^{u\dagger}\epsilon_R^u\, m_W\,, \qquad \mathcal{M}^d \simeq \frac{1}{x_d^2}\epsilon_L^{d\dagger}\epsilon_R^d\, m_W\,. \tag{63}$$

The physical quark masses emerging after canonical normalization are then found to be

$$m^u \simeq \sqrt{2}\sin\alpha_L^u \sin\alpha_R^u\, m_W\,, \qquad m^d \simeq \sin\alpha_L^d \sin\alpha_R^d\, m_W\,, \tag{64}$$

where now

$$\begin{aligned} \alpha_L^u &= \arctan\sqrt{\frac{(\epsilon_L^u)^2/x_u^2}{1+(\epsilon_L^d)^2/x_d^2}}\,, \qquad & \alpha_L^d &= \arctan\sqrt{\frac{(\epsilon_L^d)^2/x_d^2}{1+(\epsilon_L^u)^2/x_u^2}}\,, \\ \alpha_R^u &= \arctan\sqrt{(\epsilon_R^u)^2/x_u^2}\,, & \alpha_R^d &= \arctan\sqrt{(\epsilon_R^d)^2/x_d^2}\,. \end{aligned} \tag{65}$$

In this case the quark masses are of order m_W. In this situation we can therefore achieve mass matrices with a trivial flavour structure but a sizable magnitude:

$$\frac{m^l}{m_W} \sim 1\,. \tag{66}$$

Notice also that for $\epsilon_{L,R}^l \sim 1$ the angles (65) parametrizing the brane-bulk mixings tend to the large values $\alpha_L^u \simeq \delta$, $\alpha_L^d \simeq \pi/2 - \delta$ and $\alpha_R^l \simeq \pi/2$, with $\delta = \mathrm{Arctan}(\epsilon_L^u/\epsilon_L^d x^d/x^u)$, reflecting the fact that since $\epsilon_{L,R}^l \gg x_l$ the brane-bulk mixing is maximal; the masses (64) tend then to $m^d \simeq \cos\delta\, m_W$ and $m^u \simeq \sqrt{2}\sin\delta\, m_W$.

$x_l \sim 1$

In the general case $x_l \sim 1$, the effect of the wave-function corrections on the $\mathcal{O}(1)$ numerical coefficients in the physical Yukawa couplings depends in a complicated way on the parameters x_l and $\epsilon^l_{1,2}$, and must be separately studied for each point in this parameter space. We will present the results of this general analysis in sec. 5. It is however clear that the induced masses will always have a scale that is parametrically given by m_W times some suppression factor dictated by the spontaneously broken flavor symmetry. As already mentioned in the introduction, the large top mass is therefore generically difficult to accommodate in this framework [6].

4 Generalization to arbitrary representations

In this section, we generalize the construction discussed above to arbitrary representations of the electroweak and flavour groups.

We generalize the minimal choice of ref. [6] by taking ψ^d and ψ^u to belong respectively to the $(\mathbf{n^d_W+1})(\mathbf{n^d_W+2})/\mathbf{2}$ (n^d_W times symmetric) and $(\mathbf{n^u_W+1})(\mathbf{n^u_W+2})/\mathbf{2}$ (n^u_W times symmetric) of $SU(3)_W$; the $\mathbf{3}$ and $\mathbf{6}$ that were used in ref. [6] and in the previous discussion correspond to the particular cases $n^d_W = 1$ and $n^u_W = 2$. Moreover, we take these fields to belong to the $\mathbf{2j_F+1}$ (spin-j_F) representation of $SU(2)_F$, so that there are now $2j_F+1$ replicas of them with identical $SU(2)_L \times U(1)_Y$ quantum numbers but different $U(1)_F$ charges. Summarizing, we have bulk fields in the following representations of $SU(3)_W \times SU(2)_F$:

$$\psi^l, \tilde{\psi}^l : \Big(\frac{(\mathbf{n^l_W+1})(\mathbf{n^l_W+2})}{\mathbf{2}}, \mathbf{2j_F+1}\Big)\,, \tag{67}$$

The decomposition of the above general representations of the $SU(3)_W \times SU(2)_F$ group under its $SU(2)_L \times U(1)_Y \times U(1)_F$ subgroup, which we need to determine the coupling of the bulk fields to the brane fields, has the following form:

$$\Big(\frac{(\mathbf{n^l_W+1})(\mathbf{n^l_W+2})}{\mathbf{2}}, \mathbf{2j_F+1}\Big) \to \bigoplus_{j_W=0}^{n^l_W/2} \bigoplus_{m_{j_F}=-j_F}^{j_F} (\mathbf{2j_W+1})_{j_W - n^l_W/3, m_{j_F}}\,. \tag{68}$$

We get therefore a set of representations of $SU(2)_L$ with half-integer spins j_W ranging from 0 to $n^l_W/2$, canonically normalized $U(1)_Y$ charge equal to $j_W - n^l_W/3$ and $U(1)_F$ charges m_{j_F} ranging from $-j_F$ to j_F. The only components that have the right quantum numbers to couple to the brane fermions are the $SU(2)_L$ doublets and singlets with $j_W = 1/2$ and $j_W = 0$, which have $U(1)_Y$ charge[3] equal to $1/2 - n^l_W/3$ and $-n^l_W/3$, and $U(1)_F$ charges ranging from $-j_F$ to j_F.

