

Advanced Quantum Field Theory

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1 Basic formalism for interacting theories

1.1 Perturbative approach

Consider an interacting theory with fields $\phi(x)$ and split the Hamiltonian into a free part and an interaction part depending on some small coupling λ :

$$H = H_0 + H_{\text{int}}. \quad (1.1)$$

Suppose then that H_{int} can be treated as a perturbation with respect to H_0 , in the sense that the spectrum of H is qualitatively similar to that of H_0 . Call then:

$$\begin{cases} |0\rangle : \text{vacuum of } H_0 \text{ with } E_0 = 0, \\ |n\rangle : \text{particle states of } H_0 \text{ with } E_n > 0, \end{cases} \quad (1.2)$$

and

$$\begin{cases} |\Omega\rangle : \text{vacuum of } H \text{ with } E_\Omega \neq 0, \\ |N\rangle : \text{particle states of } H \text{ with } E_N > E_\Omega. \end{cases} \quad (1.3)$$

Starting from the free theory and switching on the interactions, there can be transitions occurring between states with non-zero probability. However, also the states themselves are changed by the interactions.

Perturbation theory is set up in the interaction picture, which is defined from the Heisenberg picture through a time-dependent unitary transformation:

$$U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}. \quad (1.4)$$

At the reference time t_0 , the two pictures coincide, but the evolution to $t \neq t_0$ of the operators and the states is different in the two pictures, although all the expectation values are the same. Denote then the interaction picture fields, whose evolution is controlled by the free Hamiltonian H_0 , as

$$\begin{aligned} \phi_I(t, \vec{x}) &= e^{iH_0(t-t_0)} \phi(t_0, \vec{x}) e^{-iH_0(t-t_0)} \\ &= \text{field depending on free creation and annihilator operators,} \end{aligned} \quad (1.5)$$

and the Heisenberg picture fields, whose evolution is controlled by the full Hamiltonian H , as

$$\begin{aligned} \phi(t, \vec{x}) &= e^{iH(t-t_0)} \phi(t_0, \vec{x}) e^{-iH(t-t_0)} \\ &= \text{field depending on the full complication of interactions.} \end{aligned} \quad (1.6)$$

The relation between these fields is:

$$\phi(t, \vec{x}) = U^\dagger(t, t_0) \phi_I(t, \vec{x}) U(t, t_0). \quad (1.7)$$

The states evolve with the operator $U(t, t_0)$ in the interaction picture and are time independent in the Heisenberg picture.

In order to be able to use efficiently the interaction picture, we need in practice to express the evolution operator $U(t, t_0)$ in terms of the fields $\phi_I(x)$. To do so, we start from the differential equation that this operator satisfies, which reads:

$$i \frac{\partial}{\partial t} U(t, t_0) = H_I(t) U(t, t_0), \quad (1.8)$$

in terms of the interaction Hamiltonian in the interaction picture, which is a function of the fields $\phi_I(x)$:

$$H_I(t) = e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH_0(t-t_0)}. \quad (1.9)$$

We can now solve this equation with the boundary condition $U(t_0, t_0) = 1$ by first converting it to an integral equation and then solving this by infinite iteration. The result can be expressed in a very compact form as a time-ordered product:

$$U(t, t_0) = T \exp \left\{ -i \int_{t_0}^t dt' H_I(t') \right\}. \quad (1.10)$$

This can be generalized to $U(t_2, t_1)$ with arbitrary times, defined as

$$U(t_2, t_1) = e^{iH_0(t_2-t_0)} e^{-iH(t_2-t_1)} e^{-iH_0(t_1-t_0)}. \quad (1.11)$$

This satisfies the same differential equation with the boundary condition $U(t, t) = 1$, and its form in terms of the fields $\phi_I(x)$ reads:

$$U(t_2, t_1) = T \exp \left\{ -i \int_{t_1}^{t_2} dt' H_I(t') \right\}. \quad (1.12)$$

1.2 Correlation functions

The basic objects that one wants to compute are the correlation functions defined by the vacuum expectation value of time-ordered products of fields:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \langle \Omega | T \phi(x_1) \cdots \phi(x_n) | \Omega \rangle. \quad (1.13)$$

In order to compute these objects, we have to relate them to expectation values of the fields $\phi_I(x)$ on the free vacuum $|0\rangle$, which can be easily computed. We need then to express the fields $\phi(x)$ in terms of the field $\phi_I(x)$, and find a way to relate the vacuum $|\Omega\rangle$ in terms of the free vacuum $|0\rangle$. This can be done by using the trick of acting on $|\Omega\rangle$ with the Heisenberg evolution operator e^{-iHT} at very large times T , in a slightly imaginary direction, in such a way to select its component along the lowest-lying free state $|0\rangle$. More precisely, we take:

$$T \rightarrow +\infty(1 - i\epsilon). \quad (1.14)$$

Using the completeness of the set of states $|N\rangle$ and the fact that $|\Omega\rangle$ is among these the one with lowest energy, one deduces that in this limit $e^{-iHT} = e^{-iE_\Omega T} |\Omega\rangle \langle \Omega|$. Applying this kind of result to $|0\rangle$, one finds that:

$$|\Omega\rangle = \frac{e^{iE_\Omega(t_0+T/2)}}{\langle \Omega|0\rangle} U(t_0, -T/2) |0\rangle, \quad \langle \Omega| = \frac{e^{iE_\Omega(T/2-t_0)}}{\langle 0|\Omega\rangle} \langle 0| U(T/2, t_0). \quad (1.15)$$

One computes then:

$$\langle \Omega | \Omega \rangle = \frac{e^{iE_\Omega T}}{|\langle \Omega | 0 \rangle|^2} \langle 0 | U(T/2, -T/2) | 0 \rangle, \quad (1.16)$$

and

$$\langle \Omega | T \phi(x_1) \cdots \phi(x_n) | \Omega \rangle = \frac{e^{iE_\Omega T}}{|\langle \Omega | 0 \rangle|^2} \langle 0 | T \phi_I(x_1) \cdots \phi_I(x_n) U(T/2, -T/2) | 0 \rangle. \quad (1.17)$$

Using the normalization condition $\langle \Omega | \Omega \rangle = 1$, one finally deduces that the correlation functions can be expressed as:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\langle 0 | T \phi_I(x_1) \cdots \phi_I(x_n) \exp \left\{ -i \int_{-T/2}^{T/2} dt H_I(t) \right\} | 0 \rangle}{\langle 0 | T \exp \left\{ -i \int_{-T/2}^{T/2} dt H_I(t) \right\} | 0 \rangle}. \quad (1.18)$$

Note moreover that the value of the vacuum energy E_Ω can be deduced from the T -dependent exponent in the expression for $\langle \Omega | \Omega \rangle$, and takes the form:

$$E_\Omega = \frac{i}{T} \log \langle 0 | T \exp \left\{ -i \int_{-T/2}^{T/2} dt H_I(t) \right\} | 0 \rangle. \quad (1.19)$$

1.3 Diagrammatics

Expanding the exponential of the evolution operator and using Wick's theorem, one can evaluate the correlation functions in a perturbative expansion in powers of the coupling λ , leading to Feynman diagrams. The result is that:

$$\langle 0 | T \exp \left\{ -i \int_{-T/2}^{T/2} dt H_I(t) \right\} | 0 \rangle = \exp \left(\begin{array}{c} \text{sum of} \\ \text{disc. diag.} \end{array} \right), \quad (1.20)$$

$$\langle 0 | T \phi_I(x_1) \cdots \phi_I(x_n) \exp \left\{ -i \int_{-T/2}^{T/2} dt H_I(t) \right\} | 0 \rangle = \left(\begin{array}{c} \text{sum of conn.} \\ \text{diag. with } x_i \end{array} \right) \exp \left(\begin{array}{c} \text{sum of} \\ \text{disc. diag.} \end{array} \right).$$

It follows then that

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \left(\begin{array}{c} \text{sum of conn.} \\ \text{diag. with } x_i \end{array} \right), \quad (1.21)$$

and also that

$$E_\Omega = \frac{i}{T} \left(\begin{array}{c} \text{sum of} \\ \text{disc. diag.} \end{array} \right). \quad (1.22)$$

1.4 Asymptotic states and S-matrix

In order to compute cross sections, we need to define asymptotic in and out states corresponding to isolated particles of definite momentum and polarization in the far past

and far future of a collision event. In the Heisenberg description, these states are time-independent, but they are labelled by the eigenvalues of time-dependent operators like \vec{P} or \vec{S} . In the interaction picture, instead, these states are genuinely time-dependent, but their labels are not. In any case, there can be a non-trivial overlap between given in and out states, and this can be viewed as the matrix element of a unitary scattering operator S between states defined at a common time:

$$\text{out}\langle p_1, \dots, p_n | k_1, \dots, k_m \rangle_{\text{in}} = \langle p_1, \dots, p_n | S | k_1, \dots, k_m \rangle. \quad (1.23)$$

One usually parametrizes the S operator as

$$S = 1 + iT. \quad (1.24)$$

The transition amplitudes $M(k_i, p_f)$ are then defined by excluding the trivial identity part and factorizing out the δ -function enforcing energy-momentum conservation:

$$\langle p_1, \dots, p_n | iT | k_1, \dots, k_m \rangle = iM(k_i, p_f) (2\pi)^4 \delta^{(4)} \left(\sum_f p_f - \sum_i k_i \right). \quad (1.25)$$

Finally, observable quantities like decay rates and cross sections, are defined out of the square of the sum of all the relevant amplitudes M , with a suitable kinematical factor.

In order to compute these S -matrix elements, one has to somehow relate asymptotic states to free particle states. More precisely, they are identified with free states but with intrinsic parameters describing the particles that are modified by interactions. In particular, one needs to introduce a field strength normalization constant $Z \neq 1$ and a corrected mass $m \neq m_0$, as well as a corrected coupling $\lambda \neq \lambda_0$. These are most properly defined by the spectral decomposition of the exact Fourier-transformed Feynman propagator. This representation of $\tilde{D}(p)$ is obtained by inserting a complete set of states in its definition and rewriting it as

$$\tilde{D}(p) = \int_0^{+\infty} \frac{dM^2}{2\pi} \frac{i\rho(M^2)}{p^2 - M^2 + i\epsilon}, \quad (1.26)$$

in terms of the following spectral function describing the distribution of the states $|N\rangle$ with energies m_N^2 in the rest frame:

$$\rho(M^2) = \sum_N (2\pi) \delta(M^2 - m_N^2) |\langle N | \phi(0) | \Omega \rangle|^2. \quad (1.27)$$

This function has a very particular analytic structure. It has an isolated pole in correspondence of the single-particle state and a branch cut starting at the threshold for the continuum of multi-particle states, so that:

$$\begin{aligned} \tilde{D}(p) &= \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{m_T^2}^{+\infty} \frac{dM^2}{2\pi} \frac{i\rho(M^2)}{p^2 - M^2 + i\epsilon} \\ &\xrightarrow{p^2 \rightarrow m^2} \frac{iZ}{p^2 - m^2 + i\epsilon}, \end{aligned} \quad (1.28)$$

where

$$\begin{aligned} Z &= \left| \langle 1 | \phi(0) | \Omega \rangle \right|^2 \\ &= \text{probability that } \phi \text{ creates the particle from the vacuum.} \end{aligned} \quad (1.29)$$

This means that asymptotic states are defined by including the resummed effects of 1PI insertions in the propagator. More precisely:

$$\begin{aligned}
\text{---} \bigcirc \text{---} &= \text{---} \text{---} + \text{---} \textcircled{\text{1PI}} \text{---} + \text{---} \textcircled{\text{1PI}} \textcircled{\text{1PI}} \text{---} + \dots \\
&= \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2} (-i) \Pi(p^2) \frac{i}{p^2 - m_0^2} + \dots \\
&= \frac{i}{p^2 - m_0^2 - \Pi(p^2)}. \tag{1.30}
\end{aligned}$$

In order for this to reproduce the structure of the 1-particle pole in the propagator, one identifies then:

$$m^2 = m_0^2 + \Pi(m^2), \quad Z = \frac{1}{1 - \Pi'(m^2)}. \tag{1.31}$$

One can now obtain the S -matrix elements between m and n particle states from a suitable limit of the $(m+n)$ -point correlation function in momentum space. More precisely, taking the limit where all the momenta go on the mass-shell of one-particle states, one finds:

$$\begin{aligned}
\langle \tilde{\phi}(p_1) \cdots \tilde{\phi}(p_n) \tilde{\phi}(-k_1) \cdots \tilde{\phi}(-k_m) \rangle &\xrightarrow[k_i^2 \rightarrow m_i^2, p_f^2 \rightarrow m_f^2]{} \prod_i \frac{i\sqrt{Z_i}}{k_i^2 - m_i^2 + i\epsilon} \prod_f \frac{i\sqrt{Z_f}}{p_f^2 - m_f^2 + i\epsilon} \\
&\langle p_1, \dots, p_f | iT | k_1, \dots, k_i \rangle. \tag{1.32}
\end{aligned}$$

Reversing this relation and going back to coordinate space, one obtains then the LSZ reduction formula

$$\begin{aligned}
\langle p_1, \dots, p_n | iT | k_1, \dots, k_m \rangle &= \prod_i \int d^4 x_i e^{-ik_i x_i} \frac{\square_{x_i} + m_i^2}{\sqrt{Z_i}} \prod_f \int d^4 y_f e^{ip_f y_f} \frac{\square_{y_f} + m_f^2}{\sqrt{Z_f}} \\
&i^{m+n} \langle \phi(y_1) \cdots \phi(y_n) \phi(x_1) \cdots \phi(x_m) \rangle. \tag{1.33}
\end{aligned}$$

The S -matrix depends thus only on the most singular part of the corresponding correlation function, which coincides with the fully connected part. It is in fact equal to the residue of the term involving all the external particle poles, times a numerical factor given by the product of the square-roots of the field-strength normalization factors for all the external particles. This corresponds, modulo this field-strength normalization factor, to the amputated correlation function where 1PI corrections on the external legs, which have already been included in the definition of asymptotic states, is discarded. Finally, we have thus:

$$\langle p_1, \dots, p_n | iT | k_1, \dots, k_m \rangle = \prod_i \sqrt{Z_i} \prod_f \sqrt{Z_f} \left(\text{sum of fully conn. and amput. diag. with } k_i, p_f \right). \tag{1.34}$$

1.5 Renormalization

In order to cope with UV divergences, one needs to regularize the theory by introducing some finite cut-off. One needs then to renormalize the theory by reexpressing all the physical amplitudes in terms of some physical parameters, defined out of the 1PI correlation functions. These physical parameters depend on the bare parameters as well as the cut-off. In renormalizable theories, this procedure allows to eliminate any explicit dependence on the cut-off in the physical amplitudes. It is then possible to formally remove the cut-off by keeping the physical parameters fixed to values that are matched with experimental inputs. The crucial parameters include the field-strength normalization Z , the mass m and all the interaction couplings λ . The non-trivial Z can then be reabsorbed in the definition of a renormalized field, which has a propagator with a pole at the physical mass m with unit residue:

$$\phi = Z^{1/2} \phi_r. \quad (1.35)$$

The S -matrix elements are then given by the fully connected and amputated correlation functions of renormalized fields, without any field-strength factor.

In practical calculations, it is very useful to use the so-called renormalized perturbation theory. This is defined by starting from the bare Lagrangian, which for the simplest case of self-interacting scalar field theories is of the type:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_0^2 \phi^2 - \frac{1}{4!} \lambda_0 \phi^4. \quad (1.36)$$

One reexpresses then the bare field ϕ in terms of the renormalized field ϕ_r , and splits the Lagrangian into a part with the same form as the original but with field-strength normalized to 1 and physical mass m and coupling λ , plus a residual set of counter-terms. More precisely, one gets:

$$\mathcal{L} = \mathcal{L}_r + \Delta\mathcal{L}, \quad (1.37)$$

with

$$\mathcal{L}_r = \frac{1}{2} \partial_\mu \phi_r \partial^\mu \phi_r - \frac{1}{2} m^2 \phi_r^2 - \frac{1}{4!} \lambda \phi_r^4, \quad (1.38)$$

$$\Delta\mathcal{L} = \frac{1}{2} \Delta_Z \partial_\mu \phi_r \partial^\mu \phi_r - \frac{1}{2} \Delta_m \phi_r^2 - \frac{1}{4!} \Delta_\lambda \phi_r^4, \quad (1.39)$$

where the counter-terms take the form:

$$\Delta_Z = Z - 1, \quad \Delta_m = m_0^2 Z - m^2, \quad \Delta_\lambda = \lambda_0 Z^2 - \lambda. \quad (1.40)$$

These counter-terms are finally determined order by order in perturbation theory to enforce the renormalization conditions defining the physical mass m and couplings λ , and allow to reabsorb all the divergences.

A first wide class of regularizations is based on the idea of cutting off divergent integrals at some very large momentum scale Λ . The simplest way is to sharply cut off the integrals at Λ , but there exist refinements of this method, like the Pauli-Villars regularization,

which consist in smoothly deforming the propagators appearing in the integrands in such a way to give them a fastly vanishing behavior beyond Λ . One finds then two types of integrals, after analytic continuation to Euclidean space by Wick rotation. The first are the omnipresent logarithmically divergent integrals, for example of the type:

$$\begin{aligned}\int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{(p_E^2 + \Delta)^2} &= \frac{1}{(2\pi)^4} \frac{1}{2} \int d\Omega_4 \int_0^{\Lambda^2} dp_E^2 \frac{p_E^2}{(p_E^2 + \Delta)^2} = \frac{1}{(2\pi)^4} \pi^2 \left(\log \frac{\Lambda^2}{\Delta} - 1 \right) \\ &= \frac{1}{(4\pi)^2} \left(\log \frac{\Lambda^2}{\Delta} - 1 \right).\end{aligned}\quad (1.41)$$

There are also quadratically divergent integrals, which give:

$$\begin{aligned}\int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{(p_E^2 + \Delta)} &= \frac{1}{(2\pi)^4} \frac{1}{2} \int d\Omega_4 \int_0^{\Lambda^2} dp_E^2 \frac{p_E^2}{(p_E^2 + \Delta)} = \frac{1}{(2\pi)^4} \pi^2 \left(\Lambda^2 - \Delta \log \frac{\Lambda^2}{\Delta} \right) \\ &= \frac{1}{(4\pi)^2} \left(\Lambda^2 - \Delta \log \frac{\Lambda^2}{\Delta} \right).\end{aligned}\quad (1.42)$$

Another convenient regularization is based on the idea of analytically continuing the momentum integrals to a lower dimension $d = 4 - 2\epsilon$, where they are convergent, and take then ϵ to be small. It is often convenient to compensate the change in the dimensions of fields and couplings through some arbitrary energy scale $\tilde{\mu}$, in such a way to recover the usual dimensional analysis. The logarithmically divergent integral gives then:

$$\begin{aligned}\int \frac{d^d p_E}{(2\pi)^d} \frac{\tilde{\mu}^{4-d}}{(p_E^2 + \Delta)^2} &= \frac{\tilde{\mu}^{4-d}}{2(2\pi)^d} \int d\Omega_d \int dp_E^2 \frac{p_E^{d-2}}{(p_E^2 + \Delta)^2} = \frac{\tilde{\mu}^{4-d}}{(2\pi)^d} \frac{\pi^{d/2}}{\Gamma(d/2)} \frac{\Gamma(d/2)\Gamma(2-d/2)}{\Delta^{2-d/2}} \\ &= \frac{1}{(4\pi)^{d/2}} \Gamma(2-d/2) \left(\frac{\Delta}{\tilde{\mu}^2} \right)^{d/2-2} \\ &= \frac{1}{(4\pi)^2} \Gamma(\epsilon) \left(\frac{\Delta}{4\pi\tilde{\mu}^2} \right)^{-\epsilon}.\end{aligned}\quad (1.43)$$

The quadratically divergent integral yields instead:

$$\begin{aligned}\int \frac{d^d p_E}{(2\pi)^d} \frac{\tilde{\mu}^{4-d}}{(p_E^2 + \Delta)} &= \frac{\tilde{\mu}^{4-d}}{2(2\pi)^d} \int d\Omega_d \int dp_E^2 \frac{p_E^{d-2}}{(p_E^2 + \Delta)} = \frac{\tilde{\mu}^{4-d}}{(2\pi)^d} \frac{\pi^{d/2}}{\Gamma(d/2)} \frac{\Gamma(d/2)\Gamma(1-d/2)}{\Delta^{1-d/2}} \\ &= \frac{1}{(4\pi)^{d/2}} \Delta \Gamma(1-d/2) \left(\frac{\Delta}{\tilde{\mu}^2} \right)^{d/2-2} \\ &= \frac{1}{(4\pi)^2} \Delta \Gamma(-1+\epsilon) \left(\frac{\Delta}{4\pi\tilde{\mu}^2} \right)^{-\epsilon}.\end{aligned}\quad (1.44)$$

The function $\Gamma(x)$ has poles at $x = 0, -1, -2, \dots$, and around 0 and -1 one has:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma, \quad \Gamma(-1+\epsilon) = -\left(\frac{1}{\epsilon} - \gamma + 1 \right).\quad (1.45)$$

It follows then that:

$$\begin{aligned}\int \frac{d^d p_E}{(2\pi)^d} \frac{\tilde{\mu}^{4-d}}{(p_E^2 + \Delta)^2} &= \frac{1}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma + \log \frac{4\pi\tilde{\mu}^2}{\Delta} \right) \\ &= \frac{1}{(4\pi)^2} \left(\log \frac{\tilde{\Lambda}^2}{\Delta} \right),\end{aligned}\quad (1.46)$$

and

$$\begin{aligned} \int \frac{d^d p_E}{(2\pi)^d} \frac{\tilde{\mu}^{4-d}}{(p_E^2 + \Delta)} &= -\frac{1}{(4\pi)^2} \Delta \left(\frac{1}{\epsilon} - \gamma + 1 + \log \frac{4\pi\tilde{\mu}^2}{\Delta} \right) \\ &= -\frac{1}{(4\pi)^2} \Delta \left(\log \frac{\tilde{\Lambda}^2}{\Delta} + 1 \right), \end{aligned} \quad (1.47)$$

where

$$\tilde{\Lambda} = \sqrt{4\pi} e^{-\gamma/2} e^{1/(2\epsilon)} \tilde{\mu}. \quad (1.48)$$

The above formulae show how dimensional regularization can be compared to cut-off regularizations. It works more or less like if there was an effective cut-off given by $\tilde{\Lambda}$. However, whereas the logarithmic divergences take exactly the same form as with a true cut-off regularization, the quadratic divergences disappear.

1.6 Vacuum amplitude and generating functional

Consider the vacuum to vacuum amplitude in the presence of an external current $J(x)$ acting as a source for every field $\phi(x)$:

$$\hat{Z}[J] = \langle \Omega | \Omega \rangle_J = \langle \Omega | T \exp \left\{ i \int d^4 x J(x) \phi(x) \right\} | \Omega \rangle. \quad (1.49)$$

More explicitly, this can be computed as:

$$\hat{Z}[J] = \frac{\langle 0 | T \exp \left\{ -i \int d^4 x \left(\mathcal{H}_I(x) - J(x) \phi_I(x) \right) \right\} | 0 \rangle}{\langle 0 | T \exp \left\{ -i \int d^4 x \mathcal{H}_I(x) \right\} | 0 \rangle}. \quad (1.50)$$

One can then consider the numerator of this expression on its own, since the denominator is given by the same expression with $J = 0$, and finally define:

$$Z[J] = \langle 0 | T \exp \left\{ -i \int d^4 x \left(\mathcal{H}_I(x) - J(x) \phi_I(x) \right) \right\} | 0 \rangle. \quad (1.51)$$

It is obvious that this quantity is by construction the generating functional of all the general correlation functions:

$$Z[J] = \text{generating functional for general correlation functions}. \quad (1.52)$$

This means that:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{(-i)^n}{Z[J]} \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \Big|_{J=0}. \quad (1.53)$$

1.7 Vacuum energy and connected generating functional

Consider next the functional defined by the phase of the vacuum to vacuum amplitude $Z[J]$, namely:

$$W[J] = -i \log Z[J]. \quad (1.54)$$

This is interpreted as the energy of the vacuum in the presence of the source $J(x)$, times the total evolution time T :

$$E_\Omega[J] = -\frac{1}{T} W[J]. \quad (1.55)$$

In particular, the vacuum energy for the original theory is given by:

$$E_\Omega = -\frac{1}{T} W[J] \Big|_{J=0}. \quad (1.56)$$

The first functional derivative of $W[J]$ defines the classical expectation value of the field $\phi(x)$ in the presence of the source $J(x)$:

$$\begin{aligned} \frac{\delta W[J]}{\delta J(x)} &= -\frac{i}{Z[J]} \frac{\delta Z[J]}{\delta J(x)} = \frac{\langle \Omega | \phi(x) | \Omega \rangle_J}{\langle \Omega | \Omega \rangle_J} = \frac{\langle \Omega | T \phi(x) \exp \left\{ i \int d^4x J(x) \phi(x) \right\} | \Omega \rangle}{\langle \Omega | T \exp \left\{ i \int d^4x J(x) \phi(x) \right\} | \Omega \rangle} \\ &= \phi_{\text{cl}}(x). \end{aligned} \quad (1.57)$$

Setting $J = 0$, one obtains the 1-point function:

$$\left. \frac{\delta W[J]}{\delta J(x)} \right|_{J=0} = \langle \phi(x) \rangle. \quad (1.58)$$

The second derivative produces:

$$\begin{aligned} \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} &= -\frac{i}{Z[J]} \frac{\delta^2 Z[J]}{\delta J(x) \delta J(y)} + \frac{i}{Z[J]^2} \frac{\delta Z[J]}{\delta J(x)} \frac{\delta Z[J]}{\delta J(y)} \\ &= i \left(\frac{\langle \Omega | \phi(x) \phi(y) | \Omega \rangle_J}{\langle \Omega | \Omega \rangle_J} - \frac{\langle \Omega | \phi(x) | \Omega \rangle_J}{\langle \Omega | \Omega \rangle_J} \frac{\langle \Omega | \phi(y) | \Omega \rangle_J}{\langle \Omega | \Omega \rangle_J} \right). \end{aligned} \quad (1.59)$$

Setting $J = 0$, one obtains the connected 2-point function, which is identified with the exact propagator:

$$\begin{aligned} \left. \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} \right|_{J=0} &= i \left(\langle \phi(x) \phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle \right) = i \langle \phi(x) \phi(y) \rangle_{\text{conn}} \\ &= i D(x, y). \end{aligned} \quad (1.60)$$

Proceeding in a similar way for higher-order derivatives, it turns out that $W[J]$ can be identified as the generating functional of all the fully connected correlation functions:

$$W[J] = \text{generating functional for connected correlation functions}. \quad (1.61)$$

More precisely, this means that:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle_{\text{conn}} = -i^{n+1} \left. \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0}. \quad (1.62)$$

1.8 Effective action and 1PI generating functional

Consider finally the Legendre transform of the vacuum energy $W[J]$ with respect to $\phi_{\text{cl}}(x)$ viewed as a function of $J(x)$ obtained by inverting the relation $\phi_{\text{cl}}(x) = \delta W[J]/\delta J(x)$. In order for this inverse relation to exist at least as a formal series expansion, we assume that for $J = 0$ the connected 1-point function $\delta W[J]/\delta J(x)$ takes some value $\langle\phi(x)\rangle$ whereas the connected 2-point function $\delta^2 W[J]/(\delta J(x)\delta J(y))$ takes a non-zero value $iD(x, y) \neq 0$. The effective action is then defined as:

$$\Gamma[\phi_{\text{cl}}] = W[J] - \int d^4x J(x)\phi_{\text{cl}}(x), \quad (1.63)$$

where

$$\phi_{\text{cl}}(x) = \frac{\delta W[J]}{\delta J(x)}. \quad (1.64)$$

With this definition, the first functional derivative of $\Gamma[\phi_{\text{cl}}]$ gives the current:

$$\begin{aligned} \frac{\delta \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x)} &= \frac{\delta W[J]}{\delta \phi_{\text{cl}}(x)} - \int d^4y \frac{\delta J(y)}{\delta \phi_{\text{cl}}(x)} \phi_{\text{cl}}(y) - J(x) \\ &= \int d^4y \frac{\delta W[J]}{\delta J(y)} \frac{\delta J(y)}{\delta \phi_{\text{cl}}(x)} - \int d^4y \frac{\delta J(y)}{\delta \phi_{\text{cl}}(x)} \phi_{\text{cl}}(y) - J(x) \\ &= -J(x). \end{aligned} \quad (1.65)$$

Setting now $J = 0$, which by the above definitions implies $\phi_{\text{cl}} = \langle\phi\rangle$, one deduces that the expectation value $\langle\phi\rangle$ satisfies the Euler-Lagrange equations derived by extremizing the effective action Γ :

$$\left. \frac{\delta \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x)} \right|_{\phi_{\text{cl}}=\langle\phi\rangle} = 0. \quad (1.66)$$

The second derivative of $\Gamma[\phi_{\text{cl}}]$ also has a special meaning. Indeed, one has:

$$\begin{aligned} \int d^4z \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)} \frac{\delta^2 \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(z)\delta \phi_{\text{cl}}(y)} &= \int d^4z \frac{\delta \phi_{\text{cl}}(z)}{\delta J(x)} \frac{\delta^2 \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(z)\delta \phi_{\text{cl}}(y)} \\ &= \frac{\delta^2 \Gamma[\phi_{\text{cl}}]}{\delta J(x)\delta \phi_{\text{cl}}(y)} = -\frac{\delta J(y)}{\delta J(x)} \\ &= -\delta^{(4)}(x - y). \end{aligned} \quad (1.67)$$

Setting then $J = 0$ and $\phi_{\text{cl}} = \langle\phi\rangle$, the factor $\delta^2 W[J]/(\delta J(x)\delta J(z))$ becomes the connected 2-point function, which is identified with the exact propagator, and this relations becomes:

$$\int d^4z D(x, z) \frac{\delta^2 \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(z)\delta \phi_{\text{cl}}(y)} \Big|_{\phi_{\text{cl}}=\langle\phi\rangle} = i \delta^{(4)}(x - y). \quad (1.68)$$

This means that the second derivative of the effective action yields the inverse of the propagator, which is the exact kinetic operator and can be identified with the 1PI two-point function:

$$\left. \frac{\delta^2 \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x)\delta \phi_{\text{cl}}(y)} \right|_{\phi_{\text{cl}}=\langle\phi\rangle} = i D^{-1}(x, y). \quad (1.69)$$

Proceeding in a similar way for higher-order derivatives, it turns out that the effective action Γ is actually recognized to be the generating functional of all the 1PI correlation functions:

$$\Gamma[\phi_{\text{cl}}] = \text{generating functional for 1PI correlation functions.} \quad (1.70)$$

More precisely, this means that:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle_{\text{1PI}} = i \frac{\delta^n \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x_1) \cdots \delta \phi_{\text{cl}}(x_n)} \Big|_{\phi_{\text{cl}} = \langle \phi \rangle}. \quad (1.71)$$

1.9 Path-integral representation

The path-integral representation of quantum field theory is defined in the Schrödinger picture, as in quantum mechanics, by treating the field at each spatial point as an independent canonical variable, which becomes an operator $\phi_S(\vec{x})$. In the coordinate representation, one considers the eigenstates $|\varphi\rangle$ of the field operators $\phi_S(\vec{x})$ with eigenvalues $\varphi(\vec{x})$. The time evolution of states is determined by

$$U(t) = e^{-iHt}. \quad (1.72)$$

This is also the time-dependent unitary transformation that relates the Schrödinger and the Heisenberg pictures, and the fields in the Heisenberg picture are obtained from those of the Schrödinger picture as:

$$\phi(t, \vec{x}) = U(t)^\dagger \phi_S(\vec{x}) U(t). \quad (1.73)$$

The basic object to consider is the evolution kernel $\langle \varphi_b | U(t_b - t_a) | \varphi_a \rangle$. This can be computed as a functional integral over all the possible paths for the field variable $\phi(x)$, weighted by a phase involving the classical action and with the boundary conditions that $\phi(t_a, \vec{x}) = \varphi_a(\vec{x})$ and $\phi(t_b, \vec{x}) = \varphi_b(\vec{x})$:

$$\langle \varphi_b | U(t_b - t_a) | \varphi_a \rangle = \int \mathcal{D}\phi \exp \left\{ i \int_{t_a}^{t_b} dt' L(t') \right\}. \quad (1.74)$$

To find the representation of the vacuum expectation values and correlation functions, we use the same trick as in the operatorial formulation and consider the evolution over very large and slightly imaginary times:

$$T \rightarrow +\infty(1 - i\epsilon). \quad (1.75)$$

The vacuum states can then be obtained as

$$|\Omega\rangle = \frac{e^{iE_\Omega T/2}}{\langle \Omega | \varphi_a \rangle} U(T/2) | \varphi_a \rangle, \quad \langle \Omega | = \frac{e^{iE_\Omega T/2}}{\langle \varphi_b | \Omega \rangle} \langle \varphi_b | U(T/2). \quad (1.76)$$

One computes then:

$$\langle \Omega | \Omega \rangle = \frac{e^{iE_\Omega T}}{\langle \Omega | \varphi_a \rangle \langle \varphi_b | \Omega \rangle} \langle \varphi_b | U(T) | \varphi_a \rangle, \quad (1.77)$$

and

$$\langle \Omega | T \phi(x_1) \cdots \phi(x_n) | \Omega \rangle = \frac{e^{iE_\Omega T}}{\langle \Omega | \varphi_a \rangle \langle \varphi_b | \Omega \rangle} \langle \varphi_b | T \phi_S(x_1) \cdots \phi_S(x_n) U(T) | \varphi_a \rangle. \quad (1.78)$$

Using the normalization condition $\langle \Omega | \Omega \rangle = 1$, one finally deduces that the correlation functions can be expressed as:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) \exp \left\{ i \int_{-T/2}^{T/2} dt L(t) \right\}}{\int \mathcal{D}\phi \exp \left\{ i \int_{-T/2}^{T/2} dt L(t) \right\}}. \quad (1.79)$$

The vacuum energy, defined by the T -dependent exponent in the expression for $\langle \Omega | \Omega \rangle$, becomes instead:

$$E_\Omega = \frac{i}{T} \log \int \mathcal{D}\phi \exp \left\{ i \int_{-T/2}^{T/2} dt L(t) \right\}. \quad (1.80)$$

Finally, the basic generating functional $Z[J]$ can be computed as:

$$Z[J] = \int \mathcal{D}\phi \exp \left\{ i \int d^4x \left(\mathcal{L}(x) + J(x)\phi(x) \right) \right\}. \quad (1.81)$$

2 Path integral and quantum effective action

2.1 Saddle-point evaluation of the effective action

The effective action $\Gamma[\phi_{\text{cl}}]$ contains the full information about the quantum dynamics of a theory, in the sense that all the correlation functions of the quantum theory, with their full loop corrections, are reproduced as simple tree-level correlations computed from it. It is therefore of great importance to understand the systematics of its computation. This proceeds by first evaluating $Z[J]$, then deducing its phase $W[J]$, and finally finding the Legendre transform with respect to $\phi_{\text{cl}}(x) = \delta W[J]/\delta J(x)$ to find $\Gamma[\phi_{\text{cl}}]$.

The starting point is the classical Lagrangian, rewritten in terms of the renormalized field ϕ_r . This can be split into a renormalized part \mathcal{L}_r involving the physical parameters and a counter-term part $\Delta\mathcal{L}_r$ containing the counter-terms:

$$\mathcal{L}[\phi_r] = \mathcal{L}_r[\phi_r] + \Delta\mathcal{L}[\phi_r]. \quad (2.1)$$

We then introduce the external current J and similarly split it into a first term J_r and an additional counter-term ΔJ :

$$J(x) = J_r(x) + \Delta J(x). \quad (2.2)$$

The part J_r is defined to enforce the definition of ϕ_{cl} at the lowest order in perturbation theory, namely:

$$\left. \frac{\delta \mathcal{S}_r[\phi_r]}{\delta \phi_r(x)} \right|_{\phi_r = \phi_{\text{cl}}} = -J_r(x). \quad (2.3)$$

The counter-term ΔJ is instead fixed by enforcing order by order in perturbation theory the definition of ϕ_{cl} , namely $\phi_{\text{cl}}(x) = \langle \phi(x) \rangle_{J_r + \Delta J}$.

To proceed, we consider the functional $Z[J]$ and write it as follows in terms of the above decompositions:

$$Z[J] = \int \mathcal{D}\phi \exp \left\{ i \int d^4x \left(\mathcal{L}_r[\phi_r(x)] + J_r(x)\phi_r(x) + \Delta\mathcal{L}[\phi_r(x)] + \Delta J(x)\phi_r(x) \right) \right\}. \quad (2.4)$$

The leading contribution to this path-integral comes from the classical field configuration $\phi_{\text{cl}}(x)$. We can then evaluate the exact integral as a saddle-point expansion, corresponding to a loop expansion in powers of \hbar , by writing:

$$\phi_r(x) = \phi_{\text{cl}}(x) + \eta(x). \quad (2.5)$$

We can now expand in powers of the fluctuation $\eta(x)$ the action arising in the path integral.

For the first two terms in the action, we find:

$$\begin{aligned}
\int d^4x \left(\mathcal{L}_r[\phi_r(x)] + J_r(x)\phi_r(x) \right) &= \int d^4x \left(\mathcal{L}_r[\phi_{\text{cl}}(x)] + J_r(x)\phi_{\text{cl}}(x) \right) \\
&+ \int d^4x \eta(x) \left(\left. \frac{\delta S_r[\phi_r]}{\delta \phi_r(x)} \right|_{\phi_r=\phi_{\text{cl}}} + J_r(x) \right) \\
&+ \frac{1}{2} \int d^4x \int d^4y \eta(x)\eta(y) \left. \frac{\delta^2 S_r[\phi_r]}{\delta \phi_r(x)\delta \phi_r(y)} \right|_{\phi_r=\phi_{\text{cl}}} \\
&+ \dots .
\end{aligned} \tag{2.6}$$

The term linear in η cancels by the classical equation of motion, and one is thus left with a quadratic term plus self-interaction vertices for the field η :

$$\begin{aligned}
\int d^4x \left(\mathcal{L}_r[\phi_r(x)] + J_r(x)\phi_r(x) \right) &= \int d^4x \left(\mathcal{L}_r[\phi_{\text{cl}}(x)] + J_r(x)\phi_{\text{cl}}(x) \right) \\
&+ \int d^4x \int d^4y \frac{1}{2} \eta(x) \left(\frac{\delta^2 S_r}{\delta \phi_r^2}[\phi_{\text{cl}}](x, y) \right) \eta(y) \\
&+ \text{self-interaction vertices in } \eta .
\end{aligned} \tag{2.7}$$

For the last two terms in the action, which represent the various counter-terms, we have similarly:

$$\begin{aligned}
\int d^4x \left(\Delta \mathcal{L}[\phi_r(x)] + \Delta J(x)\phi_r(x) \right) &= \int d^4x \left(\Delta \mathcal{L}[\phi_{\text{cl}}(x)] + \Delta J(x)\phi_{\text{cl}}(x) \right) \\
&+ \int d^4x \eta(x) \left(\left. \frac{\delta \Delta S_r[\phi_r]}{\delta \phi_r(x)} \right|_{\phi_r=\phi_{\text{cl}}} + \Delta J(x) \right) \\
&+ \frac{1}{2} \int d^4x \int d^4y \eta(x)\eta(y) \left. \frac{\delta^2 \Delta S_r[\phi_r]}{\delta \phi_r(x)\delta \phi_r(y)} \right|_{\phi_r=\phi_{\text{cl}}} \\
&+ \dots .
\end{aligned} \tag{2.8}$$

The linear term in η represents a tadpole and must be adjusted to vanish, in such a way that $\langle \eta(x) \rangle_J = 0$ and therefore $\langle \phi_r \rangle_J = \phi_{\text{cl}}$, as assumed. The other terms act as counter-terms for the self-interaction vertices and thus:

$$\begin{aligned}
\int d^4x \left(\Delta \mathcal{L}[\phi_r(x)] + \Delta J(x)\phi_r(x) \right) &= \int d^4x \left(\Delta \mathcal{L}[\phi_{\text{cl}}(x)] + \Delta J(x)\phi_{\text{cl}}(x) \right) \\
&+ \text{counter-terms for vertices in } \eta .
\end{aligned} \tag{2.9}$$

Putting everything together, one arrives finally at the following expression for the generating functional $Z[J]$:

$$\begin{aligned}
Z[J] &= \exp \left\{ i \int d^4x \left(\mathcal{L}_r[\phi_{\text{cl}}(x)] + J_r(x)\phi_{\text{cl}}(x) + \Delta \mathcal{L}[\phi_{\text{cl}}(x)] + \Delta J(x)\phi_{\text{cl}}(x) \right) \right\} \\
&\int \mathcal{D}\eta \exp \left\{ i\tilde{S}[\eta] + i\Delta\tilde{S}[\eta] \right\} ,
\end{aligned} \tag{2.10}$$

where:

$$\tilde{S}[\eta] = \int d^4x \int d^4y \frac{1}{2} \eta(x) \left(\frac{\delta^2 S_r}{\delta \phi_r^2} [\phi_{\text{cl}}](x, y) \right) \eta(y) + \text{vertices}, \quad (2.11)$$

$$\Delta \tilde{S}[\eta] = \text{counter-terms}. \quad (2.12)$$

At this stage, we are left with the evaluation of a path-integral for a field η with a quadratic term plus self-interaction vertices. The inverse of the operator appearing in the quadratic term defines a propagator for the field η , which is given by:

$$D(x, y) = -i \left(\frac{\delta^2 S_r}{\delta \phi_r^2} \right)^{-1} [\phi_{\text{cl}}](x, y). \quad (2.13)$$