[3]Notice that these have automatically the right hypercharge to couple to the standard left-handed doublets and right-handed singlets only in the special case $n^d_W = 1$ and $n^u_W = 2$ chosen in ref. [6]. For

The action of the orbifold projection and the SS twist on the bulk fermion fields can be easily deduced by using some simple group-theoretical techniques. In the electroweak sector, the completely symmetric representations of $SU(3)_W$ we are considering contain states with values of the two Cartan generators T^3_W and $2\,T^8_W/\sqrt{3}$ that fill an equilateral triangle in the corresponding plane. This triangle is oriented with its tip at the bottom and one of his sides at the top and horizontal. It can be sliced in essentially two different ways in a sum of lines, corresponding to decompositions with respect to nonequivalent but isomorphic maximal subgroups. Slicing the $SU(3)_W$ representation horizontally in rows, one obtains the decomposition with respect to the $SU(2)_L \times U(1)_Y$ preserved by the orbifold projection, with generators $T_W^{1,2,3}$ and $T^8_W/\sqrt{3}$. It is then clear that the generator $T^8_W/\sqrt{3}$ appearing in the orbifold projection has a definite value for each $SU(2)_L \times U(1)_Y$ representation appearing in the decomposition (68). More precisely, it acts as $j_W - n^l_W/3$ on the component with $SU(2)_L$ spin j_W. In matrix form, where these components are ordered in block with a fixed j_W ranging from $n^l_W/2$ to 0 in decreasing order and sub-entries corresponding to m_{j_W} ranging from $-j_W$ to j_W in increasing order[4], the orbifold twist has therefore the following form:

$$P_W^{\mathbf{(n^l_W+1)(n^l_W+2)/2}} = \mathrm{diag}(\underbrace{(-1)^{n^l_W},\ldots,(-1)^{n^l_W}}_{n^l_W+1 \;\text{ times}};\ldots;1,1,1;-1,-1;1)\,. \tag{69}$$

Slicing the $SU(3)_W$ representation diagonally, that is parallel to one of the two non-horizontal sides of the triangle, one obtains the decomposition with respect to a different $SU(2)' \times U(1)'$ subgroup associated to the Scherk-Schwarz twist, with generators $T^{6,7}_W$, $(-T^3_W + \sqrt{3}\,T^8_W)/2$ and $(-T^3_W - T^8_W/\sqrt{3})/2$. For each state of the original representation, the $SU(2)'$ spin j' and its third component $m_{j'}$ are related to the the $SU(2)_L$ spin j_W and its third component m_{j_W} by the relations $j' = (n^l_W - j_W - m_{j_W})/2$ and $m_{j'} = (-n^l_W + 3j_W - m_{j_W})/2$. This decomposition is useful to determine the action of the generator T^6_W appearing in the Scherk-Schwarz twist. Indeed, one can rewrite $T^6_W = (T^+_W + T^-_W)/2$ in terms of the raising and lowering operators $T^\pm_W = T^6_W \pm iT^7_W$ of the $SU(2)'$ subgroup. These leave j' unchanged and raise/lower $m_{j'}$ by 1 unit, or equivalently, they raise/lower j_W by 1/2 unit and lower/raise m_{j_W} by 1/2 unit. The generator T^6_W acts in a non-diagonal way on the decomposition (68), but its matrix elements can be easily determined using the standard SU(2) results. Its diagonal form is also easily derived, thanks to the fact that any generator of an SU(2) group has the same diagonal form, due to the fact that there is only one Cartan

more general values of $n^d_W \neq 1$ and $n^u_W \neq 2$, one needs to assign to the bulk fields a non-vanishing charge under the extra $U(1)'$ factor that is needed to tune the weak mixing angle, which is equal to $(n^d_W - 1)/3$ for $\psi^d, \tilde{\psi}^d$ and $(n^u_W - 2)/3$ for $\psi^u, \tilde{\psi}^u$. Notice however that unless these charges are opposite to each other, that is if $n^u_W + n^d_W = 3$, two different fields are needed to give mass to the u and the d quarks, due to the restrictions set by the $U(1)'$-invariance of the coupling to the left-handed quarks.

[4]This ordering of the states differs from the one used for the particular example of section 3.

generator. In our case, the diagonal form of T_W^6 must coincide in from with the generator $(-T_W^3 + \sqrt{3}T_W^8)/2$ representing the third component of the $SU(2)'$ spin. In terms of the quantum numbers defined by the decomposition (68), the latter acts as $(-n_W^l + 3j_W - m_{j_W})/2$ on the m_{j_W}-th element of the spin j_W component. In the same matrix notation as above, this means

$$t_W^{(\mathbf{n_W^l+1})(\mathbf{n_W^l+2})/2} = \mathrm{diag}(0, \frac{1}{2}, \ldots, \frac{n_W^l}{2}; \ldots; -\frac{n_W^l}{2} + \frac{1}{2}, -\frac{n_W^l}{2} + 1; -\frac{n_W^l}{2}) \,. \tag{70}$$

In the flavour sector, the situation is similar but much simpler, since we start with an $SU(2)_F$ group. The generator T_F^3 appearing in the orbifold projection is just the third component of the $SU(2)_F$ spin, and acts therefore as m_{j_F} on the m_{j_F}-th component of the decomposition (68). One then finds that the projection matrix P_F acts as $(-1)^{j_F - m_{j_F}}$ on the m_{j_F}-th component of the representation. In matrix notation, where these components are ordered with decreasing m_{j_F} ranging from j_F to $-j_F$[5], the orbifold twist has therefore the following form:

$$P_F^{\mathbf{2j_F+1}} = \mathrm{diag}(1, -1, 1, -1, \ldots) \,. \tag{71}$$

The generator T_F^1 appearing in the Scherk-Schwarz twist can be written more usefully as $T_F^1 = (T_F^+ + T_F^-)/2$ in terms of the raising and lowering operators $T_W^\pm = T_F^1 \pm iT_W^2$ of the $SU(2)_F$ subgroup. This allows to compute in a simple way any of its matrix elements. Its diagonal form must coincide with that of the Cartan generator T_F^3, which acts as m_{j_F} on the m_{j_F}-th component of the decomposition (68). The diagonal form of the twist is therefore given by

$$t_F^{\mathbf{2j_F+1}} = \mathrm{diag}\Big(j_F, j_F - 1, \ldots, -j_F + 1, -j_F\Big) \,. \tag{72}$$

We now describe the general situation that can be achieved in this more generic setting, in order to illustrate the basic features of the construction and its peculiarities compared to standard 4D flavour models.

4.1 Lagrangian

The structure of the Lagrangian is the same as in the previous section. The couplings of the family triplets of left- and right-handed brane fields $\phi = Q_L, u_R, d_R$ and their conjugates $\phi^c = Q_R^c, \text{-}u_L^c, d_L^c$ to the bulk fields ψ^l or $\tilde{\psi}^l$ are parametrized by family triplets of couplings e_1^l and e_2^l with mass-dimension 1/2, in each sector $l = u, d$. Each ϕ or ϕ^c can couple either to ψ^l or $\tilde{\psi}^l$, and has therefore only one relevant coupling. To write these couplings more explicitly, it is convenient to embed the fields ϕ and ϕ^c into new fields $\Phi = Q, u, d, \tilde{Q}, \tilde{u}, \tilde{d}$ and their conjugates $\Phi^c = Q^c, u^c, d^c, \tilde{Q}^c, \tilde{u}^c, \tilde{d}^c$, which have the same matrix structure as the representations of $SU(3)_W \times SU(2)_F$ to

[5] Again, this ordering differs from the canonical one used for the particular example of section 3.

which the bulk fields they couple to belong, the extra entries being filled with zeroes[6]. The untilded and tilded fields in Φ or Φ^c contain those SM fermions ϕ or ϕ^c that have the right quantum numbers to couple to ψ^l and $\tilde{\psi}^l$ respectively. With this notation, which is the appropriate generalization of the one used to deal with the particular example of sec. 3, the Lagrangian is obtained from eq. (17) by replacing the localized terms with

$$\begin{aligned}
\mathcal{L}^0 &= i\bar{Q}\gamma^\mu D_\mu Q + i\bar{\tilde{Q}}\gamma^\mu D_\mu \tilde{Q} \\
&\quad + \Big[\bar{Q}\, \hat{e}_1^{d\dagger}\psi^d + \bar{\tilde{Q}}\, \hat{e}_1^{d\dagger}\tilde{\psi}^d + \bar{Q}^c\, \hat{e}_1^{u\dagger}\psi^u + \bar{\tilde{Q}}^c\, \hat{e}_1^{u\dagger}\tilde{\psi}^u + \text{h.c.}\Big]\,, \qquad (73) \\
\mathcal{L}^{\pi R} &= i\bar{u}^c\gamma^\mu D_\mu u^c + i\bar{\tilde{u}}^c\gamma^\mu D_\mu \tilde{u}^c + i\bar{\tilde{d}}\gamma^\mu D_\mu \tilde{d} + i\bar{d}\gamma^\mu D_\mu d \\
&\quad + \Big[\bar{d}\, \hat{e}_2^{d\dagger}\psi^d + \bar{\tilde{d}}\, \hat{e}_2^{d\dagger}\tilde{\psi}^d + \bar{u}^c\, \hat{e}_2^{u\dagger}\psi^u + \bar{\tilde{u}}^c\, \hat{e}_2^{u\dagger}\tilde{\psi}^u + \text{h.c.}\Big]\,. \qquad (74)
\end{aligned}$$