Neglecting the interactions completely, the path-integral over η is Gaussian and can be computed explicitly. It yields a negative or positive power of the determinant of the kinetic operator, depending on whether the fields are bosonic or fermionic, which corresponds to the resummation of all the 1-loop diagrams:

$$\det^{\mp 1/2} \left(-\frac{\delta^2 S_r}{\delta \phi_r^2} [\phi_{\text{cl}}] \right) = \exp \left\{ \mp \frac{1}{2} \text{tr} \log \left(-\frac{\delta^2 S_r}{\delta \phi_r^2} [\phi_{\text{cl}}] \right) \right\}. \quad (2.14)$$

Treating the interactions in perturbation theory, one finds then an expansion in Feynman diagrams, starting with 2-loop vacuum bubbles. These can be shown to exponentiate, as usual, so that finally one obtains:

$$\int \mathcal{D}\eta \exp \left\{ i\tilde{S}[\eta] + i\Delta\tilde{S}[\eta] \right\} = \exp \left\{ \mp \frac{1}{2} \text{tr} \log \left(-\frac{\delta^2 S_r}{\delta \phi_r^2} [\phi_{\text{cl}}] \right) + \left(\text{sum of conn. diag.} \right) \right\} \quad (2.15)$$

Using the above result, the final form of the generating functional $Z[J]$ takes explicitly the form $Z[J] = \exp\{iW[J]\}$, and $W[J]$ can be identified with:

$$\begin{aligned} W[J] &= \int d^4x \left(\mathcal{L}_r[\phi_{\text{cl}}(x)] + J_r(x)\phi_{\text{cl}}(x) + \Delta\mathcal{L}[\phi_{\text{cl}}(x)] + \Delta J(x)\phi_{\text{cl}}(x) \right) \\ &\quad \pm \frac{i}{2} \text{tr} \log \left(-\frac{\delta^2 S_r}{\delta \phi_r^2} [\phi_{\text{cl}}] \right) - i \left(\text{sum of conn. diag.} \right). \end{aligned} \quad (2.16)$$

Finally, to compute the effective action we need to perform the Legendre transform of this expression:

$$\Gamma[\phi_{\text{cl}}] = W[J] - \int d^4x J(x)\phi_{\text{cl}}(x). \quad (2.17)$$

Recalling that $J = J_r + \Delta J$, this just cancels all the terms of $W[J]$ that depend explicitly on the external current, and one is left with the following simple result:

$$\Gamma[\phi_{\text{cl}}] = S_r[\phi_{\text{cl}}] \pm \frac{i}{2} \text{tr} \log \left(-\frac{\delta^2 S_r}{\delta \phi_r^2} [\phi_{\text{cl}}] \right) - i \left(\text{sum of conn. diag.} \right) + \Delta S[\phi_{\text{cl}}]. \quad (2.18)$$

As expected, the effective action does finally depend explicitly only on ϕ_{cl} . It is given by the sum of the tree-level classical action, a 1-loop correction written in closed form, plus and infinite series of higher-loop corrections that can only be computed diagrammatically, using the propagator and the vertices derived above for the fluctuation field η .

2.2 Effective vertices and effective potential

The general form of the effective action consists of the standard two-derivative kinetic term multiplied by some non-trivial wave-function factor, an effective potential without derivatives, and in general also an infinite series of higher-derivative corrections:

$$\Gamma[\phi_{\text{cl}}] = \int d^4x \left(Z_{\text{eff}}[\phi_{\text{cl}}] \partial_\mu \phi_{\text{cl}} \partial^\mu \phi_{\text{cl}} - V_{\text{eff}}[\phi_{\text{cl}}] + \text{higher-der.} \right). \quad (2.19)$$

The effective potential V_{eff} is a particularly interesting quantity, especially for the issue of symmetries, since constant vacuum expectation values are determined by minimizing it. It can be deduced from the effective action computed for constant ϕ_{cl} , which is proportional to it times the total volume of space-time VT :

$$V_{\text{eff}}[\phi_{\text{cl}}] = - \frac{1}{VT} \Gamma[\phi_{\text{cl}}] \Big|_{\phi_{\text{cl}}=\text{const.}}. \quad (2.20)$$

When computing $\Gamma[\phi_{\text{cl}}]$ for constant ϕ_{cl} , the trace over states also involves an integral over space-time of the zero-modes implied by translational invariance, which provides a factor of the space-time volume VT which cancels the one in the above formula.

The effective action encodes the full quantum dynamics of the theory in a classical language, in the sense that the full loop-corrected value of any S -matrix element can be obtained by using the effective action Γ and computing only tree-level diagrams. This is a consequence of the fact that the effective action Γ is the generating functional of all the 1PI correlation functions. Calling these $\Gamma^{(n)}(x_1, \dots, x_n)$, this means that:

$$\begin{aligned} \Gamma[\phi_{\text{cl}}] &= \frac{i}{2} \int d^4x_1 \int d^4x_2 \Gamma^{(2)}(x_1, x_2) \phi_{\text{cl}}(x_1) \phi_{\text{cl}}(x_2) \\ &\quad + \frac{i}{3!} \int d^4x_1 \int d^4x_2 \int d^4x_3 \Gamma^{(3)}(x_1, x_2, x_3) \phi_{\text{cl}}(x_1) \phi_{\text{cl}}(x_2) \phi_{\text{cl}}(x_3) \\ &\quad + \dots \end{aligned} \quad (2.21)$$

For the effective potential, corresponding to the zero-derivative term in the effective action, one finds a similar expansion, with constant effective vertices $V_{\text{eff}}^{(n)}$:

$$V_{\text{eff}}[\phi_{\text{cl}}] = \frac{1}{2} \int d^4x V_{\text{eff}}^{(2)} \phi_{\text{cl}}^2(x) + \frac{1}{3!} \int d^4x V_{\text{eff}}^{(3)} \phi_{\text{cl}}^3(x) + \dots \quad (2.22)$$

In other words, the effective action and the effective potential contain all the effective vertices among the fields ϕ_{cl} which are induced by quantum fluctuations through loops, respectively at any momentum and at zero-momentum.

2.3 Symmetry breaking and Goldstone theorem

As already mentioned, the effective potential is the quantity controlling the way in which a global symmetry of the theory is realized. This may be respected by the vacuum, and thus linearly realized, or spontaneously broken by the vacuum, and therefore non-linearly realized. The parameter distinguishing between the two options is simply the vacuum

expectation value of the field ϕ_{cl} obtained by minimizing V_{eff} . If this leads to a non-zero value of the field transformation law, then the symmetry is broken, whereas otherwise, the symmetry is preserved.

Using the effective potential, it is possible to prove Goldstone's theorem in full generality at the quantum level, in a very simple and intuitive way which parallels the classical argumentation. Consider for this an infinitesimal symmetry transformation of the type:

$$\phi_{\text{cl}} \rightarrow \phi_{\text{cl}} + \alpha \Delta(\phi_{\text{cl}}). \quad (2.23)$$

If this is a symmetry of the theory, the effective potential must be invariant:

$$V_{\text{eff}}(\phi_{\text{cl}} + \alpha \Delta(\phi_{\text{cl}})) = V_{\text{eff}}(\phi_{\text{cl}}). \quad (2.24)$$

For infinitesimal α , this implies that at any point ϕ_{cl} one should have:

$$\Delta(\phi_{\text{cl}}) V'(\phi_{\text{cl}}) = 0. \quad (2.25)$$

Differentiating then this equation with respect to ϕ_{cl} , one obtains a condition involving the second derivative of the potential:

$$\Delta(\phi_{\text{cl}}) V''(\phi_{\text{cl}}) + \Delta'(\phi_{\text{cl}}) V'(\phi_{\text{cl}}) = 0. \quad (2.26)$$

Consider now the particular point $\langle \phi_{\text{cl}} \rangle$ which minimizes $V(\phi_{\text{cl}})$. At that point the potential is by definition stationary, and thus:

$$V'(\langle \phi_{\text{cl}} \rangle) = 0. \quad (2.27)$$

The second derivative of the potential at this point gives instead the squared mass for small fluctuations around the vacuum, which must be semi-positive definite for metastability:

$$V''(\langle \phi_{\text{cl}} \rangle) = m^2 \geq 0. \quad (2.28)$$

Finally, the quantity $\Delta(\langle \phi_{\text{cl}} \rangle)$ is the order parameter controlling the spontaneous breaking of the symmetry by the vacuum, since it is non-zero if and only if the vacuum is not invariant under symmetry transformations:

$$\Delta(\langle \phi_{\text{cl}} \rangle) = \rho. \quad (2.29)$$

Applying the above-derived general relation involving the second derivative of the potential at the particular point $\langle \phi_{\text{cl}} \rangle$, one finally finds:

$$\rho m^2 = 0. \quad (2.30)$$

This means that whenever $\rho \neq 0$, the squared mass matrix m^2 must necessarily have a flat direction, corresponding to a massless mode. Schematically:

$$\rho \neq 0 \Rightarrow m = 0. \quad (2.31)$$

For more general situations involving several symmetries, the mass matrix must have one independent flat direction for each independent continuous symmetry that is spontaneously broken, leading to equally many massless Goldstone bosons.

It may happen that the tree-level potential has a minimum which preserves the symmetry, but that taking into account quantum corrections one finds an effective potential whose minimum breaks the symmetry. One then says that spontaneous symmetry breaking occurs radiatively, in the sense that it is induced by quantum effects.

2.4 Leading quantum corrections and determinants

We have seen that the leading quantum corrections to the effective action are encoded in the spectrum of the operator governing the propagation of fluctuations around the configuration $\phi_{\text{cl}}(x)$. To illustrate how this correction can be computed, let us first consider simple scalar field theories with non-derivative interactions. The leading classical part of the action has then the form:

$$\Gamma^0[\phi_{\text{cl}}] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi_{\text{cl}}(x) \partial^\mu \phi_{\text{cl}}(x) - \frac{1}{2} m^2 \phi_{\text{cl}}^2(x) - \frac{1}{4!} \lambda \phi_{\text{cl}}^4(x) \right). \quad (2.32)$$

The 1-loop correction to the effective action is then

$$\begin{aligned} \Gamma^1[\phi_{\text{cl}}] &= \frac{i}{2} \text{tr} \log \left(\square + m^2 + \frac{\lambda}{2} \phi_{\text{cl}}^2 \right) + \Delta^1 S \\ &= \frac{i}{2} \text{tr} \log \left[\left(\square + m^2 \right) \left(1 + \frac{\lambda}{2} (\square + m^2)^{-1} \phi_{\text{cl}}^2 \right) \right] + \Delta^1 S \\ &= \frac{i}{2} \text{tr} \log \left(\square + m^2 \right) + \frac{i}{2} \text{tr} \log \left(1 + \frac{\lambda}{2} D \phi_{\text{cl}}^2 \right) + \Delta^1 S. \end{aligned} \quad (2.33)$$

In order to evaluate these traces, we can consider the continuous basis of plane waves with definite momentum p , and view the operators as infinite-dimensional matrices in this space. The first term is the Gaussian path-integral for a free particle. This amounts to an irrelevant constant field-independent contribution, which can be dropped. The second term can be expanded by using:

$$\log(1 - x) = - \sum_n \frac{x^n}{n}. \quad (2.34)$$

In this way one finds, making more explicit also the structure of the counter-terms:

$$\begin{aligned} \Gamma^1[\phi_{\text{cl}}] &= \text{const.} - \sum_n \frac{i}{2n} \text{tr} \left(-\frac{\lambda}{2} D \phi_{\text{cl}}^2 \right)^n \\ &\quad + \frac{1}{2} \Delta_Z^1 \partial_\mu \phi_{\text{cl}} \partial^\mu \phi_{\text{cl}} - \frac{1}{2} \Delta_m^1 \phi_{\text{cl}}^2 - \frac{1}{4!} \Delta_\lambda^1 \phi_{\text{cl}}^4. \end{aligned} \quad (2.35)$$

Diagrammatically, this corresponds to sum up all the 1-loop diagrams with an arbitrary number n of vertices between the fluctuation field η and the background field ϕ_{cl} , which is $\lambda/2 \phi_{\text{cl}}^2 \eta^2$, including counter-terms:

$$\begin{aligned}
\Gamma^1[\phi_{\text{cl}}] = & \text{const.} + \text{diag}_1 + \text{diag}_2 + \dots \\
& + \text{diag}_3 + \text{diag}_4 + \text{diag}_5.
\end{aligned} \tag{2.36}$$

The n -th diagram has a symmetry factor $1/n$, correctly reproducing the coefficient in the Taylor expansion of the logarithm, because one can rotate the n interactions cyclically without changing the diagram. The only divergences come from the diagrams with 1 and 2 vertices, and can be reabsorbed in the counter-terms. The diagrams with 3 and more vertices are instead all finite.

The effective potential is computed by considering the effective action for constant configurations for ϕ_{cl} , and is given by

$$V_{\text{eff}}^1(\phi_{\text{cl}}) = -\frac{i}{2} \text{tr}' \log \left(\square + M^2(\phi_{\text{cl}}) \right) + \Delta^1 V. \tag{2.37}$$

where tr' denotes the trace over non-zero modes and $M(\phi_{\text{cl}})$ is a mass depending on the constant background field:

$$M^2(\phi_{\text{cl}}) = m^2 + \frac{\lambda}{2} \phi_{\text{cl}}^2. \tag{2.38}$$

This can be evaluated explicitly, because the trace over plane wave states boils down to a simple momentum integral, as a consequence of the lack of any coordinate dependence. It is convenient to perform an analytic continuation to Euclidean space and use dimensional regularization. Recalling that for small α one has $\Gamma(\alpha) \sim \alpha^{-1}$, one finds:

$$\begin{aligned}
\text{tr}' \log \left(\square + M^2(\phi_{\text{cl}}) \right) &= i \int \frac{d^d p_E}{(2\pi)^d} \log \left(p_E^2 + M^2(\phi_{\text{cl}}) \right) \\
&= -i \frac{\partial}{\partial \alpha} \int \frac{d^d p_E}{(2\pi)^d} \left(p_E^2 + M^2(\phi_{\text{cl}}) \right)^{-\alpha} \Big|_{\alpha=0}
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
&= -i \frac{\partial}{\partial \alpha} \left(\frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)} \left(M^2(\phi_{\text{cl}}) \right)^{d/2 - \alpha} \right) \Big|_{\alpha=0} \\
&= -i \frac{\Gamma(-d/2)}{(4\pi)^{d/2}} \left(M^2(\phi_{\text{cl}}) \right)^{d/2}.
\end{aligned} \tag{2.40}$$

One can use this result to write down the 1-loop contribution to the effective potential. Introducing an arbitrary scale $\tilde{\mu}$ to restore ordinary dimensions and setting $d = 4 - 2\epsilon$, one finds:

$$\begin{aligned}
V_{\text{eff}}^1(\phi_{\text{cl}}) &= -\frac{1}{2} \tilde{\mu}^{4-d} \frac{\Gamma(-d/2)}{(4\pi)^{d/2}} \left(M^2(\phi_{\text{cl}}) \right)^{d/2} + \frac{1}{2} \Delta_m^1 \phi_{\text{cl}}^2 + \frac{1}{4!} \Delta_\lambda^1 \phi_{\text{cl}}^4 \\
&= -\frac{1}{2(4\pi)^2} M^4(\phi_{\text{cl}}) \Gamma(-2 + \epsilon) \left(\frac{M^2(\phi_{\text{cl}})}{4\pi \tilde{\mu}^2} \right)^{-\epsilon} + \frac{1}{2} \Delta_m^1 \phi_{\text{cl}}^2 + \frac{1}{4!} \Delta_\lambda^1 \phi_{\text{cl}}^4.
\end{aligned} \tag{2.41}$$

Taking finally the limit $\epsilon \rightarrow 0$ and using:

$$\Gamma(-2 + \epsilon) = \frac{1}{2} \left(\frac{1}{\epsilon} - \gamma + \frac{3}{2} \right), \tag{2.42}$$

one finds:

$$\begin{aligned} V_{\text{eff}}^1(\phi_{\text{cl}}) &= -\frac{1}{4(4\pi)^2} M^4(\phi_{\text{cl}}) \left(\frac{1}{\epsilon} - \gamma + \frac{3}{2} + \log \frac{4\pi\tilde{\mu}^2}{M^2(\phi_{\text{cl}})} \right) + \frac{1}{2} \Delta_m^1 \phi_{\text{cl}}^2 + \frac{1}{4!} \Delta_\lambda^1 \phi_{\text{cl}}^4 \\ &= -\frac{1}{4(4\pi)^2} M^4(\phi_{\text{cl}}) \left(\frac{3}{2} + \log \frac{\tilde{\Lambda}^2}{M^2(\phi_{\text{cl}})} \right) + \frac{1}{2} \Delta_m^1 \phi_{\text{cl}}^2 + \frac{1}{4!} \Delta_\lambda^1 \phi_{\text{cl}}^4, \end{aligned} \quad (2.43)$$

where

$$\tilde{\Lambda} = \sqrt{4\pi} e^{-\gamma/2} e^{1/(2\epsilon)} \tilde{\mu}. \quad (2.44)$$

2.5 World-line formalism

It is possible to rewrite the expression for the 1-loop contribution to the effective action in a way which allows a very useful interpretation in terms of first quantized relativistic particles. We will illustrate the point for scalar fields, but the same idea can be generalized to any kind of field.

Consider a free spin-less relativistic point-particle of mass m . This is described by a world-line action proportional to the relativistic interval accumulated along the particle trajectory in space-time:

$$S = -m \int d\tau \sqrt{\eta_{\mu\nu} \dot{q}^\mu \dot{q}^\nu}. \quad (2.45)$$

This expression is, as it should, invariant under reparametrizations of the proper-time, $\tau \rightarrow f(\tau)$, and depends only on the length measured with the Minkowski metric. In practice, however, it is not very convenient to use, because of the square root. Moreover, the canonical momentum is given by:

$$\pi_\mu = -m \frac{\dot{q}_\mu}{\sqrt{\eta_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta}}. \quad (2.46)$$

This leads to a constraint:

$$-\pi_\mu \pi^\mu + m^2 = 0. \quad (2.47)$$

As a result, the naive Hamiltonian is trivial, $H = \pi_\mu \dot{q}^\mu - L = 0$, and the dynamics is actually governed by the constraint. At the quantum level, where $\pi_\mu \rightarrow i\partial_\mu$, this correctly leads to the Klein-Gordon equation for the wave-function.

A more convenient formulation of the same theory can be obtained by introducing an auxiliary field e to rewrite the action in a quadratic form:

$$S = -\frac{1}{2} \int d\tau \left[e^{-1} \eta_{\mu\nu} \dot{q}^\mu \dot{q}^\nu + e m^2 \right]. \quad (2.48)$$

The field e appears without derivative in the action, and it is therefore an auxiliary field. Its value is completely fixed by its equation of motion, and yields back the original action:

$$e = \frac{\sqrt{\eta_{\mu\nu} \dot{q}^\mu \dot{q}^\nu}}{m}. \quad (2.49)$$

One has exactly the same dynamics as before. The constraint comes now from the equation of motion of the auxiliary field, which before simplification has the form $-\pi_\mu \pi^\mu + m^2 = 0$ with $\pi_\mu = -e^{-1} \dot{q}_\mu$. In this new formulation, the reparametrization symmetry is realized in a more geometric way, with the field e playing the role of a world-line metric. The corresponding transformations are $\tau \rightarrow f(\tau)$, $q^\mu(\tau) \rightarrow q^\mu(f(\tau))$ and $e(\tau) \rightarrow \dot{f}(\tau)e(f(\tau))$, and leave the action invariant. The corresponding conserved quantity is a sort of world-line energy and reads $T = -\pi_\mu \pi^\mu + m^2$. The fact that $\dot{T} = 0$ ensures that the constraint $T = 0$ is preserved by the dynamics.