To be more precise about the embeddings, let us denote $SU(2)_L\times U(1)_Y$ and family indices by $\alpha,\beta,\ldots$ and $I,J,\ldots=1,2,3$, and $SU(3)_W$ and $SU(2)_F$ indices by $i,j,\ldots$ and $a,b,\ldots$. The embedding of each field is then specified by some $(n^l_W+1)(n^l_W+2)/2$ by $2j_W+1$ matrix $(\mathcal{I}_W)_{i\alpha}$ for gauge indices, where j_W is 0 for singlets and 1/2 for doublets, and similarly by some $2j_F+1$ by 3 matrix $(\mathcal{I}_F)_{aI}$ for flavour indices. The position of each field ϕ or ϕ^c in Φ or Φ^c is uniquely determined by its $SU(2)_L\times U(1)_Y$ and $U(1)_F$ quantum numbers in the gauge and flavour sectors respectively. For the couplings, the embedding is trivial for gauge indices and is determined in an obvious way in terms of that of the fields for flavour indices: it is a diagonal $2j_F+1$ by $2j_F+1$ matrix whose non-zero entries are the couplings that are relevant for each field, in the corresponding positions.

The embedding in the gauge sector generalizes the one used in ref. [6]. Rather than reporting the matrices $\mathcal{I}_W$ for each field, we can exhibit the same information by reporting the expressions of the fields $\Phi_W=\mathcal{I}_W^\Phi\phi$ and $\Phi^c_W=\mathcal{I}_W^{\Phi^c}\phi^c$. These are $(n^l_W+1)(n^l_W+2)/2$-dimensional vectors will all the entries set to zero apart from the last three, which host the SM fields:

$$Q_W=\tilde{Q}_W=\begin{pmatrix}0\\ \vdots\\ 0\\ u_L\\ d_L\\ 0\end{pmatrix}\,,\qquad d_W=\tilde{d}_W=\begin{pmatrix}0\\ \vdots\\ 0\\ 0\\ 0\\ d_R\end{pmatrix}\,, \qquad (75)$$

[6]We denote the new embedded fields with the same letter as the original ones, but drop the L,R subscripts to them.

$$Q^c_W = \tilde{Q}^c_W = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d^c_R \\ \text{-}u^c_R \\ 0 \end{pmatrix}, \qquad u^c_W = \tilde{u}^c_W = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \text{-}u^c_L \end{pmatrix}. \tag{76}$$

The embedding in the flavour sector is done in a similar way and depends on the choice of flavour quantum numbers. Again, rather than reporting the matrices $\mathcal{I}_F$ for each field, one can consider directly the redefined fields $\Phi_F = \mathcal{I}^{\Phi}_F \phi$ and $\Phi^c_F = \mathcal{I}^{\Phi^c}_F \phi^c$. For Φ_F, each SM fermion ϕ is embedded at the $(j_F - q_F + 1)$-th entry if its flavour charge is q_F, and appears only in the untilded or tilded redefined fields if $j_F - q_F$ is respectively even or odd. Similarly, for the conjugate Φ^c_F, each conjugate SM fermion ϕ^c is embedded at the $(j_F + q_F + 1)$-th entry if its flavour charge is $-q_F$, and appears only in the untilded or tilded redefined fields if $j_F - q_F$ is respectively even or odd. As a consequence, for the embedding of the SM fields ϕ in Φ, only the odd and even entries of respectively the untilded and the tilded redefined fields are relevant, all the other being always zero; for the embedding of the conjugate SM fields ϕ^c in Φ^c, the situation is similar, and Φ^c is obtained from Φ through a reflection. Schematically, the structure is as follows, with at most three non-vanishing entries for each vector:

$$Q_F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (Q_L)_{I_1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{Q}_F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (Q_L)_{\tilde{I}_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad d_F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (d_L)_{J_1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{d}_F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (d_L)_{\tilde{J}_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{77}$$

$$Q^c_F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (Q^c_R)_{I_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \tilde{Q}^c_F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (Q^c_R)_{\tilde{I}_1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad u^c_F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (\text{-}u^c_L)_{K_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \tilde{u}^c_F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (\text{-}u^c_L)_{\tilde{K}_1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \tag{78}$$

In this expressions, I_1, J_1, K_1 and $\tilde{I}_1$, $\tilde{J}_1$, $\tilde{K}_1$ are restricted family indices running respectively over those families for which the left-handed doublets, the right-handed down singlets and the right-handed up singlets are embedded in untilded and tilded vectors.

Finally, the brane-bulk couplings are correspondingly embedded into diagonal matrices $\hat{e}^l_{1,2}$ at those entries that correspond to a non-vanishing entry of the redefined fields. They have the following schematic form, with three non-vanishing entries labeled by a family index M:

$$\hat{e}^l_{1,2} = \mathrm{diag}(0, \ldots, 0, (e^l_{1,2})_{M_1}, 0, \ldots, 0, (e^l_{1,2})_{M_2}, 0, \ldots, 0, (e^l_{1,2})_{M_3}, 0, \ldots, 0)\,. \tag{79}$$

4.2 Fermion masses and mixings

From the Lagrangian above one can proceed exactly as in Sec. 3 to derive the effective Lagrangian for the SM fermions, which still has the form of eq. (48). The general expressions of $\mathcal{M}$ and $\mathcal{Z}$ depend on the matrix elements of the generator T^6_W implementing the electroweak symmetry breaking and those of arbitrary powers of the generator T^1_F implementing the flavour symmetry breaking, which appear in the functions of eqs. (45). The relevant matrix element of T^6_W is universal and can be computed in general. It is the one connecting the next-to-last element of the embedding vector of the left-handed fields and their conjugates, that is the $m_{j_W} = -1/2$ component of the doublet with $j_W = 1/2$, and the last element of the embedding vector of the right-handed fields and their conjugates, that is the singlet with $m_{j_W} = 0$ and $j_W = 0$. As already explained, this can be easily evaluated by rewriting $T^6_W = (T^+_W + T^-_W)/2$ in terms of the raising and lowering operators $T^\pm_W = T^6_W \pm i T^7_W$ of the $SU(2)'$ subgroup defined by the twist, which have non-vanishing matrix elements between neighbour states, namely $\sqrt{(j' \mp m_{j'})(j' \pm m_{j'} + 1)}$. The matrix element we are interested in is therefore an ordinary transition from the component with $m_{j'} = -n^l_W/2$ to the component with $m_{j'} = -n^l_W/2 + 1$ of an $SU(2)'$ representation of spin $j' = n^l_W/2$, and gives a factor $\sqrt{n^l_W}/2$. The matrix elements of a generic power of T^1_F can be computed similarly. Here we simply rewrite the flavour traces in terms of the $2j_F + 1$ by 3 matrices $\mathcal{I}_F$ defining how the family triplet of each SM field is embedded into an $(2j_F + 1)$-dimensional flavour vector. The results are given by the following expressions:

$$\begin{aligned}
\mathcal{Z}^d_L &= \mathbf{1} + \mathcal{E}^{d\dagger}_1 \Big[\mathcal{I}^{Q\dagger}_F f(x_d, T^1_F\beta)\, \mathcal{I}^Q_F + \mathcal{I}^{\tilde{Q}\dagger}_F f(x_d, T^1_F\beta)\, \mathcal{I}^{\tilde{Q}}_F\Big] \mathcal{E}^d_1 \\
&\quad + \mathcal{E}^{u\dagger}_1 \Big[\mathcal{I}^{Q^c\dagger}_F f(x_u, T^1_F\beta)\, \mathcal{I}^{Q^c}_F + \mathcal{I}^{\tilde{Q}^c\dagger}_F f(x_u, T^1_F\beta)\, \mathcal{I}^{\tilde{Q}^c}_F\Big] \mathcal{E}^u_1\,, \\
\mathcal{Z}^u_L &= \mathbf{1} + \mathcal{E}^{u\dagger}_1 \Big[\mathcal{I}^{Q^c\dagger}_F f(x_u, T^1_F\beta)\, \mathcal{I}^{Q^c}_F + \mathcal{I}^{\tilde{Q}^c\dagger}_F f(x_u, T^1_F\beta)\, \mathcal{I}^{\tilde{Q}^c}_F\Big] \mathcal{E}^u_1 \\
&\quad + \mathcal{E}^{d\dagger}_1 \Big[\mathcal{I}^{Q\dagger}_F f(x_d, T^1_F\beta)\, \mathcal{I}^Q_F + \mathcal{I}^{\tilde{Q}\dagger}_F f(x_d, T^1_F\beta)\, \mathcal{I}^{\tilde{Q}}_F\Big] \mathcal{E}^d_1\,, \\
\mathcal{Z}^d_R &= \mathbf{1} + \mathcal{E}^{d\dagger}_2 \Big[\mathcal{I}^{d\dagger}_F f(x_d, T^1_F\beta)\, \mathcal{I}^d_F + \mathcal{I}^{\tilde{d}\dagger}_F f(x_d, T^1_F\beta)\, \mathcal{I}^{\tilde{d}}_F\Big] \mathcal{E}^d_2\,, \\
\mathcal{Z}^u_R &= \mathbf{1} + \mathcal{E}^{u\dagger}_2 \Big[\mathcal{I}^{u^c\dagger}_F f(x_u, T^1_F\beta)\, \mathcal{I}^{u^c}_F + \mathcal{I}^{\tilde{u}^c\dagger}_F f(x_u, T^1_F\beta)\, \mathcal{I}^{\tilde{u}^c}_F\Big] \mathcal{E}^u_2\,,
\end{aligned} \tag{80}$$