We can now use the new action in a different way, by eliminating the auxiliary field e through a gauge fixing of the reparametrization symmetry, rather than through its equations of motion. In this way, no constraint is left and both the Lagrangian and the Hamiltonian become non-trivial and quadratic. A convenient gauge choice for our purposes here is to set

$$e = 2. \tag{2.50}$$

Plugging this into the Lagrangian, and performing an analytic continuation to Euclidean space, we find then the following very simple Lagrangian:

$$\hat{L} = \frac{1}{4} \dot{q}_E^2 - m^2. \tag{2.51}$$

This resembles now to the action for a non-relativistic particle in 4 Euclidean space dimensions, of the form $\hat{L} = (\hat{m}/2) \dot{q}_E^2 - \hat{V}(q_E)$, with a fixed mass parameter $\hat{m} = 1/2$ and a constant potential $\hat{V} = m^2$. The corresponding Hamiltonian is then given simply by $\hat{H} = p_E^2/(2\hat{m}) + \hat{V}$, or:

$$\hat{H} = -\square_E + m^2. \tag{2.52}$$

Consider now the evolution operator $U(\tau) = e^{-i\tau\hat{H}}$ associated to this free particle and construct the corresponding evolution kernel, or propagator. This is given by a path-integral with suitable boundary conditions $q_E(\tau_a) = q_{Ea}$ and $q_E(\tau_b) = q_{Eb}$:

$$\langle q_{Eb} | e^{-i(\tau_b - \tau_a)\hat{H}} | q_{Ea} \rangle = \int_{\text{b.c.}} \mathcal{D}q_E \exp \left\{ i \int_{\tau_a}^{\tau_b} d\tau' \hat{L}[q_E(\tau')] \right\}. \tag{2.53}$$

It is now clear that the object that must be relevant for a closed loop is the sum of all the diagonal elements of this propagator, or in other terms the trace of the evolution operator. This is given by a path-integral with periodic boundary conditions $q_E(-\tau/2) = q_E(\tau/2)$. Making also an analytic continuation to Euclidean time $\tau \rightarrow -i\tau$, one finds then:

$$\text{tr} \left[e^{-\tau\hat{H}} \right] = \int_{\text{p.b.c.}} \mathcal{D}q_E \exp \left\{ - \int_{-\tau/2}^{\tau/2} d\tau' \left(\frac{1}{4} \dot{q}_E^2(\tau') + m^2 \right) \right\}. \tag{2.54}$$

This is the amplitude for the particle to freely propagate around a closed loop with Euclidean proper time τ . Clearly, in order to relate this to the 1-loop effective action of the corresponding quantum field theory, it is necessary to figure out how the external field ϕ_{cl} affects the propagation of the particle, and also somehow to integrate over all possible

$\tau \in [0, \infty[$. But how this should be done precisely can of course not be inferred from this simple first quantized approach

In order to find a precise representation of the 1-loop contribution to the effective action in terms of the partition function $Z[\tau]$ of a first quantized particle, we can use the following identity:

$$\log(x) = \int_0^\infty \frac{d\tau}{\tau} (e^{-\tau} - e^{-\tau x}). \quad (2.55)$$

Using this representation, and making an analytic continuation to Euclidean space, we can then rewrite the 1-loop contribution to the effective action as

$$\begin{aligned} \Gamma^1[\phi_{\text{cl}}] &= -\frac{1}{2} \text{tr} \log \left(-\square_E + m^2 + \frac{\lambda}{2} \phi_{\text{cl}}^2 \right) + \Delta^1 S \\ &= \text{const.} + \frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \text{tr} \exp \left\{ -\tau \left(-\square_E + m^2 + \frac{\lambda}{2} \phi_{\text{cl}}^2 \right) \right\} + \Delta^1 S. \end{aligned} \quad (2.56)$$

With this trick, we have thus transformed the logarithm into a suitable integrated exponential, and we are left with the problem of evaluating the trace of the exponential of an operator. This can be interpreted as the evolution operator for a first-quantized point particle, as expected. More precisely, we see that we can rewrite the result as:

$$\Gamma^1[\phi_{\text{cl}}] = \text{const.} + \frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} Z[\tau] + \Delta^1 S, \quad (2.57)$$

in terms of the partition function

$$Z[\tau] = \text{tr} e^{-\tau \hat{H}}, \quad (2.58)$$

defined with the Hamiltonian

$$\hat{H} = -\square_E + m^2 + \frac{\lambda}{2} \phi_{\text{cl}}^2(q_E). \quad (2.59)$$

The appropriate integration measure is thus found to be $d\tau/\tau$. This reproduces the logarithmic structure of the quantum field theory effective action, which as we have seen is linked to the peculiar symmetry factors associated to the diagrams that are resummed. The appropriate Hamiltonian has instead a field dependent mass. This reproduces the correct vertex with the external field.

According to our general discussion, we would now like to interpret this partition function in terms of a non-relativistic point particle in 4 Euclidean space dimensions. Comparing the above expression with the general form of the Hamiltonian $\hat{H} = -\square_E/(2\hat{m}) + \hat{V}(q_E)$ for a particle of mass \hat{m} subject to a potential $\hat{V}(q_E)$, we deduce that the mass and the potential that should be attributed to the first-quantized particle are given by $\hat{m} = 1/2$ and $\hat{V}(q_E) = m^2 + (\lambda/2)\phi_{\text{cl}}^2(q_E)$. The corresponding classical Lagrangian $\hat{L} = (\hat{m}/2)\dot{q}_E^2 - \hat{V}(q_E)$ reads then:

$$\hat{L} = \frac{1}{4} \dot{q}_E^2 - m^2 - \frac{\lambda}{2} \phi_{\text{cl}}^2(q_E). \quad (2.60)$$

The partition function $Z[\tau] = \text{tr}[e^{-\tau\hat{H}}]$ can be computed either operatorially, from the spectrum of the Hamiltonian, or with a functional integral with Euclidean time and periodic boundary conditions $q(-\tau/2) = q(\tau/2)$:

$$Z[\tau] = \int_{\text{p.b.c.}} \mathcal{D}q \exp \left\{ - \int_{-\tau/2}^{\tau/2} d\tau' \left(\frac{1}{4} \dot{q}_E^2(\tau') + m^2 + \frac{\lambda}{2} \phi_{\text{cl}}^2(q_E(\tau')) \right) \right\}. \quad (2.61)$$

For arbitrary coordinate-dependent field configurations for ϕ_{cl} , this partition function can not be computed exactly, but this formulation can still be useful, because it becomes possible to use semi-classical approximation methods, like the WKB method.

For constant coordinate-independent field configurations ϕ_{cl} , we can look at the effective potential, which is given by:

$$\begin{aligned} V_{\text{eff}}^1[\phi_{\text{cl}}] &= \frac{1}{2} \text{tr}' \log \left(-\square_E + M^2(\phi_{\text{cl}}) \right) + \Delta^1 V \\ &= \text{const.} - \frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \text{tr}' \exp \left\{ -\tau \left(-\square_E + M^2(\phi_{\text{cl}}) \right) \right\} + \Delta^1 V, \end{aligned} \quad (2.62)$$

where

$$M^2(\phi_{\text{cl}}) = m^2 + \frac{\lambda}{2} \phi_{\text{cl}}^2. \quad (2.63)$$

Performing the same manipulations as before, this can be rewritten as:

$$V^1[\phi_{\text{cl}}] = \text{const.} - \frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} Z'[\tau] + \Delta^1 V, \quad (2.64)$$

in terms of the partition function

$$Z'[\tau] = \text{tr}' e^{-\tau\hat{H}}, \quad (2.65)$$

defined with the Hamiltonian

$$\hat{H} = -\square_E + M^2(\phi_{\text{cl}}). \quad (2.66)$$

The effect of the constant external field is therefore accounted by a deformed mass for the point particle, and the corresponding Lagrangian is:

$$\hat{L} = \frac{1}{4} \dot{q}_E^2 - M^2(\phi_{\text{cl}}^2). \quad (2.67)$$

The partition function $Z'[\tau] = \text{tr}'[e^{-\tau\hat{H}}]$ can then be computed either operatorially or through a path-integral with Euclidean time, periodic boundary conditions and zero-modes omitted:

$$Z'[\tau] = \int_{\substack{\text{p.b.c.} \\ \text{no z.m.}}} \mathcal{D}q \exp \left\{ - \int_{-\tau/2}^{\tau/2} d\tau' \left(\frac{1}{4} \dot{q}_E^2(\tau') + M^2(\phi_{\text{cl}}) \right) \right\}. \quad (2.68)$$

In this case, the computation is very simple. Proceeding operatorially with a basis of plane waves with definite momentum, one finds:

$$\begin{aligned} Z'[\tau] &= \int \frac{d^d p_E}{(2\pi)^d} \exp \left\{ -\tau \left(p_E^2 + M^2(\phi_{\text{cl}}) \right) \right\} \\ &= (4\pi\tau)^{-d/2} e^{-\tau M^2(\phi_{\text{cl}})}. \end{aligned} \quad (2.69)$$

It follows that:

$$\begin{aligned} \int_0^\infty \frac{d\tau}{\tau} Z'[\tau] &= (4\pi)^{-d/2} \int_0^\infty d\tau \tau^{-1-d/2} e^{-\tau M^2(\phi_{\text{cl}})} \\ &= \frac{\Gamma(-d/2)}{(4\pi)^{d/2}} \left(M^2(\phi_{\text{cl}}) \right)^{d/2}. \end{aligned} \quad (2.70)$$

Finally, introducing as before an arbitrary scale $\tilde{\mu}$ to switch back to ordinary dimensions, we recover the correct expression for the effective potential:

$$V_{\text{eff}}^1(\phi_{\text{cl}}) = -\frac{1}{2} \tilde{\mu}^{4-d} \frac{\Gamma(-d/2)}{(4\pi)^{d/2}} \left(M^2(\phi_{\text{cl}}) \right)^{d/2} + \frac{1}{2} \Delta_m^1 \phi_{\text{cl}}^2 + \frac{1}{4!} \Delta_\lambda^1 \phi_{\text{cl}}^4. \quad (2.71)$$

As we already saw, setting $d = 4 - 2\epsilon$, taking $\epsilon \rightarrow 0$ and defining $\tilde{\Lambda} = \sqrt{4\pi} e^{-\gamma/2} e^{1/(2\epsilon)} \tilde{\mu}$, one can finally rewrite this result as:

$$V_{\text{eff}}^1(\phi_{\text{cl}}) = -\frac{1}{4(4\pi)^2} M^4(\phi_{\text{cl}}) \left(\frac{3}{2} + \log \frac{\tilde{\Lambda}^2}{M^2(\phi_{\text{cl}})} \right) + \frac{1}{2} \Delta_m^1 \phi_{\text{cl}}^2 + \frac{1}{4!} \Delta_\lambda^1 \phi_{\text{cl}}^4. \quad (2.72)$$

3 Renormalization group and running couplings

3.1 Renormalization at an arbitrary scale

In renormalized perturbation theory, the bare Lagrangian is split into a renormalized Lagrangian plus a set of counter-terms. These are then fixed order by order in perturbation theory by imposing some renormalization conditions on 1PI correlation functions corresponding to potentially divergent quantities. For generality, we shall impose such conditions at some arbitrary renormalization scale μ , assigning a particular value to these basic 1PI couplings at the point where all the momenta p_i are space-like and of order μ . The counter-terms will then consist of a divergent part, which compensates the divergences arising in the loop integrals of the other diagrams, and a finite part depending on μ . Our basic aim is to study the dependence of physical quantities on this scale μ .

As the simplest example, we shall consider as usual a self-interacting scalar theory. The renormalized Lagrangian is defined in terms of the renormalized field

$$\phi = Z^{1/2} \phi_r, \quad (3.1)$$

and takes the form:

$$\mathcal{L} = \mathcal{L}_r + \Delta\mathcal{L}, \quad (3.2)$$

with

$$\mathcal{L}_r = \frac{1}{2} \partial_\mu \phi_r \partial^\mu \phi_r - \frac{1}{2} m^2 \phi_r^2 - \frac{1}{4!} \lambda \phi_r^4, \quad (3.3)$$

$$\Delta\mathcal{L} = \frac{1}{2} \Delta_Z \partial_\mu \phi_r \partial^\mu \phi_r - \frac{1}{2} \Delta_m \phi_r^2 - \frac{1}{4!} \Delta_\lambda \phi_r^4, \quad (3.4)$$

where the counter-terms have the following structure:

$$\Delta_Z = Z - 1, \quad \Delta_m = m_0^2 Z - m^2, \quad \Delta_\lambda = \lambda_0 Z^2 - \lambda. \quad (3.5)$$

In this simple example, the relevant 1PI couplings are the 2-point function $\Pi(p^2)$ and the 4-point function $\Gamma(s, t, u)$, where $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$ are the usual Mandelstam variables:

$$\begin{array}{c} \text{---} \textcircled{\text{1PI}} \text{---} \\ p \qquad p \end{array} = -i \Pi(p), \quad \begin{array}{c} p_2 \quad p_4 \\ \diagdown \quad \diagup \\ \textcircled{\text{1PI}} \\ \diagup \quad \diagdown \\ p_1 \quad p_3 \end{array} = -i \Gamma(s, t, u). \quad (3.6)$$

One can then impose the following general renormalization conditions:

$$\Pi(-\mu^2) = 0, \quad (3.7)$$

$$\Pi'(-\mu^2) = 0, \quad (3.8)$$

$$\Gamma(-\mu^2, -\mu^2, -\mu^2) = \lambda. \quad (3.9)$$

These conditions define the renormalized theory at the scale μ by imposing that at that momentum scale the propagator takes the form $i/(p^2 - m^2)$ whereas the quartic coupling is equal to λ . Notice that with these generalized renormalization conditions, the pole of the propagator is no longer at $p^2 = m^2$, and moreover the residue at the pole is no longer equal to 1. This means that the parameter m differs now from the physical mass, although it can be related to it order by order in perturbation theory. Moreover, when extracting S -matrix elements from renormalized Green function by LSZ reduction, one must take into account the non-trivial residue at the pole.

Notice that one can in principle perform any additional finite rescaling ξ of the renormalized fields, without affecting the physical S -matrix elements. Indeed, under the transformation $\phi_r \rightarrow \xi\phi_r$, a generic n -point Green function will simply get rescaled as $G^{(n)} \rightarrow \xi^n G^{(n)}$. Moreover, the residue of the pole in the propagator will be rescaled by ξ^2 . Then, when computing S -matrix elements as amputated Green-function obtained by factorizing two-point functions for each external leg near the mass-shell, and multiplying by the square-root of the pole residues, all the ξ factors cancel out. As a result, the S -matrix is independent of ξ .

3.2 Dimensionless couplings

Consider first the case of a renormalizable theory with massless fields and interactions governed by dimensionless couplings. At the classical level, such a theory is scale invariant. At the quantum level, however, it ceases to be so. This is due to the fact that in order to define the theory we have to regulate it, and this unavoidably involves the introduction of some mass scale which breaks this symmetry. In other words, the regularization does not preserve the symmetry, and as a results the physical predictions of the theory depend somehow on the scale. This phenomenon falls into the general category of quantum anomalies, which we will study later on.

The starting point of the analysis is the observation that the renormalization conditions that we have imposed involve an arbitrary scale μ , a corresponding numerical coupling λ and implicitly also some canonical field-strength normalization factor ξ , taken to be 1. We could then have equally well defined the theory at a different scale μ' , with a different numerical coupling λ' and a different field-strength normalization ξ' . In order for this to be the same theory, however, the relation between λ' , ξ' and λ , ξ is fixed by the relation between μ' and μ , since the 1PI vertices used in the renormalization conditions depend on the momentum scale. This suggests that a given theory is associated with a continuous family of equivalent (μ, λ, ξ) . Correspondingly, the renormalized Green functions will have a dependence on the quantities μ , λ and ξ that is actually constrained.

The scaling identity satisfied by renormalized correlation functions can be easily deduced by recalling their definition and their relation to the bare correlation functions. The renormalized Green functions are defined with the renormalized fields and depend on the scale μ and the coupling λ :

$$G_r(x_1, \dots, x_n; \mu, \lambda) = \langle \phi_r(x_1) \cdots \phi_r(x_n) \rangle. \quad (3.10)$$

The bare Green functions, on the other hand, are defined with the bare fields and depend on the bare coupling λ_0 :

$$G(x_1, \dots, x_n; \lambda_0) = \langle \phi(x_1) \cdots \phi(x_n) \rangle. \quad (3.11)$$

The relation between bare and renormalized fields is $\phi(x) = Z^{1/2} \phi_r(x)$, where Z depends on the scale μ . One deduces then that:

$$G_r(x_1, \dots, x_n; \mu, \lambda) = Z^{-n/2} G(x_1, \dots, x_n; \lambda_0). \quad (3.12)$$

The apparently trivial relation contains in fact all the information we are after. The left-hand side depends on the renormalization scale μ , the coupling λ and the field-strength normalization ξ , and is finite. The right hand side, on the other hand, depends instead on the bare coupling λ_0 and the field-strength renormalization Z , which depend on the cut-off. It also depends on the scale μ , but only through Z . Their equality fixes then the structure of the relative dependence on the parameters (μ, λ, ξ) resulting from the renormalization procedure of pushing all of the cut-off dependence into the bare parameters. More precisely, the trajectories of parameters (μ, λ, ξ) defining the same theory through different renormalization conditions are determined by requiring that the bare Green-function should be invariant under infinitesimal transformations $\mu \rightarrow \mu + \delta\mu$, $\lambda \rightarrow \lambda + \delta\lambda$ combined with $Z \rightarrow Z + \delta Z$, done at fixed values for the bare coupling λ_0 . Starting from the inverted relation

$$G(x_1, \dots, x_n; \lambda_0) = Z^{n/2} G_r(x_1, \dots, x_n; \mu, \lambda), \quad (3.13)$$

one deduces then that:

$$\left(\delta\mu \frac{\partial}{\partial\mu} + \delta\lambda \frac{\partial}{\partial\lambda} + \delta Z \frac{\partial}{\partial Z} \right) \left(Z^{n/2} G_r(x_1, \dots, x_n; \mu, \lambda) \right) = 0. \quad (3.14)$$

Working out the derivative with respect to Z and multiplying by $Z^{-n/2}$, one arrives at:

$$\left(\delta\mu \frac{\partial}{\partial\mu} + \delta\lambda \frac{\partial}{\partial\lambda} + \frac{n}{2} \frac{\delta Z}{Z} \right) G_r(x_1, \dots, x_n; \mu, \lambda) = 0. \quad (3.15)$$

Finally, multiplying by $\mu/\delta\mu$, and rewriting ratios of differentials as derivatives taken at constant value for the bare coupling, one arrives at the Callan-Symanzik equation:

$$\left(\mu \frac{\partial}{\partial\mu} + \beta \frac{\partial}{\partial\lambda} + n\gamma \right) G_r(x_1, \dots, x_n; \mu, \lambda) = 0, \quad (3.16)$$

where:

$$\beta = \mu \frac{\partial\lambda}{\partial\mu} = \frac{\partial\lambda}{\partial \log \mu}, \quad (3.17)$$

$$\gamma = \frac{1}{2} \frac{\mu}{Z} \frac{\partial Z}{\partial\mu} = \frac{\partial \log \sqrt{Z}}{\partial \log \mu}. \quad (3.18)$$

The quantities β and γ are the same for every n , and must be independent of the points x_i . They must also be finite and independent of the cut-off, since the equation applies to

renormalized Green functions which are finite. By dimensional analysis, they can then only be some universal functions of the coupling λ . In fact, we see from the above equations that they can be interpreted as the rate of change in λ and $\log \sqrt{Z}$ which are needed to compensate the effect of a variation of the scale μ .

The above equation can be generalized in a straightforward way to any renormalizable theory involving several fields ϕ_i and only dimensionless couplings λ_a . For a correlation functions involving n generically different fields one finds then:

$$\left(\mu \frac{\partial}{\partial \mu} + \sum_a \beta_a \frac{\partial}{\partial \lambda_a} + \sum_{k=1}^n \gamma_{i_k} \right) G_r^{i_1 \dots i_n}(x_1, \dots, x_n; \mu, \lambda_a) = 0. \quad (3.19)$$

where the β -function of each coupling and the γ -function of each field are defined as before:

$$\beta_a = \mu \frac{\partial \lambda_a}{\partial \mu} = \frac{\partial \lambda_a}{\partial \log \mu}, \quad (3.20)$$

$$\gamma_i = \frac{1}{2} \frac{\mu}{Z_i} \frac{\partial Z_i}{\partial \mu} = \frac{\partial \log \sqrt{Z_i}}{\partial \log \mu}. \quad (3.21)$$

3.3 Computation of the renormalization group functions

Let us now illustrate how the functions β and γ associated to each coupling and to each field can be computed in perturbation theory as power series in the renormalized couplings. In general, these can be extracted from the counter-terms, which are computed order by order by imposing the renormalization conditions. What matters is the μ -dependence arising from the finite parts of these terms. However, this is essentially fixed by the coefficient of μ -independent divergences, due to the logarithmic nature of divergences.

At a given order n in perturbation theory, the contribution to β is suppressed by at least one power of the coupling with respect to the contribution to γ . One can then first deduce the order- n contribution to γ by considering the connected 2 point function for the field and using the Callan-Symanzik equation for it where β is approximated with its expression at the previous order $n - 1$. Once the order n value of γ has been determined in this way, one can deduce the order n contribution to β by considering the connected correlation function corresponding to the vertex and applying the Callan-Symanzik equation for it with the previously computed order n value of γ .

At the leading order in perturbation theory, the procedure is particularly simple. It is best illustrated by considering the simplest case of a self-interacting scalar theory. This has actually an accidental simplification, due to the fact that the only 1-loop diagram contributing to the 2-point function is a tadpole which turns out to be a momentum-independent constant. This implies that there is no need of wave-function counter-term at this order and that the function γ actually vanishes at leading order. But we shall ignore this here and display only the structure of the results. We will also assume that $m = 0$, since we want to study only the dimensionless coupling λ . This condition needs to be enforced order by order by adjusting the mass counter-term Δ_m , but in the following discussion we will completely ignore anything having to do with m .

Now, no matter what regularization is used, the dependence on any combination Δ of squared momentum can only arise in combination with the cut-off Λ , and is actually proportional to $\log(\Lambda^2/\Delta)$. It is then possible to derive β and γ from the coefficients of these logarithmic divergences, which are easier to compute than full finite amplitudes. In dimensional regularization, this statement is also true, with the cut-off being effectively identified with $\tilde{\Lambda}$, and the coefficients of the terms $\log(\tilde{\Lambda}^2/\Delta)$ are seen to be simply the coefficients of the $1/\epsilon$ poles.

To determine the function γ , we consider the connected 2-point function. This has the following structure, where $A, C \sim \mathcal{O}(\lambda)$:

$$\begin{aligned}
G_r^{(2)}(p) &= \frac{1}{p} \frac{1}{p} + \frac{1}{p} \textcircled{1L} \frac{1}{p} + \frac{\Delta_Z}{p} \bullet \frac{1}{p} \\
&= \frac{i}{p^2} + \frac{i}{p^2} \left(-i A p^2 \log \frac{\Lambda^2}{-p^2} - i C p^2 \right) \frac{i}{p^2} + \frac{i}{p^2} \left(i p^2 \Delta_Z \right) \frac{i}{p^2} \\
&= \frac{i}{p^2} \left[1 + \left(A \log \frac{\Lambda^2}{-p^2} + C - \Delta_Z \right) \right] \\
&= \frac{i}{p^2} \left[1 + \frac{1}{p^2} \Pi(p^2) \right].
\end{aligned} \tag{3.22}$$

The renormalization condition $\Pi'(-\mu^2) = 0$ fixes the counter-term Δ_Z to:

$$\Delta_Z = A \log \frac{\Lambda^2}{\mu^2} + C - A. \tag{3.23}$$

Using the definition of γ with $Z = 1 + \Delta_Z$, one finds:

$$\gamma = \frac{1}{2} \mu \frac{\partial \Delta_Z}{\partial \mu} = -A. \tag{3.24}$$

Finally, the finite renormalized 2-point function can be rewritten in terms of γ and is given by the following expression:

$$\begin{aligned}
G_r^{(2)}(p, \mu, \lambda) &= \frac{i}{p^2} \left(1 - A \log \frac{-p^2}{\mu^2} + A \right) \\
&= \frac{i}{p^2} \left(1 + \gamma \log \frac{-p^2}{\mu^2} - \gamma \right).
\end{aligned} \tag{3.25}$$

It satisfies the Callan-Symanzik equation at leading order in λ :

$$\left(\mu \frac{\partial}{\partial \mu} + 2\gamma \right) G_r^{(2)}(p, \mu, \lambda) = 0. \tag{3.26}$$

To determine the function β , we consider next the connected 4-point function. Let us focus for simplicity on the dependence on the overall scale \bar{p}^2 of the three invariants s , t and u , and discard the dependence on their ratios, which affects only finite terms. One finds then the following structure, where $A, C \sim \mathcal{O}(\lambda)$ and $B, D \sim \mathcal{O}(\lambda^2)$:

$$\begin{aligned}
G_r^{(4)}(p_i) &= \begin{array}{c} p_2 \quad p_4 \\ \diagdown \quad / \\ p_1 \quad p_3 \end{array} + \begin{array}{c} p_2 \quad p_4 \\ \diagdown \quad / \\ \textcircled{1L} \\ / \quad \diagdown \\ p_1 \quad p_3 \end{array} + \begin{array}{c} p_2 \quad p_4 \\ \diagdown \quad / \\ \bullet \\ / \quad \diagdown \\ p_1 \quad p_3 \end{array} \\
&+ \begin{array}{c} p_2 \quad p_4 \\ \diagdown \quad / \\ \textcircled{1L} \\ / \quad \diagdown \\ p_1 \end{array} + \begin{array}{c} p_2 \quad p_4 \\ \diagdown \quad / \\ \bullet \\ / \quad \diagdown \\ p_1 \end{array} + \text{perm.} \\
&= \prod_i \frac{i}{p_i^2} \left[-i\lambda + \left(-iB \log \frac{\Lambda^2}{-p^2} - iD \right) - i\Delta_\lambda \right. \\
&\quad \left. - i\lambda \sum_i \left(A \log \frac{\Lambda^2}{-p_i^2} + C \right) - i\lambda \sum_i \left(-\Delta_Z \right) \right] \quad (3.27) \\
&= -i \prod_i \frac{1}{p_i^2} \left[\lambda + \left(B \log \frac{\Lambda^2}{-p^2} + D + \Delta_\lambda \right) + \lambda \sum_i \left(A \log \frac{\Lambda^2}{-p_i^2} + C - \Delta_Z \right) \right] \\
&= -i \prod_i \frac{1}{p_i^2} \left[\Gamma(s, t, u) + \lambda \sum_i \frac{1}{p_i^2} \Pi(p_i^2) \right].
\end{aligned}$$

The renormalization condition $\Gamma(-\mu^2, -\mu^2, -\mu^2) = \lambda$ fixes the counter-term Δ_λ to:

$$\Delta_\lambda = -B \log \frac{\Lambda^2}{\mu^2} - D. \quad (3.28)$$

Using then the definition of β with $\lambda = \lambda_0 - \Delta_\lambda + 2\lambda_0\Delta_Z$, one finds:

$$\beta = -\mu \frac{\partial \Delta_\lambda}{\partial \mu} + 2\lambda_0 \mu \frac{\partial \Delta_Z}{\partial \mu} = -2B - 4\lambda A. \quad (3.29)$$

Finally, the finite renormalized 4-point function can be expressed in terms of γ and β , and is given by the following expression:

$$\begin{aligned}
G_r^{(4)}(p_i, \mu, \lambda) &= -i \prod_i \frac{1}{p_i^2} \left(\lambda - B \log \frac{-p^2}{\mu^2} - \lambda A \sum_i \log \frac{-p_i^2}{\mu^2} + 4\lambda A \right) \\
&= -i \prod_i \frac{1}{p_i^2} \left(\lambda + \frac{\beta - 4\lambda\gamma}{2} \log \frac{-p^2}{\mu^2} + \lambda\gamma \sum_i \log \frac{-p_i^2}{\mu^2} - 4\lambda\gamma \right). \quad (3.30)
\end{aligned}$$

It is straightforward to verify that it satisfies the Callan-Symanzik equation at leading order in λ^2 :

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} + 4\gamma \right) G_r^{(4)}(p_i, \mu, \lambda) = 0. \quad (3.31)$$

3.4 Running couplings

The Callan-Symanzik equation for the n -point function $G^{(n)}(p_i)$ can be formally solved in an exact way. This allows to obtain very useful information on the exact trajectory of the renormalization flow. It will be useful to parametrize the flow from the renormalization

scale μ at which the theory is defined to some arbitrary typical momentum scale p in terms of a scale factor defined as:

$$t = \log \frac{p}{\mu}. \quad (3.32)$$

The point $t = 0$ corresponds to the renormalization scale μ at which the couplings λ is defined, whereas $t > 0$ and $t < 0$ correspond respectively to higher and lower energy scales.

Consider the connected n -point function in momentum space, with all the momenta taken to be space-like and of the order of p . Taking into account the n Fourier transforms and the momentum conservation δ -function in its definition from the coordinate space, this has dimension $d_n = n - 4n + 4 = 4 - 3n$. We can then rewrite it as a prefactor involving d_n powers of the momentum and a function of the dimensionless scale factor t and the dimensionless coupling λ :

$$G_r^{(n)}(p, \lambda) = p^{d_n} g^{(n)}(t, \lambda). \quad (3.33)$$

The Callan-Symanzik equation can then be rewritten for $g^{(n)}$ with the term $\mu \partial/\partial\mu$ becoming now $-\partial/\partial t$:

$$\left(\frac{\partial}{\partial t} - \beta(\lambda) \frac{\partial}{\partial \lambda} - n \gamma(\lambda) \right) g^{(n)}(t, \lambda) = 0. \quad (3.34)$$

The general form of the solution of this equation can now be found in two steps, with the method of characteristics, by first solving the equation without the term depending on γ and then constructing the general solution of the full equation in terms of an arbitrary solution of the simpler equation.

It turns out that the basic building blocks of the general solution depend on a running coupling $\bar{\lambda}(t)$ and a running field-strength normalization $\bar{\xi}(t)$. These are defined as the solutions of the ordinary differential equations

$$\frac{d\bar{\lambda}(t)}{dt} = \beta(\bar{\lambda}(t)), \quad \frac{d \log \bar{\xi}(t)}{dt} = \gamma(\bar{\lambda}(t)), \quad (3.35)$$

with the boundary conditions

$$\bar{\lambda}(0) = \lambda, \quad \bar{\xi}(0) = 1. \quad (3.36)$$

The implicit solution for $\bar{\lambda}(t)$ can be obtained by integrating the corresponding equation between the initial point $(0, \lambda)$ and an arbitrary point $(t, \bar{\lambda})$. This gives:

$$\int_{\lambda}^{\bar{\lambda}} \frac{d\lambda'}{\beta(\lambda')} = \int_0^t dt' \Rightarrow \bar{\lambda}(t) = \bar{\lambda}(t, \lambda). \quad (3.37)$$

Once this is known, the solution for $\bar{\xi}(t)$ is obtained by simply integrating the corresponding equation between $(0, 1)$ and $(t, \bar{\xi}(t))$:

$$\bar{\xi}(t) = \exp \left\{ \int_0^t dt' \gamma(\bar{\lambda}(t')) \right\}. \quad (3.38)$$

Let us now see how the general solution of the Callan-Symanzik equation can be constructed by using these two functions.

The first step is to solve the simpler and universal equation obtained by dropping the non-derivative term involving γ :

$$\left(\frac{\partial}{\partial t} - \beta(\lambda)\frac{\partial}{\partial \lambda}\right)f(t, \lambda) = 0. \quad (3.39)$$

Now, it turns out that $\bar{\lambda}(t) = \bar{\lambda}(t, \lambda)$ satisfies this equation when considered as a function of both t and λ . This can be easily verified by taking derivatives with respect to t and λ of the integrated equation for $\bar{\lambda}$. As a consequence, the general solution of the above equation is a generic function depending on t and λ only through $\bar{\lambda}(t)$:

$$f(t, \lambda) = f(\bar{\lambda}(t)). \quad (3.40)$$

The second step is to construct the general solution of the full equation including the term depending on γ . This turns out to be the product of an arbitrary solution $f^{(n)}(\bar{\lambda}(t))$ of the simpler equation that we just studied and the n -th power of the running scale factor $\bar{\xi}(t)$:

$$\begin{aligned} g^{(n)}(t, \lambda) &= f^{(n)}(\bar{\lambda}(t)) \bar{\xi}(t)^n \\ &= f^{(n)}(\bar{\lambda}(t)) \exp\left\{n \int_0^t dt' \gamma(\bar{\lambda}(t'))\right\}. \end{aligned} \quad (3.41)$$

Indeed, since both $f^{(n)}(\bar{\lambda}(t))$ and $\gamma(\bar{\lambda}(t))$ satisfy the simpler equation, one computes

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \beta(\lambda)\frac{\partial}{\partial \lambda}\right)g^{(n)}(t, \lambda) &= n g^{(n)}(t, \lambda) \left(\frac{\partial}{\partial t} - \beta(\lambda)\frac{\partial}{\partial \lambda}\right) \int_0^t dt' \gamma(\bar{\lambda}(t')) \\ &= n g^{(n)}(t, \lambda) \left(\gamma(\bar{\lambda}(t)) - \int_0^t dt' \beta(\lambda)\frac{\partial \gamma}{\partial \lambda}(\bar{\lambda}(t'))\right) \\ &= n g^{(n)}(t, \lambda) \left(\gamma(\bar{\lambda}(t)) - \int_0^t dt' \frac{\partial \gamma}{\partial t'}(\bar{\lambda}(t'))\right) \\ &= n g^{(n)}(t, \lambda) \gamma(\lambda). \end{aligned} \quad (3.42)$$

This implies finally that the Callan-Symanzik equation is indeed satisfied:

$$\left(\frac{\partial}{\partial t} - \beta(\lambda)\frac{\partial}{\partial \lambda} - n\gamma(\lambda)\right)g^{(n)}(t, \lambda) = 0. \quad (3.43)$$

The above general solution of the Callan-Symanzik equation for Green functions restricts the way in which the exact Green function changes when the momentum scale at which it is evaluated is changed from the renormalization scale $p = \mu$ to an arbitrary momentum scale $p' = \mu e^t$. From the form of the general solution, we see that the effect of the rescaling is extraordinarily simple, and can be entirely summarized in two effects. The first is that the coupling λ should be replaced everywhere with the new coupling $\lambda' = \bar{\lambda}(t)$ at the momentum scale under consideration. The second is that the field-strength normalization $\xi = 1$ should be replaced with the new value $\xi' = \bar{\xi}(t)$ at the new scale. Finally, we therefore have:

$$G_r^{(n)}(p, \mu, \lambda) = p^{dn} f^{(n)}(\bar{\lambda}(t)) \bar{\xi}(t)^n. \quad (3.44)$$

Since for theories with only dimensionless couplings there are no other scales in the problem than the typical momentum scale p and the renormalization scale μ , the scaling behavior in these two different variables can be identified. This means that the running coupling and the running field-strength factor $\bar{\xi}(t)$ also represent the trajectories of equivalent renormalization conditions, in the sense that points (μ', λ', ξ') that are equivalent to (μ, λ, ξ) are parametrized by

$$\mu' = \mu e^t, \quad \lambda' = \bar{\lambda}(t), \quad \xi' = \bar{\xi}(t). \quad (3.45)$$

In more general theories with several dimensionless couplings λ_a and several fields ϕ_i , the situation is similar. The solution of the Callan-Symanzik equation can be constructed exactly in the same way in terms of the running couplings and scale factors, defined by the coupled differential equations:

$$\frac{d\bar{\lambda}_a(t)}{dt} = \beta_a(\bar{\lambda}_b(t)), \quad \frac{d\log \bar{\xi}_i(t)}{dt} = \gamma_i(\bar{\lambda}_b(t)). \quad (3.46)$$

3.5 Scheme dependence

The functions β and γ depend in principle on the renormalization scheme that is used to define the theory. However, different renormalization schemes must correspond to a change of parametrization in the couplings that are used to define the theory, and once the functional relation between the couplings in two different schemes is known, the corresponding β and γ functions can also be related.

An important result is that the coefficients of the first two terms in the perturbative expansion of β are actually independent of the scheme choice. To show this, suppose that we start from a scheme with coupling λ and β -function satisfying:

$$\beta(\lambda) = \frac{\partial \lambda}{\partial \log \mu}. \quad (3.47)$$

Switching to a different scheme corresponds to using a new coupling λ' , which is a function of the original one: $\lambda' = \lambda'(\lambda)$. The new β' -function will then be:

$$\begin{aligned} \beta'(\lambda') &= \frac{\partial \lambda'}{\partial \log \mu} = \frac{\partial \lambda'}{\partial \lambda} \frac{\partial \lambda}{\partial \log \mu} \\ &= \frac{\partial \lambda'}{\partial \lambda} \beta(\lambda). \end{aligned} \quad (3.48)$$

At weak coupling, the relation between the couplings in the two renormalization schemes may be expanded as:

$$\lambda' = \lambda + a\lambda^2 + \mathcal{O}(\lambda^3). \quad (3.49)$$

The inverse relation has then the form:

$$\lambda = \lambda' - a\lambda'^2 + \mathcal{O}(\lambda'^3). \quad (3.50)$$

It follows that the transformation matrix has the form:

$$\frac{\partial \lambda'}{\partial \lambda} = 1 + 2a\lambda + \mathcal{O}(\lambda^2) = 1 + 2a\lambda' + \mathcal{O}(\lambda'^2). \quad (3.51)$$

Suppose now that the β -function in the original scheme has a weak coupling expansion with first few terms given by:

$$\beta(\lambda) = \beta_1\lambda^2 + \beta_2\lambda^3 + \mathcal{O}(\lambda^4). \quad (3.52)$$

The β' -function in the new scheme is then given by:

$$\begin{aligned} \beta'(\lambda') &= \frac{\partial \lambda'}{\partial \lambda} \beta(\lambda) = \left[1 + 2a\lambda + \mathcal{O}(\lambda^2)\right] \left[\beta_1\lambda^2 + \beta_2\lambda^3 + \mathcal{O}(\lambda^4)\right] \\ &= \left[1 + 2a\lambda' + \mathcal{O}(\lambda'^2)\right] \left[\beta_1\lambda'^2 + (\beta_2 - 2a\beta_1)\lambda'^3 + \mathcal{O}(\lambda'^4)\right] \\ &= \beta_1\lambda'^2 + \beta_2\lambda'^3 + \mathcal{O}(\lambda'^4). \end{aligned} \quad (3.53)$$

This has indeed the same form as in the original scheme, as far as the first two leading terms are concerned, independently of the parameter a describing the change of scheme at leading order.

3.6 Effect of mass parameters

Let us now generalize the above analysis to theories involving dimensionful coupling constants. We shall in particular focus on the effect of mass terms. Proceeding exactly as before, we can use the relation between renormalized and bare Green-functions to deduce the equation defining the continuous family of equivalent parameters (μ, λ, m^2, ξ) defining the same theory within a given renormalization scheme. The result is the generalized Callan-Symanzik equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} + \gamma_m m^2 \frac{\partial}{\partial m^2} + n\gamma\right) G_r(x_1, \dots, x_n; \mu, \lambda, m^2) = 0. \quad (3.54)$$

where β and γ are given by the same expressions as before and

$$\gamma_m = \frac{\mu}{m^2} \frac{\partial m^2}{\partial \mu} = \frac{\partial \log m^2}{\partial \log \mu}. \quad (3.55)$$

The dimensionless quantities β , γ and γ_m must again be universal finite functions. However, in this case they depend in general not only on λ but also on m^2/μ^2 . We see from its definition that γ_m can be interpreted as the rate of change in $\log m^2$ which is needed to compensate the effect of a variation of the scale μ . This looks similar to the fact that γ is the rate of change needed in $\log \sqrt{Z}$. In fact, one can view m^2 as being a field-strength renormalization associated to the composite operator $\mathcal{O}_m = \phi^2$, much in the same way as \sqrt{Z} in the field-strength renormalization for the elementary field ϕ . The effect of the mass parameter is then similar to the effect of a field-strength renormalization. This becomes clear by imagining to expand the functional integral defining the correlation functions in powers of m^2 . The correlation functions of the massive theory can then be computed

as a sum of correlation functions of the massless theory involving a growing number of additional insertions of the operator \mathcal{O}_m . The differential operator $m^2\partial/\partial m^2$ counts then simply the number of insertions of the mass operator \mathcal{O}_m .