and

$$
\begin{aligned}
\mathcal{M}^d &= \sqrt{n_W^d}\,\mathcal{E}_1^{d\dagger}\Big[\mathcal{I}_F^{Q\dagger} g_1(x_d,T_F^1\beta)\,\mathcal{I}_F^d - \mathcal{I}_F^{\tilde{Q}\dagger} g_1(x_d,T_F^1\beta)\,\mathcal{I}_F^{\tilde{d}} \\
&\qquad + \mathcal{I}_F^{Q\dagger} g_2(x_d,T_F^1\beta)\,\mathcal{I}_F^{\tilde{d}} + \mathcal{I}_F^{\tilde{Q}\dagger} g_2(x_d,T_F^1\beta)\,\mathcal{I}_F^d\Big]\mathcal{E}_2^d\, m_W\,, \\
\mathcal{M}^u &= \sqrt{n_W^u}\,\mathcal{E}_1^{u\dagger}\Big[\mathcal{I}_F^{Q^c\dagger} g_1(x_u,T_F^1\beta)\,\mathcal{I}_F^{u^c} - \mathcal{I}_F^{\tilde{Q}^c\dagger} g_1(x_u,T_F^1\beta)\,\mathcal{I}_F^{\tilde{u}^c} \\
&\qquad + \mathcal{I}_F^{Q^c\dagger} g_2(x_u,T_F^1\beta)\,\mathcal{I}_F^{\tilde{u}^c} + \mathcal{I}_F^{\tilde{Q}^c\dagger} g_2(x_u,T_F^1\beta)\,\mathcal{I}_F^{u^c}\Big]\mathcal{E}_2^u\, m_W\,.
\end{aligned}
\tag{81}
$$

Once the above quantities have been computed, the physical implications of the Lagrangian (48) are uniquely determined and can be analyzed as follows. First, one performs a suitable redefinition of the fermions fields to reabsorb the non-trivial wave function factor and canonically normalize their kinetic terms. In this process, the mass matrices will however be changed to new matrices $\hat{\mathcal{M}}^l$. Second, one proceeds as in the SM and diagonalizes these two mass matrices through some unitary transformations $\mathcal{U}_{L,R}$ and $\mathcal{D}_{L,R}$ in the u and d sectors. This will then induce a CKM mixing matrix given by $V_{CKM} = \mathcal{U}_L^\dagger \mathcal{D}_L$.

The above procedure is complicated by the non-diagonal field redefinition that is required to get read of the wave function. One might fear that the new mass matrices $\hat{\mathcal{M}}^l$ that are generated after wave-function renormalization might have hierarchical structures in powers of λ that are messed up compared to those of $\mathcal{M}^l$. However, as shown in general in Sec. 3.4, this is not the case: at most the order one coefficients multiplying the powers of λ in the various entries are changed. We now generalize the discussion of Sec. 3.7 for generic representations.

$\mathbf{x_l \gg 1}$

In the limit of $x_l \gg 1$, the functions in eqs. (46) simplify to the form in eq. (49). As we have seen, at leading order in e^{-x_l} the wave functions reduce to diagonal constants:

$$
\mathcal{Z}_L^l \simeq \mathbf{1} + \frac{1}{x_d}\mathcal{E}_L^{d\dagger}\mathcal{E}_L^d + \frac{1}{x_u}\mathcal{E}_L^{u\dagger}\mathcal{E}_L^u\,, \qquad \mathcal{Z}_R^l \simeq \mathbf{1} + \frac{1}{x_l}\mathcal{E}_R^{l\dagger}\mathcal{E}_R^l\,. \tag{82}
$$

The masses $\mathcal{M}^l$ take instead the form

$$
\mathcal{M}^l \simeq \sqrt{n_W^l}\, e^{-x_l}\mathcal{E}_L^{l\,\dagger}\widetilde{Y}^l\mathcal{E}_R^l\, m_W\,, \tag{83}
$$

where $\widetilde{Y}^l$ are two 3×3 matrices that are functions of λ and carry all the information about the group-theoretical details of the flavour sector. Assuming that $\lambda \ll 1$, they have the form (21), but with completely fixed numerical coefficients, which can be easily computed using the standard realization of the $SU(2)$ algebra in terms of raising and lowering operators. Further assuming, for simplicity and without loss of generality, that the flavour charges l_I of the right-handed fields are larger than the charges q_I of the left-handed fields, and recalling that $\lambda = \pi\beta$, the result is, modulo a sign:

$$\widetilde{Y}^l_{IJ} \simeq \prod_{k=1}^{l_J - q_I} \sqrt{1 + \frac{j_F - l_J}{k}} \sqrt{1 + \frac{j_F + q_I}{k}}\, \lambda^{l_J - q_I}\,. \tag{84}$$

The first subleading corrections to these expressions can be easily evaluated using again creation and annihilation operators. The relative effect represented by these corrections is of order λ^2, and its precise expression, modulo a sign, is given by

$$\frac{\Delta\widetilde{Y}^l_{IJ}}{\widetilde{Y}^l_{IJ}} \simeq \sum_{k=0}^{l_J - q_I + 1} \frac{(j_F + q_I + k)(j_F - q_I - k + 1)}{(l_J - q_I + 1)(l_J - q_I + 2)}\, \lambda^2\,. \tag{85}$$

From this expression it is clear that there is an obstruction against increasing too much the spin j_F of the representation of the bulk mediators at fixed flavour charges for the brane fields. Indeed, doing so increases the relative impact of the subleading terms and puts a limit on how large the parameter λ can be at fixed j_F, or viceversa how large j_F can be at fixed λ, without spoiling the simple idea that the Yukawa texture is fixed by the leading terms with powers of λ fixed by the charges. Notice for instance that in the extreme limit in which j_F is much larger than all of the charges, one finds that the leading term (84) goes like $j_F/(l_J - q_I)!\, \lambda^{l_J - q_I}$ if $l_J \neq q_I$ and 1 if $l_J = q_I$, whereas the relative subleading correction (85) goes like $j_F^2/(l_J - q_I + 1)\, \lambda^2$. Requiring that the latter be much smaller than 1 then implies that $j_F \ll \sqrt{l_J - q_I + 1}/\lambda$. For $\lambda \sim 10^{-1}$ and reasonable charges, one must then take $j_F \ll 10$. For $j_F \sim 3 - 4$, as in the examples that we shall study below, the subleading corrections represent therefore a significant error of about 10%.