One can reach an alternative treatment of the effect of the mass by translating the dimensionful coupling m^2 into a dimensionless coupling λ_m defined as:

$$\lambda_m = \frac{m^2}{\mu^2}. \quad (3.56)$$

With this change of variables, the dependence on m^2 is traded with a dependence on the dimensionless coupling λ_m plus some additional dependence on μ . More precisely, $m^2\partial/\partial m^2$ becomes $\lambda_m\partial/\partial\lambda_m$ and $\mu\partial/\partial\mu$ becomes $\mu\partial/\partial\mu - 2\lambda_m\partial/\partial\lambda_m$. The Callan-Symanzik equation can then be rewritten as

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \beta_m\frac{\partial}{\partial\lambda_m} + n\gamma\right)G_r(x_1, \dots, x_n; \mu, \lambda, \lambda_m) = 0, \quad (3.57)$$

where

$$\beta_m = (\gamma_m - 2)\lambda_m. \quad (3.58)$$

Using the definition of γ_m with $m^2 = \lambda_m\mu^2$, one can rewrite $\gamma_m = 2 + \partial\log\lambda_m/\partial\log\mu$, and therefore the β_m is as expected given by:

$$\beta_m = \mu\frac{\partial\lambda_m}{\partial\mu} = \frac{\partial\lambda_m}{\partial\log\mu}. \quad (3.59)$$

The dimensionless quantities β , γ and β_m are now universal finite functions of the two dimensionless couplings λ and λ_m . The function β_m is interpreted as the rate of change in the dimensionless coupling λ_m that is needed to compensate a change in μ , on the same footing as β is the rate of change needed in λ .

It is now pretty obvious that the general solution to the Callan-Symanzik equation in the presence of mass parameters will involve not only a running coupling and a running field-strength normalization, but also a running mass parameter. The derivation of the form of this solution is somewhat complicated by the fact that in general the functions β , γ and γ_m depend both on the coupling λ and on the mass parameter through the ratio m^2/μ^2 . Using the formulation in which the mass is translated into a dimensionless coupling λ_m besides λ , we see however that this simply amounts to having two running couplings $\bar{\lambda}(t)$ and $\bar{\lambda}_m(t)$ whose evolutions mix, with beta functions β and β_m which depend on both of the couplings. The running mass will then simply be given by $\bar{m}^2(t) = \bar{\lambda}_m(t)\mu^2$.

3.7 Minimal subtraction schemes

Recall that in dimensional regularization, with $d = 4 - 2\epsilon$, an arbitrary scale $\tilde{\mu}$ is naturally introduced to compensate the slight change in dimensionality of couplings and fields. In this way, the regularized loop integrals have the usual dimensions even for non-zero ϵ . As

a result, a 1-loop logarithmically divergent integral depending on some combination Δ of squared momenta results in factors like:

$$C_{\text{divergent}} = \frac{1}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma + \log(4\pi) + \log \frac{\tilde{\mu}^2}{\Delta} \right) = \frac{1}{(4\pi)^2} \log \frac{\tilde{\Lambda}^2}{\Delta}, \quad (3.60)$$

where

$$\tilde{\Lambda} = \sqrt{4\pi} e^{-\gamma/2} e^{1/(2\epsilon)} \tilde{\mu}. \quad (3.61)$$

Imposing a renormalization condition at the scale μ , meaning $\Delta = \mu^2$, fixes then the counter-terms in such a way that in the renormalized Green function the above divergent quantity will be turned into:

$$C_{\text{subtracted}} = \frac{1}{(4\pi)^2} \log \frac{\mu^2}{\Delta} + \text{constant}. \quad (3.62)$$

We observe now that the same result can be obtained without imposing any precise renormalization condition at the scale μ , but rather using the prescription of fixing the arbitrary scale $\tilde{\mu}$ to be of the order of μ and subtracting the $1/\epsilon$ pole. More precisely, the $\overline{\text{MS}}$ scheme is defined by taking $\mu = \tilde{\mu}$ and removing the $1/\epsilon$ factor, whereas the $\overline{\text{MS}}$ scheme is defined by taking $\mu = \tilde{\mu}$ and removing instead the quantity $1/\epsilon - \gamma + \log(4\pi)$. These renormalization schemes are very convenient to perform computations. However, due to the fact that they are defined by a subtraction prescription and not true renormalization conditions, they imply usually some extra work to relate the renormalized mass and coupling to corresponding quantities defined in a more physical renormalization scheme.

For minimal subtraction schemes, the renormalization group equations can be defined in a way which parallels the discussion done for more physically defined renormalization schemes. The basic point is again that renormalized Green functions depend on μ but are related to the bare Green-functions which do not, and it must therefore be possible to compensate any change in μ by a change in λ , m^2 and ξ . A important point is that in these schemes the renormalization group functions β , γ and γ_m turn out to depend only on λ . This is because they correspond to using counter-terms which have only the minimal structure needed to cancel divergence and which do not carry any dependence on the mass m . As we have seen, this property simplifies the solution of the Callan-Symanzik equation.

3.8 Resummation of leading logarithms

One of the most important applications of the renormalization group differential equations is the resummation of higher-powers of logarithms occurring at higher-order in perturbation theory. These effects become important when evaluating amplitudes at typical momenta p which are much higher or much lower than the renormalization scale μ , since in that regime the logarithm $t = \log(p/\mu)$ becomes large in absolute value. More precisely, it turns out that using the running coupling obtained by solving its defining equation with a perturbative approximation to the function β is equivalent to reorganize perturbation theory in such a way to resum a series of dominant logarithmic corrections.

To understand the point, let us consider again a prototype theory with a single coupling λ . At the 1-loop level, the beta function has the structure:

$$\beta(\lambda) = \frac{b}{(4\pi)^2} \lambda^2. \quad (3.63)$$

Using this approximate result, the running coupling is found to be given by the following expression:

$$\bar{\lambda}(t) = \frac{\lambda}{1 - b \lambda / (4\pi)^2 t}. \quad (3.64)$$

Recalling the definition of $t = \log(p/\mu)$, this means that:

$$\begin{aligned} \bar{\lambda}(p) &= \frac{\lambda}{1 - b \lambda / (4\pi)^2 \log(p/\mu)} = \lambda + \sum_{n=1}^{\infty} b^n \frac{\lambda^{n+1}}{(4\pi)^{2n}} \log^n \frac{p}{\mu} \\ &= \lambda + b \frac{\lambda^2}{(4\pi)^2} \log \frac{p}{\mu} + b^2 \frac{\lambda^3}{(4\pi)^4} \log^2 \frac{p}{\mu} + \dots \end{aligned} \quad (3.65)$$

The first constant term in this expansion is just the scale-independent vertex, whereas the first simple logarithm is recognized to capture the momentum dependence coming from the 1-loop correction to the 1PI 4-point function. Clearly, the other terms involving multiple logarithms must have something to do with higher loops. In fact, an n -loop diagram involves $n + 1$ vertices, giving a factor λ^{n+1} , and n momentum integrals, giving a factor $(4\pi)^{-2n}$. Moreover, a momentum dependence of the type $\log^n(p/\mu)$ can arise from regions of the integrals where all the n loops momenta have different scales. Evidently, the 1-loop approximation to the running coupling resums the leading logarithmic behavior of all the loop diagrams.

The remarkable thing about this resummation is that it improves significantly the perturbative expansion for momenta p departing significantly from the scale μ at which the coupling parameter λ is defined. Indeed, already at the 1-loop level, we see that the effective coupling or expansion parameter is no longer λ by actually $\lambda \log(p/\mu)$. This means that even if λ was small at the scale μ , $\lambda \log(p/\mu)$ might become big at the scale p if this is not comparable to μ . The fact that the 1-loop running coupling resums a geometric series in powers of this effective coupling opens up the possibility that $\bar{\lambda}(p)$ might actually be small, even if $\lambda \log(p/\mu)$ was big, providing us then with a better expansion parameter. In fact, we see that when $b > 0$, $\bar{\lambda}(p)$ stays small for $p < \mu$, even if $\lambda \log(p/\mu)$ grows large. Vice versa, when $b < 0$, $\bar{\lambda}(p)$ stays small for $p > \mu$, even if $\lambda \log(p/\mu)$ grows large.

The interpretation of the running coupling as a way to resum certain classes of large logarithms can be extended to higher orders. It is possible to show that the running coupling obtained by solving the renormalization group equations with the function β approximated with its k -loop expression resums not only the leading logarithm $\lambda^{n+1} \log^n(p/\mu)$ arising at each loop order n , but also the first $k - 1$ subleading logarithms of the form $\lambda^{n+1} \log^{n-k+1}(p/\mu)$.

It is possible to understand rather intuitively the reason why a different coupling $\bar{\lambda}(p)$ is better suited at a scale p which is different from the scale μ at which the theory

is defined by imposing renormalization conditions. In order to use the renormalization conditions to re-express bare quantities in terms of the renormalized quantities, we need to evaluate the amplitude on which these conditions are imposed at the renormalization scale μ . Since all the external momenta are of order μ , the loop integrals are effectively cut-off at momentum scales μ , if one ignores the intrinsic divergences that can appear. Forgetting these for a moment, since the physical interpretation of renormalization should be independent of them, it is then clear that the renormalization conditions are sensitive to degrees of freedom with a certain range of frequencies. Changing the renormalization scale μ implies thus effectively a change in this range of wave-lengths. The lesson of the renormalization group is that in order to avoid large logarithms, we should choose μ to be of the order of the typical energy of the process to be studied. This means then that we should effectively include in the definition of the theory degrees of freedom with frequency up to μ . This is at the root of the connection with the Wilsonian approach to renormalization.

The running coupling analysis shows that there exist two very different possible asymptotic behaviors, depending on the sign of β at weak-coupling. If $\beta > 0$ for small coupling, then $\bar{\lambda}$ is small in the IR and grows in the UV. But when the coupling increases, β may possibly change behavior and go back to cross 0 at some value λ^* of the coupling. When increasing the scale from IR to UV, $\bar{\lambda}$ slows then down and asymptotically stops its growth when approaching this value λ^* . This corresponds to a UV fixed point. If instead $\beta < 0$ for small coupling, then $\bar{\lambda}$ is small in the UV and grows in the IR. But again, when the coupling increases, β may possibly change behavior and cross back 0 at some value λ^* of the coupling. When decreasing the scale from UV to IR, $\bar{\lambda}$ flows then to the value λ^* . This corresponds to an IR fixed point.

Close to a fixed-point λ^* where $\beta(\lambda^*) = 0$, correlation functions display a critical behavior with scaling laws that are characterized by the slope of β and the value of γ . More precisely, for $\bar{\lambda}$ close to λ^* , one has:

$$\beta(\bar{\lambda}) \simeq \beta'^*(\bar{\lambda} - \lambda^*), \quad \gamma(\bar{\lambda}) \simeq \gamma^*. \quad (3.66)$$

In that region, the approximate solution to the differential equation defining the running coupling is given by:

$$\bar{\lambda}(t) - \lambda^* \sim e^{\beta'^* t}. \quad (3.67)$$

This implies that when p is very large or very small, depending on whether the fixed-point is a UV or IR fixed-point, the running coupling $\bar{\lambda}$ flows to λ^* with a rate which is governed by β'^* :

$$\bar{\lambda}(p) - \lambda^* \sim \left(\frac{p}{\mu}\right)^{\beta'^*}. \quad (3.68)$$

Moreover, in this asymptotic region where p is respectively very large or very small, the integral yielding the exponent of the wave-function normalization factor $\bar{\xi}$ is dominated by the region where $\bar{\lambda}$ is very close to λ^* , and $\gamma(\bar{\lambda})$ is close to γ^* , giving:

$$\bar{\xi}(t) \sim e^{\gamma^* t}. \quad (3.69)$$

This means that the fields scale with an anomalous dimension given by γ^* :

$$\bar{\xi}(p) \sim \left(\frac{p}{\mu}\right)^{\gamma^*} \quad (3.70)$$

As a consequence of these two results, the form implied by the Callan-Symanzik equation for the connected n -point function becomes a simple power law for asymptotic p corresponding to the fixed-point:

$$G^{(n)}(p, \lambda) \sim p^{d_n + n\gamma^*}. \quad (3.71)$$

In particular, the 2-point function behaves like

$$G^{(2)}(p) \sim p^{-2(1-\gamma^*)}. \quad (3.72)$$

3.9 Effective action

It is straightforward to derive scaling equations also for the effective action $\Gamma[\phi_{\text{cl}}]$. To do so, recall that it is actually the generating functional of the renormalized 1PI correlation functions:

$$\Gamma[\phi_{\text{cl}}] = \sum_{n=2}^{\infty} \frac{i}{n!} \int d^4x_1 \cdots \int d^4x_n \Gamma_r^{(n)}(x_1, \dots, x_n) \phi_{\text{cl}}(x_1) \cdots \phi_{\text{cl}}(x_n). \quad (3.73)$$

The 1PI correlation functions, on the other hand, satisfy a Callan-Symanzik equation which is very similar to the one satisfied by the connected Green functions. The only difference is the power of Z entering the relation between bare and renormalized correlation functions. Recalling the relation between connected and 1PI correlation functions, one can verify that in this case one has:

$$\Gamma_r^{(n)}(x_1, \dots, x_n; \mu, \lambda) = Z^{+n/2} \Gamma^{(n)}(x_1, \dots, x_n; \lambda_0). \quad (3.74)$$

The Callan-Symanzik equation is thus the same as before, except for a sign in the anomalous dimension term:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} - n\gamma\right) \Gamma_r^{(n)}(x_1, \dots, x_n, \mu, \lambda) = 0. \quad (3.75)$$

Now, the effective action $\Gamma[\phi_{\text{cl}}]$ is a sum of terms where $\Gamma^{(n)}$ appears together with n powers of ϕ_{cl} . As a consequence, it also satisfies a Callan-Symanzik equation, with a field counting factor in front of the anomalous dimension that is given by the functional derivative with respect to ϕ_{cl} . More precisely, one has:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} - \gamma \int d^4x \phi_{\text{cl}}(x) \frac{\delta}{\delta \phi_{\text{cl}}(x)}\right) \Gamma[\phi_{\text{cl}}, \mu, \lambda] = 0. \quad (3.76)$$

Notice finally that the renormalization conditions are conditions on the value at the momentum scale μ of the lowest 1PI correlation functions, and these can be turned into conditions on the value of some derivatives of the effective action at the field scale μ .

4 Symmetry breaking and quantum corrections

4.1 The $O(N)$ sigma model

Consider the theory of N scalar fields ϕ^i with a positive square mass $m^2 > 0$ and a quartic coupling λ respecting an $O(N)$ rotation symmetry. The Lagrangian is given by:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu \phi^i)^2 - \frac{1}{2}m^2(\phi^i)^2 - \frac{\lambda}{4}[(\phi^i)^2]^2 \\ & + \frac{1}{2}\Delta_Z(\partial_\mu \phi^i)^2 - \frac{1}{2}\Delta_m(\phi^i)^2 - \frac{1}{4}\Delta_\lambda[(\phi^i)^2]^2. \end{aligned} \quad (4.1)$$

If $m^2 > 0$, the classical potential has a minimum at $\phi^i = 0$. The $O(N)$ symmetry is then preserved by the vacuum, and the N fluctuation fields ϕ^i around the vacuum have positive squared mass m^2 . Quantum corrections are expected to respect this symmetry, and the 3 $O(N)$ symmetric counter-terms should therefore be sufficient to renormalize the theory.

On the other hand, in the same model but with negative square mass $m^2 < 0$, there is a partial spontaneous symmetry breaking. The minimum of the classical potential is degenerate and lies along an arbitrary field direction on the surface $(\phi^i)^2 = -m^2/\lambda$. In other words, $\phi^i = v^i$ with $(v^i)^2 = -m^2/\lambda$. The $O(N)$ symmetry, which had $N(N-1)/2$ generators, is then spontaneously broken down to an $O(N-1)$ symmetry, which has only $(N-1)(N-2)/2$ generators, and there are $N-1$ spontaneously broken symmetries. The spectrum consists of $N-1$ massless modes π^a , $a = 1, \dots, N-1$, in the directions orthogonal to v^i , corresponding to the Goldstone bosons, and one massive mode σ in the direction parallel to v^i with positive squared mass $-2m^2$. Expanding the fields around their vacuum expectation values, the Lagrangian is found to be:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(m^2 + 3\lambda v^2)\sigma^2 + \frac{1}{2}(\partial_\mu \pi^a)^2 - \frac{1}{2}(m^2 + \lambda v^2)(\pi^a)^2 \\ & - v(m^2 + \lambda v^2)\sigma - \lambda v[\sigma^3 + \sigma(\pi^a)^2] + \frac{\lambda}{4}[\sigma^4 + \sigma^2(\pi^a)^2 + [(\pi^a)^2]^2] \\ & + \frac{1}{2}\Delta_Z(\partial_\mu \sigma)^2 + \frac{1}{2}\Delta_Z(\partial_\mu \pi^a)^2 - \frac{1}{2}(\Delta_m + \Delta_\lambda v^2)\sigma^2 - \frac{1}{2}(\Delta_m + 3\Delta_\lambda v^2)(\pi^a)^2 \\ & - v(\Delta_m + \Delta_\lambda v^2)\sigma - \Delta_\lambda v[\sigma^3 + \sigma(\pi^a)^2] + \frac{\Delta_\lambda}{4}[\sigma^4 + \sigma^2(\pi^a)^2 + [(\pi^a)^2]^2], \end{aligned} \quad (4.2)$$

where:

$$v = \sqrt{\frac{-m^2}{\lambda}}. \quad (4.3)$$

Due to the vacuum expectation value breaking $O(N)$ to $O(N-1)$, there is a proliferation of couplings, and correspondingly also of counter-terms. However, these couplings and counter-terms are not all independent, but rather related to each other. In this situation, only the $O(N-1)$ symmetry is manifest, and one may wonder whether the same limited set of independent counter-terms can be sufficient to renormalize the theory. It turns out that this is indeed the case, as we will see below. The reason for this is that divergences have to do with the behavior in the far UV, where the symmetry is restored independently of the IR behavior, where spontaneous symmetry breaking becomes effective.

4.2 Diagrammatic computation of β , γ and γ_m

In the symmetric model with $m^2 > 0$, the 1-loop behavior is rather easy to study. Indeed, the Feynman rules are found to be the following:

$$\begin{aligned}
i \text{ --- } j &= \frac{i}{p^2 - m^2} \delta^{ij}, \\
\begin{array}{c} k \\ \diagdown \\ i \text{ --- } j \\ \diagup \\ l \end{array} &= -2i\lambda (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}), \\
i \text{ --- } \bullet \text{ --- } j &= ip^2 \Delta_Z \delta^{ij}, \quad i \text{ --- } \bullet \text{ --- } j = -i\Delta_m \delta^{ij}, \\
\begin{array}{c} k \\ \diagdown \\ i \text{ --- } \bullet \text{ --- } j \\ \diagup \\ l \end{array} &= -2i\Delta_\lambda (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}).
\end{aligned} \tag{4.4}$$

The propagator correction is easily computed and is found to be given by the following simple diagram:

$$\begin{aligned}
i \text{ --- } \bigcirc \text{ --- } j &= \frac{1}{2} (-2i\lambda)(N+2) \delta^{ij} \int \frac{d^d p_E}{(2\pi)^d} \frac{\tilde{\mu}^{4-d}}{p_E^2 + m^2} \\
&= i \frac{\lambda(N+2)}{(4\pi)^2} m^2 \log \frac{\tilde{\Lambda}^2}{m^2} \delta^{ij} + \text{finite}.
\end{aligned} \tag{4.5}$$

It follows that the wave function and mass counter-term must have the form:

$$\Delta_Z = 0, \quad \Delta_m = \frac{\lambda(N+2)}{(4\pi)^2} m^2 \log \frac{\tilde{\Lambda}^2}{\mu^2} + \text{finite}. \tag{4.6}$$

The vertex correction involves instead three different permutations of a basic diagram. Using a Feynman parameter x and introducing $\Delta = m^2 - x(1-x)k^2$, one finds:

$$\begin{aligned}
&\begin{array}{c} k \text{ --- } l \\ \bigcirc \\ i \text{ --- } j \end{array} + \begin{array}{c} j \text{ --- } l \\ \bigcirc \\ i \text{ --- } k \end{array} + \begin{array}{c} k \text{ --- } j \\ \bigcirc \\ i \text{ --- } l \end{array} \\
&= \frac{1}{2} (-2i\lambda)^2 \left[(N+4) \delta^{ij} \delta^{kl} + 2 \delta^{ik} \delta^{jl} + 2 \delta^{il} \delta^{jk} \right] \int_0^1 dx \int \frac{d^d p_E}{(2\pi)^d} \frac{(-i) \tilde{\mu}^{4-d}}{(p_E^2 + \Delta)^2} \\
&\quad + (j \leftrightarrow k) + (j \leftrightarrow l) \\
&= 2i \frac{\lambda^2(N+8)}{(4\pi)^2} \log \frac{\tilde{\Lambda}^2}{m^2} (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) + \text{finite}.
\end{aligned} \tag{4.7}$$

This has indeed the expected tensor structure, and the needed counter-term is:

$$\Delta_\lambda = \frac{\lambda^2(N+8)}{(4\pi)^2} \log \frac{\tilde{\Lambda}^2}{\mu^2} + \text{finite}. \tag{4.8}$$

Using the above results for the dependence of the counter-terms on the renormalization scale μ , and assuming a minimal renormalization scheme where the finite terms do not depend on μ/m , one can now compute the renormalization group functions β , γ and γ_m . The results are given by:

$$\gamma = \frac{1}{2} \frac{\partial(\Delta_Z)}{\partial \log \mu} = 0, \quad (4.9)$$

$$\beta = -\frac{\partial \Delta_\lambda}{\partial \log \mu} + 2\lambda \frac{\partial \Delta_Z}{\partial \log \mu} = \frac{2\lambda^2(N+8)}{(4\pi)^2}, \quad (4.10)$$

$$\gamma_m = -\frac{\partial(\Delta_m/m^2)}{\partial \log \mu} + \frac{\partial \Delta_Z}{\partial \log \mu} = \frac{2\lambda(N+2)}{(4\pi)^2}. \quad (4.11)$$

In the asymmetric model with $m^2 < 0$, the 1-loop behavior is more cumbersome to study, due to the proliferation of interactions and counter-terms for the two types of fluctuation fields σ and π^a . It is a non-trivial exercise to check by brute force that the correlated counter-terms are still sufficient to renormalize the divergences occurring in all the correlated vertices. As expected, the result of such a diagrammatic analysis is that the counter-terms required to cancel all the divergences in the asymmetric theory correspond exactly to the values determined above for the symmetric theory. This implies that the functions β , γ and γ_m take exactly the same form.