The physical quark Yukawa couplings are obtained by redefining the quark fields to reabsorb the wave-function corrections $\mathcal{Z}^l_{L,R}$. The physical mass matrices are then found to be (see eq. (57)):

$$m^l \simeq \sqrt{n^l_W}\, x_l e^{-x_l} \sin\alpha^l_L\, \widetilde{Y}^l \sin\alpha^l_R\, m_W\,. \tag{86}$$

$\mathbf{x_l \ll 1}$

In the limit of $x_l \ll 1$, the masses in eq. (63) generalize as

$$\mathcal{M}^l \simeq \sqrt{n^l_W} \frac{1}{x_l^2} \epsilon^{l\dagger}_L \epsilon^l_R\, m_W\,. \tag{87}$$

The physical quark masses emerging after canonical normalization are then found to be

$$m^l \simeq \sqrt{n^l_W} \sin\alpha^l_L \sin\alpha^l_R\, m_W\,. \tag{88}$$

5 Model building

In this section, we apply the general construction developed so far to build viable flavour models. We present two illustrative examples that emphasize some important phenomenological aspects.

5.1 Mixing angles and mass ratios

The model presented in sec. 3 produces the correct structure of powers of λ for Yukawa couplings, but suffers from large group-theoretical coefficients that spoil the success of the chosen texture. The simplest way to solve this problem is to assign charges in such a way as no $\mathcal{O}(\lambda^0)$ term is present. Then, all entries will have comparable numerical coefficients, and the power expansion will be consistent. We can start for instance from

$$Y^d \sim \begin{pmatrix} \lambda^6 & \lambda^5 & \lambda^4 \\ \lambda^5 & \lambda^4 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda \end{pmatrix}, \qquad Y^u \sim \begin{pmatrix} \lambda^7 & \lambda^6 & \lambda^4 \\ \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^4 & \lambda^3 & \lambda \end{pmatrix}. \tag{89}$$

The simplest flavour charge assignment for the brane fermions that is compatible with these textures is given by

$$q_I = \left\{-\frac{7}{2}, -\frac{5}{2}, -\frac{1}{2}\right\}, \qquad d_I = \left\{\frac{5}{2}, \frac{3}{2}, \frac{1}{2}\right\}, \qquad u_I = \left\{\frac{7}{2}, \frac{5}{2}, \frac{1}{2}\right\}. \tag{90}$$

Since the maximal absolute value of the charge is now $7/2$, the smallest allowed representation for the bulk fermions has now spin $j_F = 7/2$.

Assuming as before $x_l \gg 1$ to simplify the analysis of the effects on order one coefficients due to wave-function corrections, the induced mass matrices $\mathcal{M}^u$ and $\mathcal{M}^d$ are given by eqs. (50) and (51) with:

$$\widetilde{Y}^d = 4\,\lambda \times \begin{pmatrix} \frac{\sqrt{7}}{4}\lambda^5 & -\frac{\sqrt{21}}{4}\lambda^4 & -\frac{\sqrt{35}}{4}\lambda^3 \\ -\frac{3}{2}\lambda^4 & \frac{5\sqrt{3}}{4}\lambda^3 & \sqrt{5}\,\lambda^2 \\ \sqrt{5}\,\lambda^2 & -\frac{\sqrt{15}}{2}\lambda & -1 \end{pmatrix}, \tag{91}$$

$$\widetilde{Y}^u = 4\,\lambda \times \begin{pmatrix} \frac{1}{4}\lambda^6 & \frac{\sqrt{7}}{4}\,\lambda^5 & -\frac{\sqrt{35}}{4}\,\lambda^3 \\ -\frac{\sqrt{7}}{4}\,\lambda^5 & -\frac{3}{2}\lambda^4 & \sqrt{5}\,\lambda^2 \\ \frac{\sqrt{35}}{4}\,\lambda^3 & \sqrt{5}\,\lambda^2 & -1 \end{pmatrix}, \tag{92}$$

The mass matrices that are obtained in this case still have the problem of a too low overall scale, but it is now possible to reproduce mass ratios and mixing angles with reasonable values of the parameters (except for the down quark mass which is too low).

5.2 Example with improved overall scale

The problem of the small overall scale can be solved by introducing, in addition to a pair of bulk fermions that are flavour-charged and induce general hierarchical mass matrices, an extra pair of bulk fermions that are flavour-neutral and contribute therefore only to the mass of flavour-neutral states. Assigning third-generation quarks a vanishing charge, neutral bulk fermions will only contribute to the $(3,3)$ entries of quark masses. If charged bulk fermions are heavier, all the other entries will be additionally suppressed by a factor $e^{-\pi R(M_l^C - M_l^N)}$, where M_l^C and M_l^N stand for the masses of charged and neutral bulk fermions respectively. It is clear that in this case the mass ratio between the third and the first two generations is not a prediction of the flavour model any more, but stems from the exponential factor $e^{-\pi R(M_l^C - M_l^N)}$. Taking into account this extra suppression, we can choose for example

$$Y^d \sim \begin{pmatrix} \lambda^5 & \lambda^4 & \lambda^3 \\ \lambda^4 & \lambda^3 & \lambda^2 \\ \lambda^2 & \lambda & 1 \end{pmatrix}, \qquad Y^u \sim \begin{pmatrix} \lambda^6 & \lambda^4 & \lambda^3 \\ \lambda^5 & \lambda^3 & \lambda^2 \\ \lambda^3 & \lambda & 1 \end{pmatrix}. \tag{93}$$

The simplest flavour charge assignment for the brane fermions that realize these is

$$q_I = \{-3, -2, 0\}, \qquad d_I = \{2, 1, 0\}, \qquad u_I = \{3, 1, 0\}. \tag{94}$$

The smallest allowed representation for the charged bulk fermions has in this case spin $j_F = 3$.

These charged states give a contribution to the mass matrices $\mathcal{M}^u$ and $\mathcal{M}^d$ given by eqs. (51) and (50) with

$$\widetilde{Y}^d \simeq \begin{pmatrix} -\sqrt{6}\,\lambda^5 & \sqrt{15}\,\lambda^4 & 2\sqrt{5}\,\lambda^3 \\ -5\,\lambda^4 & 2\sqrt{10}\,\lambda^3 & \sqrt{30}\,\lambda^2 \\ -2\sqrt{3}\,\lambda & \sqrt{30}\,\lambda & -1 \end{pmatrix}, \tag{95}$$

$$\widetilde{Y}^u = \begin{pmatrix} -\lambda^6 & \sqrt{15}\,\lambda^4 & 2\sqrt{5}\,\lambda^3 \\ -\sqrt{6}\,\lambda^5 & 2\sqrt{10}\,\lambda^3 & \sqrt{30}\,\lambda^2 \\ 2\sqrt{5}\,\lambda^3 & -2\sqrt{3}\,\lambda & -1 \end{pmatrix}. \tag{96}$$

For the corresponding flavour-neutral states, if we stick to the $SU(3)_W$ representations used in ref. [6], we still have a problem with the top mass, which remains too low. As an illustrative example, one can choose a rank 6 symmetric representation for the flavour-neutral fermion coupling to the top quark, even though one should check that the cutoff is not lowered too much by the presence of fermions in large representations of $SU(3)_W$. With this caveat, the situation improves, and we can reproduce all masses and mixing angles with reasonable values of the parameters, except again for the down quark which tends to be too light.

5.3 FCNC processes and CP violation

Since there is a mixing between brane and bulk fermions, tree-level FCNC couplings to the Z boson are expected to arise. On general grounds, they will be suppressed by α^2 and by the appropriate power of β. It remains to be seen whether in any specific model this suppression is sufficient to guarantee a successful description of FCNC phenomena: to this aim, we are presently carrying out a full one-loop phenomenological analysis.

In all the above discussions, for simplicity, we have taken the ϵ couplings to be real. In general, they are complex numbers and their phases enter the effective mass matrices and the CKM matrix. The strength of CP violation then depends on the size and phases of ϵ parameters, and can be estimated in any specific model.

6 Conclusion

We have proposed a mechanism to implement flavour symmetries in gauge-Higgs unification models. In five-dimensional orbifold constructions the only possibility consists in a flavour $SU(2)_F$ symmetry broken to $U(1)_F$ by the orbifold projection and then to nothing via a compactification twist. Assuming that the problems connected to electroweak symmetry breaking in gauge-Higgs unification were solved, our proposal can successfully predict the orders of magnitude of all mass ratios and mixing angles. Quantitative agreement can be obtained with reasonable values of all relevant parameters. We stress that this class of models is much more constrained than ordinary FN abelian flavour models because of the higher-dimensional non-Abelian nature of the flavour symmetry. We are presently investigating the phenomenology of FCNC processes in this kind of construction, both at the tree and the one-loop levels.

An interesting possibility would be to implement our idea in the framework of warped five-dimensional models or in six-dimensional orbifolds, in which electroweak symmetry breaking seems more successful (see for instance [26, 27] and [7, 8]).

Acknowledgments

We would like to thank S. Lavignac, R. Rattazzi, A. Romanino, M. Serone and A. Wulzer for useful discussions. We also thank the Theory Division of CERN for hospitality. This work has been partly supported by the European Commission through a Marie Curie fellowship, under the contract MRTN-CT-2004-005104 and through the programme "The quest for unification" under contract MRTN-CT-2004-503369, the Swiss National Science Foundation, the German Bundesministerium für Bildung und Forschung under the contract 05HT4WOA/3, and the German-Israeli Foundation under the contract G-698-22.7/2002. M.S. acknowledges MECD for financial support through FPU fellowship AP2003-1540.

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