4.3 Effective potential

It is possible to study the 1-loop behavior of this theory in a much more efficient way by using the formalism of the effective action. More precisely, one can compute the effective potential for an arbitrary value of ϕ_{cl}^i . This allows to study the behavior of the model for any value of m^2 in a single shot.

The first step is to compute the values of the masses for all the independent fluctuations around the point ϕ_{cl}^i . To do this, one decomposes:

$$\phi^i = \phi_{\text{cl}}^i + \eta^i. \quad (4.12)$$

Substituting into the Lagrangian, one finds that the quadratic terms in η^i defined by the second functional derivative of the action at the point ϕ_{cl} are given by

$$\frac{\delta^2 \mathcal{S}}{\delta \phi^i \delta \phi^j}[\phi_{\text{cl}}] = -\square \delta^{ij} - M^{2ij}(\phi_{\text{cl}}), \quad (4.13)$$

where

$$M^{2ij}(\phi_{\text{cl}}) = m^2 \delta^{ij} + \lambda \left((\phi_{\text{cl}}^k)^2 \delta^{ij} + 2 \phi_{\text{cl}}^i \phi_{\text{cl}}^j \right). \quad (4.14)$$

This gives the kinetic operator for the fluctuations η^i . It is convenient to switch to a diagonal basis of new fields where the mass matrix is diagonal. This is achieved by decomposing η^i in a component parallel to ϕ_{cl}^i and $N-1$ components orthogonal to ϕ_{cl}^i , by using the complementary projectors:

$$P_{\parallel}^{ij} = \frac{\phi_{\text{cl}}^i \phi_{\text{cl}}^j}{(\phi_{\text{cl}}^k)^2}, \quad P_{\perp}^{ij} = \delta^{ij} - \frac{\phi_{\text{cl}}^i \phi_{\text{cl}}^j}{(\phi_{\text{cl}}^k)^2}. \quad (4.15)$$

Using these expressions, the mass matrix can be rewritten as

$$M^{2ij}(\phi_{\text{cl}}) = m^2 \delta^{ij} + \lambda (\phi_{\text{cl}}^k)^2 \left(3 P_{\parallel}^{ij} + P_{\perp}^{ij} \right). \quad (4.16)$$

It is then clear that the mass eigenvalues are:

$$M^{2i}(\phi_{\text{cl}}) = \begin{cases} m^2 + 3\lambda (\phi_{\text{cl}}^k)^2, & \text{for the 1 field } \parallel \text{ to } \phi_{\text{cl}}^i, \\ m^2 + \lambda (\phi_{\text{cl}}^k)^2, & \text{for the } N-1 \text{ fields } \perp \text{ to } \phi_{\text{cl}}^i. \end{cases} \quad (4.17)$$

We can now easily compute the 1-loop contribution to the effective potential. It is simply given by the sum of N contributions of fields with squared masses given by the above $M^{2i}(\phi_{\text{cl}})$. Taking into account also the counter-terms, the result is found to be:

$$\begin{aligned} V_{\text{eff}}^1(\phi_{\text{cl}}) &= -\frac{1}{4(4\pi)^2} \sum_i M^{4i}(\phi_{\text{cl}}) \left(\frac{3}{2} + \log \frac{\tilde{\Lambda}^2}{M^{2i}(\phi_{\text{cl}})} \right) \\ &\quad + \frac{1}{2} \Delta_m (\phi_{\text{cl}}^i)^2 + \frac{1}{4} \Delta_{\lambda} [(\phi_{\text{cl}}^i)^2]^2. \end{aligned} \quad (4.18)$$

We see that the divergences consist of an irrelevant piece which is field-independent and can be dropped, a piece which is quadratic in ϕ_{cl} and can be compensated with the mass counter-term, and finally a piece which is quartic in ϕ_{cl} and can be compensated by the vertex counter-term.

4.4 Renormalization and counter-terms

A generic set of renormalization conditions defined at a scale μ requires the counter-terms to be of the following form:

$$\begin{aligned} \Delta_m &= \frac{1}{2(4\pi)^2} \sum_i \left. \frac{\partial M^{4i}(\phi_{\text{cl}})}{\partial (\phi_{\text{cl}}^k)^2} \right|_{\phi_{\text{cl}}=0} \log \frac{\tilde{\Lambda}^2}{\mu^2} + \text{finite} \\ &= \frac{\lambda(N+2)}{(4\pi)^2} m^2 \log \frac{\tilde{\Lambda}^2}{\mu^2} + \text{finite}, \end{aligned} \quad (4.19)$$

and

$$\begin{aligned} \Delta_{\lambda} &= \frac{1}{2(4\pi)^2} \sum_i \left. \frac{\partial^2 M^{4i}(\phi_{\text{cl}})}{\partial [(\phi_{\text{cl}}^k)^2]^2} \right|_{\phi_{\text{cl}}=0} \log \frac{\tilde{\Lambda}^2}{\mu^2} + \text{finite} \\ &= \frac{\lambda^2(N+8)}{(4\pi)^2} \log \frac{\tilde{\Lambda}^2}{\mu^2} + \text{finite}. \end{aligned} \quad (4.20)$$

These expressions reproduce the results derived diagrammatically for the symmetric theory with $m^2 > 0$, but apply now also to the asymmetric theory with $m^2 < 0$, and are thus universal. Their finite parts depend of course as usual on the choice of renormalization scheme.

In the \overline{MS} scheme, renormalization at a scale μ amounts simply to substitute $\tilde{\Lambda}$ with μ and drop the counter-terms. The final expression for the effective potential is then:

$$\begin{aligned} V_{\text{eff}}^1(\phi_{\text{cl}}) &= -\frac{1}{4(4\pi)^2} \sum_i M^{4i}(\phi_{\text{cl}}) \left(\frac{3}{2} + \log \frac{\mu^2}{M^{2i}(\phi_{\text{cl}})} \right) \\ &= -\frac{1}{4(4\pi)^2} \left[m^2 + 3\lambda(\phi_{\text{cl}}^k)^2 \right]^2 \left(\frac{3}{2} + \log \frac{\mu^2}{m^2 + 3\lambda(\phi_{\text{cl}}^k)^2} \right) \\ &\quad - \frac{N-1}{4(4\pi)^2} \left[m^2 + \lambda(\phi_{\text{cl}}^k)^2 \right]^2 \left(\frac{3}{2} + \log \frac{\mu^2}{m^2 + \lambda(\phi_{\text{cl}}^k)^2} \right). \end{aligned} \quad (4.21)$$

In total, adding up the tree-level potential and the 1-loop correction, one finally finds the following result for the effective potential:

$$\begin{aligned} V_{\text{eff}}(\phi_{\text{cl}}) &= \frac{1}{2} m^2 \phi_{\text{cl}}^2 + \frac{\lambda}{4} \phi_{\text{cl}}^4 + \frac{1}{4(4\pi)^2} \left(m^2 + 3\lambda \phi_{\text{cl}}^2 \right)^2 \left(\log \frac{m^2 + 3\lambda \phi_{\text{cl}}^2}{\mu^2} - \frac{3}{2} \right) \\ &\quad + \frac{N-1}{4(4\pi)^2} \left(m^2 + \lambda \phi_{\text{cl}}^2 \right)^2 \left(\log \frac{m^2 + \lambda \phi_{\text{cl}}^2}{\mu^2} - \frac{3}{2} \right), \end{aligned} \quad (4.22)$$

where

$$\phi_{\text{cl}} = \sqrt{(\phi_{\text{cl}}^k)^2}. \quad (4.23)$$

4.5 Renormalization group analysis

The effective potential that we have derived depends on the coupling λ , the mass m^2 and also on the renormalization scale μ at which these parameters are defined. We can now check that a small change in the scale μ can be compensated by a small change in the value of the parameters λ and m^2 . To do so, we must work at leading order in the coupling λ , and therefore expand the field dependence in the logarithms. We can also ignore the constant field-independent part of the potential, which is irrelevant. One is then left with the following relevant terms:

$$\begin{aligned} V_{\text{eff}}(\phi_{\text{cl}}) &\simeq \frac{1}{2} m^2 \phi_{\text{cl}}^2 \left[1 - \frac{\lambda(N+2)}{(4\pi)^2} \left(\log \frac{\mu^2}{m^2} + 1 \right) \right] \\ &\quad + \frac{\lambda}{4} \phi_{\text{cl}}^4 \left[1 - \frac{\lambda(N+8)}{(4\pi)^2} \log \frac{\mu^2}{m^2} \right], \end{aligned} \quad (4.24)$$

We see then that a transformation of the form

$$\mu \rightarrow \mu + \delta\mu, \quad \lambda \rightarrow \lambda + \delta\lambda, \quad m^2 \rightarrow m^2 + \delta m^2, \quad (4.25)$$

leaves the effective potential invariant provided that one chooses:

$$\frac{\delta\lambda}{\lambda} = \frac{\lambda(N+8)}{(4\pi)^2} \frac{\delta\mu^2}{\mu^2}, \quad \frac{\delta m^2}{m^2} = \frac{\lambda(N+2)}{(4\pi)^2} \frac{\delta\mu^2}{\mu^2}. \quad (4.26)$$

Under such a transformation the variation coming from the change in μ in the 1-loop contribution is compensated by the variation coming from the change in λ and m^2 in the tree-level contribution.

From the above results, we conclude that the trajectory of equivalent theories is as expected parametrized by the curves:

$$\frac{\delta\lambda}{\delta\log\mu} = \frac{2\lambda^2(N+8)}{(4\pi)^2} = \beta(\lambda), \quad (4.27)$$

$$\frac{\delta\log m^2}{\delta\log\mu} = \frac{2\lambda(N+2)}{(4\pi)^2} = \gamma_m(\lambda). \quad (4.28)$$

4.6 Radiative symmetry breaking

The 1-loop correction to the effective potential alters the structure of stationary points of the potential, and may therefore have a relevant effect for spontaneous symmetry breaking. To illustrate this point, let us consider the critical case where $m^2 = 0$. The effective potential is then given by:

$$V_{\text{eff}}(\phi_{\text{cl}}) = \frac{\lambda}{4}\phi_{\text{cl}}^4 + \frac{\lambda^2}{4(4\pi)^2}\phi_{\text{cl}}^4 \left[(N+8) \left(\log \frac{\lambda\phi_{\text{cl}}^2}{\mu^2} - \frac{3}{2} \right) + 9 \log 3 \right]. \quad (4.29)$$

The minimum of this effective potential does not occur at zero value of the field, but at a finite point close to the tiny value:

$$\phi_{\text{cl}} \sim \frac{\mu}{\sqrt{\lambda}} \exp \left\{ -\frac{(4\pi)^2}{2(N+8)\lambda} \right\}. \quad (4.30)$$

It seems thus that quantum corrections induce a spontaneous symmetry breaking which was not present at the classical level.

One may now wonder whether this conclusion is correct. Indeed, the above stationary point arises from a competition between the tree-level and the 1-loop corrections, and this means that at the new vacuum, quantum corrections can have an impact that is as big as the classical dynamics. Correspondingly, higher-loop corrections cannot be neglected in this situation. We conclude then that the point under consideration lies outside the range of validity where the 1-loop approximation to the effective potential can be trusted.

Comparing the tree-level potential with the 1-loop correction, we see that the expansion parameter controlling quantum loops is actually:

$$\kappa = \left| \frac{2(N+8)}{(4\pi)^2} \lambda \log \frac{\phi_{\text{cl}}}{\mu} \right|. \quad (4.31)$$

Even if λ is small, this is not necessarily small for values of ϕ_{cl} far away from the scale μ , and the 1-loop approximation can break down. In particular, for the stationary point apparently leading to spontaneous symmetry breaking, one finds $\kappa \sim 1$, and the simple 1-loop approximation is not valid.

In order to say something more, one can try to resum the leading logarithms by using the renormalization group equations. In this case, since the β function is positive, this should allow to reliably resum the logarithms for $\phi_{\text{cl}} \lesssim \mu$, and thus to shed light on the true behavior at the point where the apparent symmetry breaking arises. In order to do so, we start from the Callan-Symanzik equation for the effective potential:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} - \gamma_{\phi_{\text{cl}}} \frac{\partial}{\partial \phi_{\text{cl}}} \right) V_{\text{eff}}(\phi_{\text{cl}}, \lambda, \mu) = 0. \quad (4.32)$$

By dimensional analysis, we can now rewrite the effective potential in terms of a dimensionless function v as:

$$V_{\text{eff}} = \phi_{\text{cl}}^4 v(t, \lambda), \quad (4.33)$$

where

$$t = \log \frac{\phi_{\text{cl}}}{\mu}. \quad (4.34)$$

Using the fact that on the function v one can trade the derivative $\mu \partial / \partial \mu$ with the derivative $-\partial / \partial t$, and similarly $\phi_{\text{cl}} \partial / \partial \phi_{\text{cl}}$ with $\partial / \partial t$, the Callan-Symanzik equation becomes:

$$\left(\frac{\partial}{\partial t} - \frac{\beta}{1 + \gamma} \frac{\partial}{\partial \lambda} + \frac{4\gamma}{1 + \gamma} \right) v(t, \lambda) = 0. \quad (4.35)$$

To solve this, one can proceed in the usual way. The basic building block is in this case a running coupling depending on the field ϕ_{cl} and defined by the differential equation

$$\frac{d\bar{\lambda}(t)}{dt} = \frac{\beta(\bar{\lambda}(t))}{1 + \gamma(\bar{\lambda}(t))}, \quad (4.36)$$

with the boundary condition

$$\bar{\lambda}(0) = \lambda. \quad (4.37)$$

The general solution for v involves then an arbitrary function w and has the form:

$$v(t, \lambda) = w(\bar{\lambda}(t)) \exp \left\{ -4 \int_0^t dt' \frac{\gamma(\bar{\lambda}(t'))}{1 + \gamma(\bar{\lambda}(t'))} \right\}. \quad (4.38)$$

Recall now that in the 1-loop approximation, the anomalous dimension vanishes, $\gamma = 0$, whereas the beta function is positive $\beta > 0$. The differential equation defining the running coupling is then easily solved, and its solution is:

$$\bar{\lambda}(t) = \frac{\lambda}{1 - 2(N + 8)/(4\pi)^2 \lambda t}. \quad (4.39)$$

This means the the improved coupling dictated by the renormalization group equations depends on the field ϕ_{cl} :

$$\begin{aligned} \bar{\lambda}(\phi_{\text{cl}}) &= \frac{\lambda}{1 - 2(N + 8)/(4\pi)^2 \lambda \log(\phi_{\text{cl}}/\mu)} \\ &= \lambda + \frac{2(N + 8)}{(4\pi)^2} \lambda^2 \log \frac{\phi_{\text{cl}}}{\mu} + \dots \end{aligned} \quad (4.40)$$

The renormalization group improved 1-loop effective potential, where the leading logarithms appearing at higher-loops are resummed through the running coupling, should then have the following form:

$$V_{\text{eff}}(\phi_{\text{cl}}) = \phi_{\text{cl}}^4 w(\bar{\lambda}(\phi_{\text{cl}})). \quad (4.41)$$

The function w can now be fixed by requiring that the 1-loop result is reproduced if $\bar{\lambda}(\phi_{\text{cl}})$ is approximated by its first two terms, namely the tree-level constant contribution and the single-logarithm 1-loop correction. One finds in this way that the improved effective potential has the following form:

$$V_{\text{eff}}(\phi_{\text{cl}}) = \frac{1}{4} \phi_{\text{cl}}^4 \left\{ \bar{\lambda}(\phi_{\text{cl}}) + \frac{\bar{\lambda}^2(\phi_{\text{cl}})}{(4\pi)^2} \left[(N+8) \left(\log \bar{\lambda}(\phi_{\text{cl}}) - \frac{3}{2} \right) + 9 \log 3 \right] \right\}. \quad (4.42)$$

With this result, we can now readdress the question of whether radiative corrections really induce spontaneous symmetry breaking or not. In this model, the minimum of the improved potential lies at the point $\phi_{\text{cl}} = 0$, where no spontaneous symmetry breaking occurs. At that point, or more in general for small values of ϕ_{cl} , the improved result is reliable, because the running coupling $\bar{\lambda}$ is small.

Summarizing, we have used the renormalization group equations to resum large logarithmic effects arising for values of ϕ_{cl} much smaller than μ , where the effective expansion parameter κ of the ordinary 1-loop approximation becomes large, $\kappa \gtrsim 1$. The net effect of these logarithms is encoded in a new running coupling, which becomes small in that region of fields, $\bar{\lambda} = \lambda/(1+\kappa) \lesssim \lambda/2$. The improved 1-loop effective potential becomes then reliable even for small values of ϕ_{cl} .

Notice that the same approach does not allow to resum large logarithmic effects arising for values of ϕ_{cl} much larger than μ , where the effective expansion parameter κ of the ordinary 1-loop approximation also becomes large, $\kappa \gtrsim 1$. Indeed, in such a region the basic logarithm changes sign and the running coupling behaves as $\lambda/(1-\kappa)$, which explodes.

5 Yang-Mills gauge theories

5.1 Gauge-fixing, ghosts and Feynman rules

Consider a non-Abelian gauge theory based on an arbitrary compact group G with generators T^a satisfying the algebra

$$[T^a, T^b] = if^{abc}T^c. \quad (5.1)$$

The quantities f^{abc} are the totally antisymmetric structure constants of the Lie algebra, and satisfy the following relation implied by the Jacobi identity:

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0. \quad (5.2)$$

The quadratic combination $T^2 = T^a T^a$ is always a Casimir, as a consequence of the antisymmetry of the structure constants:

$$[T^2, T^a] = 0. \quad (5.3)$$

In any given unitary irreducible representation R of the group, the generators T^a are represented by Hermitian matrices t^a acting on a vector space of dimension $d(R)$. By Schur's Lemma, the Casimir operator $t^2 = t^a t^a$ is proportional to the identity matrix, with a coefficient $C_2(R)$ depending on the representation:

$$t^2 = C_2(R) \mathbb{1}. \quad (5.4)$$

The normalization of the generators is encoded in the quantity $\text{tr}[t^a t^b]$. It can be shown that with a suitable choice of the basis of generators, this can be made proportional to δ^{ab} , with a coefficient $C(R)$ depending on the representation:

$$\text{tr}[t^a t^b] = C(R) \delta^{ab}. \quad (5.5)$$

Now, taking $a = b$ and summing over these two equal indices, the left-hand side gives $\text{tr}[t^2] = C_2(R)d(R)$, whereas the right hand side gives $C(R)d(G)$. It follows that:

$$C_2(R)d(R) = C(R)d(G), \quad C_2(G) = C(G). \quad (5.6)$$

The gauge fields A_μ^a are in the adjoint representation, in which the a -th generator has matrix element in the b -th row and the c -th column given by the structure constants: $(t_G^a)_{bc} = -if^{abc}$. In addition, let us suppose that there are some Dirac fermions ψ^i in an arbitrary representation R of the gauge group, where the generators are represented by some matrices $(t^a)_{ij}$. The Lagrangian is then given by:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi}(i\not{D} - m_0)\psi, \quad (5.7)$$

where:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_0 f^{abc} A_\mu^b A_\nu^c, \quad (5.8)$$

$$D_\mu A_\nu^a = \partial_\mu A_\nu^a + g_0 f^{abc} A_\mu^b A_\nu^c, \quad (5.9)$$

$$D_\mu \psi = \partial_\mu \psi - ig_0 A_\mu^a t^a \psi. \quad (5.10)$$

The theory is invariant under the gauge transformations generated by the group generators T^a with arbitrary parameters α^a . The infinitesimal transformation laws are:

$$\delta A_\mu^a = g_0^{-1} D_\mu \alpha^a = g_0^{-1} \partial_\mu \alpha^a + f^{abc} A_\mu^b \alpha^c, \quad (5.11)$$

$$\delta \psi = i \alpha^a t^a \psi. \quad (5.12)$$

In order to use the theory, it is necessary to choose a gauge. In performing the gauge-fixing, however, one must be careful and properly implement the gauge-fixing with a functional integral approach through the Faddeev-Popov procedure. In general, a gauge is specified by some constraints $G(A) = 0$ on the gauge fields and their derivatives. It is however not trivial to implement this as an operator equation, because the gauge-fixing equations represent constraints on the canonical variables, which are naively incompatible with the canonical commutation relations. One has then to handle these constraints properly. In the functional integral approach, this subtlety translates into the fact that the constraint $G(A) = 0$ must be implemented through a functional δ -function, and this produces a non-trivial Jacobian in the integral over the gauge group.

To perform the gauge fixing in a precise way, we start from the gauge-invariant functional integral over the gauge fields and fermion fields:

$$Z = \int \mathcal{D}A \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} e^{iS[A, \psi, \bar{\psi}]}. \quad (5.13)$$

Because of the invariance of $S[A, \psi, \bar{\psi}]$ under gauge transformations, the part of the integral along field directions in which $S[A, \psi, \bar{\psi}]$ is constant diverges. But this is just because we are redundantly integrating over physically equivalent gauge configurations. To fix this problem, we have to implement a gauge fixing condition $G(A) = 0$ in such a way that physically equivalent configurations are counted only once. This can be done by inserting into the original gauge-invariant path-integral of the theory a factor of the form:

$$\int \mathcal{D}\alpha \delta[G(A^\alpha)] \det \frac{\delta G(A^\alpha)}{\delta \alpha} = 1. \quad (5.14)$$

This is an integral over the parameters $\alpha^a(x)$ of the gauge group, and $A_\mu^{\alpha a}(x)$ denotes the fields obtained by applying a finite gauge transformation $U(\alpha) = e^{i\alpha^a(x)T^a}$ to the gauge fields $A_\mu^a(x)$. After this manipulation, the original path-integral becomes:

$$Z = \int \mathcal{D}\alpha \int \mathcal{D}A \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} e^{iS[A, \psi, \bar{\psi}]} \delta[G(A^\alpha)] \det \frac{\delta G(A^\alpha)}{\delta \alpha}. \quad (5.15)$$

We can at this point change the dummy integration variable A_μ to its gauge-transformed A_μ^α , and similarly ψ and $\bar{\psi}$ to ψ^α and $\bar{\psi}^\alpha$. The integration measure is unchanged by this, since the gauge transformations are combinations of shifts and unitary transformations: $\mathcal{D}A^\alpha \mathcal{D}\psi^\alpha \mathcal{D}\bar{\psi}^\alpha = \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi}$. Moreover, the action is also unchanged, since it is gauge invariant: $S[A^\alpha, \psi^\alpha, \bar{\psi}^\alpha] = S[A, \psi, \bar{\psi}]$. This allows to rewrite the path-integral as:

$$Z = \int \mathcal{D}\alpha \int \mathcal{D}A \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} e^{iS[A, \psi, \bar{\psi}]} \delta[G(A)] \det \frac{\delta G(A^\alpha)}{\delta \alpha}. \quad (5.16)$$

We see that the functional integral over gauge fields has now been split into an independent and divergent integral over the gauge group, which yields an irrelevant normalization constant, and an integral over physically inequivalent gauge fields satisfying the gauge-fixing condition $G(A) = 0$, which involves a non-trivial determinant depending on A , besides the usual phase $e^{iS[A]}$. The gauge-fixing function $G(A)$ must be chosen in such a way that $\delta[G(A)]$ selects a regular slice of the gauge orbits, but is otherwise arbitrary. Notice in this respect that from any given choice $G(A)$, we can also construct a more general family of choices $G(A) - \omega$, where ω are arbitrary fields. This affects only the δ -function term, and not the determinant term. To get rid of the annoying δ -function, and turn it into some gauge fixing term to be added to the original action, we can then take arbitrary linear combinations of these equivalent path-integrals for different functions w , and weight them with a Gaussian weight:

$$\begin{aligned} \delta[G(A)] &\rightarrow \int \mathcal{D}\omega \exp \left\{ -i \int d^4x \frac{\omega^2(x)}{2\xi} \right\} \delta[G(A) - \omega] \\ &= \exp \left\{ -i \int d^4x \frac{G^2(A(x))}{2\xi} \right\}. \end{aligned} \quad (5.17)$$

Finally, the determinant term can be rewritten as a functional integral over complex anticommuting ghost fields with kinetic operator given by $\delta G(A^\alpha)/\delta\alpha$:

$$\det \frac{\delta G(A^\alpha)}{\delta\alpha} = \int \mathcal{D}c \int \mathcal{D}\bar{c} \exp \left\{ -i \int d^4x g_0 \bar{c} \frac{\delta G(A^\alpha)}{\delta\alpha} c \right\}. \quad (5.18)$$

Putting everything together, we finally arrive at the following extremely simple and manageable expression for the path-integral:

$$Z = \int \mathcal{D}A \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \int \mathcal{D}c \int \mathcal{D}\bar{c} e^{iS[A, \psi, \bar{\psi}] + iS_{\text{fix}}[A] + iS_{\text{gh}}[c, \bar{c}, A]}, \quad (5.19)$$

where:

$$S_{\text{fix}}[A] = \int d^4x \left(-\frac{1}{2\xi} G^2(A(x)) \right), \quad (5.20)$$

$$S_{\text{gh}}[c, \bar{c}, A] = \int d^4x \left(-g_0 \bar{c}(x) \frac{\delta G(A^\alpha)}{\delta\alpha} c(x) \right). \quad (5.21)$$

The ghost fields are anticommuting scalar fields, and have therefore the wrong spin-statistics relation. They do not correspond to anything that has ever been observed in particle physics. It is then assumed that they do not exist as external particle states, but only occur as virtual intermediate particles. In other words, we define the theory by projecting it onto the subspace at ghost number zero.

A particularly convenient and covariant gauge is the non-Abelian generalization of the Lorenz gauge for each generator:

$$G(A(x)) = \partial^\mu A_\mu(x). \quad (5.22)$$

For such a linear condition, the operator appearing in the Jacobian determinant is rather simple and entirely fixed by the infinitesimal transformation law of the gauge field:

$$\frac{\delta G(A^\alpha)}{\delta\alpha} = \partial^\mu \frac{\delta A_\mu^\alpha}{\delta\alpha} = g_0^{-1} \partial_\mu D^\mu. \quad (5.23)$$

The gauge-fixing and ghost Lagrangian take then the following simple forms:

$$\mathcal{L}_{\text{fix}} = -\frac{1}{2\xi} \partial^\mu A_\mu^a(x) \partial^\nu A_\nu^a(x), \quad (5.24)$$

$$\mathcal{L}_{\text{gh}} = \partial_\mu \bar{c}^a(x) (D^\mu c)^a(x). \quad (5.25)$$

Expanding all the terms, the total Lagrangian is finally given by:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} A_\mu^a \left(\square \eta^{\mu\nu} - (1 - \xi^{-1}) \partial^\mu \partial^\nu \right) A_\nu^a - \bar{c}^a \square c^a + \bar{\psi} (i \not{\partial} - m_0) \psi \\ & - g_0 f^{abc} \partial_\mu A_\nu^a A^{b\mu} A^{c\nu} + g_0 f^{abc} \bar{c}^a \partial^\mu (c^b A_\mu^c) + g_0 \bar{\psi} \gamma^\mu t^a \psi A_\mu^a \\ & - \frac{1}{4} g_0^2 f^{abe} f^{cde} A_\mu^a A_\nu^b A^{c\mu} A^{d\nu}. \end{aligned} \quad (5.26)$$

To set up renormalized perturbation theory, we define renormalized gauge, ghosts and fermion fields as:

$$A_\mu^a = Z_A^{1/2} A_{r\mu}^a, \quad c^a = Z_c^{1/2} c_r^a, \quad \psi = Z_\psi^{1/2} \psi_r. \quad (5.27)$$

We also introduce a renormalized gauge coupling g and a renormalized mass m . The parameter ξ is now introduced directly for the renormalized fields, without any additional counter-term. We rewrite then the action by splitting it in a renormalized part plus a counter-term part:

$$\mathcal{L} = \mathcal{L}_r + \Delta \mathcal{L}. \quad (5.28)$$

The renormalized part has the form:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} A_{r\mu}^a \left(\square \eta^{\mu\nu} - (1 - \xi^{-1}) \partial^\mu \partial^\nu \right) A_{r\nu}^a - \bar{c}_r^a \square c_r^a + \bar{\psi}_r (i \not{\partial} - m) \psi_r \\ & - g f^{abc} \partial_\mu A_{r\nu}^a A_r^{b\mu} A_r^{c\nu} + g f^{abc} \bar{c}_r^a \partial^\mu (c_r^b A_{r\mu}^c) + g \bar{\psi}_r \gamma^\mu t^a \psi_r A_{r\mu}^a \\ & - \frac{1}{4} g^2 f^{abe} f^{cde} A_{r\mu}^a A_{r\nu}^b A_r^{c\mu} A_r^{d\nu}. \end{aligned} \quad (5.29)$$

The counter-term part is similarly given by:

$$\begin{aligned} \Delta \mathcal{L} = & \frac{1}{2} \Delta_{Z_A} A_{r\mu}^a \left(\square \eta^{\mu\nu} - \partial^\mu \partial^\nu \right) A_{r\nu}^a - \Delta_{Z_c} \bar{c}_r^a \square c_r^a + i \Delta_{Z_\psi} \bar{\psi}_r \not{\partial} \psi_r - \Delta_m \bar{\psi}_r \psi_r \\ & - g \Delta_g^A f^{abc} \partial_\mu A_{r\nu}^a A_r^{b\mu} A_r^{c\nu} + g \Delta_g^c f^{abc} \bar{c}_r^a \partial^\mu (c_r^b A_{r\mu}^c) + g \Delta_g^\psi \bar{\psi}_r \gamma^\mu t^a \psi_r A_{r\mu}^a \\ & - \frac{1}{4} g^2 \Delta_{g^2} f^{abe} f^{cde} A_{r\mu}^a A_{r\nu}^b A_r^{c\mu} A_r^{d\nu}, \end{aligned} \quad (5.30)$$

where the counter-terms are related to the wave-function factors and the bare coupling and mass by the relations:

$$\begin{aligned} \Delta_{Z_A} &= Z_A - 1, \quad \Delta_{Z_c} = Z_c - 1, \quad \Delta_{Z_\psi} = Z_\psi - 1, \quad \Delta_m = m_0 Z_\psi - m, \\ \Delta_g^A &= \frac{g_0}{g} Z_A^{3/2} - 1, \quad \Delta_g^c = \frac{g_0}{g} Z_A^{1/2} Z_c - 1, \quad \Delta_g^\psi = \frac{g_0}{g} Z_A^{1/2} Z_\psi - 1, \\ \Delta_{g^2} &= \frac{g_0^2}{g^2} Z_A^2 - 1. \end{aligned} \quad (5.31)$$

There exist 3 relations among the above 8 counter-terms, which are dictated by gauge invariance, since they depend only on 5 independent quantities.

5.2 BRST symmetry

The full Lagrangian of the gauge-fixed non-Abelian theory described in previous section can be rewritten in the following form:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi}(i\not{D} - m)\psi - \bar{c}^a \partial^\mu D_\mu c^a + \frac{\xi}{2} B^a B^a + B^a \partial^\mu A_\mu^a. \quad (5.35)$$

The field B^a is a non-dynamical auxiliary field, which appears without derivatives in the Lagrangian. The original Lagrangian can then be recovered by eliminating this field through its equations of motion:

$$B^a = -\xi^{-1} \partial^\mu A_\mu^a. \quad (5.36)$$

It turns out that written in this form, the gauge-fixed Lagrangian has a non-trivial continuous symmetry, depending on an anticommuting parameter ϵ and acting as follows at the infinitesimal level:

$$\delta A_\mu^a = \epsilon D_\mu c^a, \quad (5.37)$$

$$\delta \psi = i g \epsilon c^a t^a \psi, \quad (5.38)$$

$$\delta \bar{\psi} = -i g \epsilon c^a \bar{\psi} t^a, \quad (5.39)$$

$$\delta c^a = -g/2 \epsilon f^{abc} c^b c^c, \quad (5.40)$$

$$\delta \bar{c}^a = \epsilon B^a, \quad (5.41)$$

$$\delta B^a = 0. \quad (5.42)$$

Note that this symmetry acts non-linearly. On the gauge fields A_μ and the fermion fields $\psi, \bar{\psi}$, it corresponds to a gauge transformation with parameter $\alpha^a = g \epsilon c^a$ depending on the ghost fields. The original Lagrangian for these fields is thus invariant by construction under these transformations. On the ghost fields c^a, \bar{c}^a and the auxiliary fields B^a , the action is defined in a somewhat asymmetric way, and turns out to leave invariant the sum of the gauge-fixing and ghost Lagrangian. More precisely, one can verify that as a consequence of the anticommuting nature of the ghost and the Jacobi identity satisfied by the structure constants one has $\delta(D_\mu c^a) = 0$. One is then left with only one term from the variation of the ghost Lagrangian, which exactly cancels the variation of the auxiliary field Lagrangian.

The above symmetry is called BRST symmetry, and is a remnant of the original gauge-symmetry that survives in the gauged-fixed theory. A remarkable feature of the above symmetry is that it is nilpotent, in the sense that applying successively twice the transformation one gets no variation at all. More precisely, writing the transformation laws in terms of an operator \hat{Q} acting on any of the fields ϕ as

$$\delta \phi = \epsilon \hat{Q} \phi, \quad (5.43)$$

one finds that:

$$\hat{Q}^2 \phi = 0. \quad (5.44)$$

This property can be verified in a straightforward way by using again the Jacobi identity satisfied by the structure constants.

By Nöther's theorem, we know that there must exist a conserved charge Q associated to the above continuous symmetry:

$$Q = \text{conserved charge} . \quad (5.45)$$

Since this is conserved and does not depend explicitly on time, it must commute with the Hamiltonian:

$$[Q, H] = 0 . \quad (5.46)$$

Moreover, the fact that the operator \hat{Q} is nilpotent implies that also the BRST charge is so, and thus:

$$Q^2 = 0 . \quad (5.47)$$

Due to these properties, the BRST charge Q divides the space \mathcal{H} of eigentates of H into three distinct subspaces \mathcal{H}_0 , \mathcal{H}_1 and \mathcal{H}_2 , defined as follows:

$$\begin{aligned} \mathcal{H}_0 &= \left\{ |\psi_0\rangle \in \mathcal{H} \left| Q|\psi_0\rangle = 0; |\psi_0\rangle \neq Q|\psi\rangle \right. \right\} \\ &= \text{closed but not exact states} , \end{aligned} \quad (5.48)$$

$$\begin{aligned} \mathcal{H}_1 &= \left\{ |\psi_1\rangle \in \mathcal{H} \left| Q|\psi_1\rangle \neq 0 \right. \right\} \\ &= \text{non-closed states} , \end{aligned} \quad (5.49)$$

$$\begin{aligned} \mathcal{H}_2 &= \left\{ |\psi_2\rangle \in \mathcal{H} \left| Q|\psi_2\rangle = 0; |\psi_2\rangle = Q|\psi\rangle \right. \right\} \\ &= \text{closed and also exact states} . \end{aligned} \quad (5.50)$$

Notice that the states in \mathcal{H}_2 are rather trivial, because they have vanishing inner product among themselves and also with states in \mathcal{H}_0 : $\langle \psi_2 | \psi_2' \rangle = \langle \psi_2 | \psi_0 \rangle = 0$.

In order to gain further insight on the above three subspaces of states, in an intuitive and not completely rigorous way, let us consider the limit of vanishing gauge coupling, $g \rightarrow 0$. In this limit, and using the on-shell value of the auxiliary field B^a , the transformation laws become:

$$\delta A_\mu^a \rightarrow \epsilon \partial_\mu c^a , \quad (5.51)$$

$$\delta c^a \rightarrow 0 , \quad (5.52)$$

$$\delta \bar{c}^a \rightarrow -\xi^{-1} \epsilon \partial_\mu A^\mu , \quad (5.53)$$

$$\delta(\partial^\mu A_\mu^a) \rightarrow 0 . \quad (5.54)$$

These relations show how asymptotic states are transformed by the BRST transformation. More precisely, decomposing each gluon field A_μ^a into two physical transverse polarizations satisfying $\partial^\mu A_\mu^a = 0$ with $A_\mu^a \neq \partial_\mu \lambda^a$, an unphysical forward polarization of the type $A_\mu^a = \partial_\mu \lambda^a$ and an unphysical backward polarization with $\partial^\mu A_\mu^a \neq 0$, one finds that:

$$Q|A_{\text{tran}}\rangle = 0 , \quad Q|A_{\text{forw}}\rangle = |c\rangle , \quad Q|A_{\text{back}}\rangle = 0 , \quad (5.55)$$

$$Q|c\rangle = 0 , \quad Q|\bar{c}\rangle = |A_{\text{back}}\rangle . \quad (5.56)$$

This means that the two physical polarizations of the gauge bosons belong to \mathcal{H}_0 , whereas the two unphysical polarizations of the gauge bosons, the ghost and the antighosts belong to $\mathcal{H}_1 \cup \mathcal{H}_2$:

$$|A_{\text{tran}}\rangle \in \mathcal{H}_0, \quad (5.57)$$

$$|A_{\text{forw}}\rangle, |\bar{c}\rangle \in \mathcal{H}_1, \quad (5.58)$$

$$|A_{\text{back}}\rangle, |c\rangle \in \mathcal{H}_2. \quad (5.59)$$

A physically sensible and unitary theory can then be defined by projecting onto physical states belonging to \mathcal{H}_0 . These are catalogued by the cohomology of the BRST charge Q , in the sense that they are closed but not exact. The projected S -matrix defined by this prescription is guaranteed to be unitary. Indeed, since Q is conserved, any closed state annihilated by Q must evolve into closed states that are still annihilated by Q . Now, even if the original states were not exact, meaning that they were not of the form of Q acting on some other state, the evolved states may contain both non-exact states and exact states. But the latter are trivial, since they have vanishing overlap with any closed state. So effectively, closed states remain closed modulo irrelevant exact states, and the S -matrix defined on the subspace \mathcal{H}_0 is unitary:

$$S|_{\text{phys}} : \text{unitary}. \quad (5.60)$$

In diagrams involving loops, the contribution of unphysical gauge bosons combines with that of ghosts, and the corresponding contributions to the imaginary part of the amplitude cancel. This is crucial to enforce the optical theorem with only physical gauge bosons as external states.

5.3 Diagrammatic computation of β .

Let us now consider the structure of divergences at the 1-loop level. In this approximation, the 3 relations that exist among the 8 counter-terms can be made more manifest and take a simple form. To derive them, we start by defining the basic counter-term that is related to the renormalization of the coupling:

$$\Delta_g = \frac{g_0}{g} - 1. \quad (5.61)$$

Working at leading order in all the Δ 's, we can then derive explicit expressions for the 4 different coupling counter-terms, in terms of this universal Δ_g and the 3 wave-function counter-terms Δ_{Z_A} , Δ_{Z_c} and Δ_{Z_ψ} . One finds:

$$\Delta_g^A = \Delta_g + \frac{1}{2}(3\Delta_{Z_A}), \quad (5.62)$$

$$\Delta_g^c = \Delta_g + \frac{1}{2}(\Delta_{Z_A} + 2\Delta_{Z_c}), \quad (5.63)$$

$$\Delta_g^\psi = \Delta_g + \frac{1}{2}(\Delta_{Z_A} + 2\Delta_{Z_\psi}), \quad (5.64)$$

$$\Delta_{g^2} = 2\Delta_g + \frac{1}{2}(4\Delta_{Z_A}). \quad (5.65)$$

Consider next the counter-term Δ_{Z_ψ} . This is extracted from the divergent part of the 1-loop correction to the 2-point function of matter fermions. There is a 1 loop diagram plus 2 counter-term diagrams. The result can be written in the following form:

$$\text{---}\text{---}\text{---} + \text{---}\overset{\Delta_{Z_\psi}}{\bullet}\text{---} + \text{---}\overset{\Delta_m}{\bullet}\text{---} = i\Sigma(\not{p}). \quad (5.71)$$

One finds, after a rather easy computation:

$$\Sigma(\not{p}) = \frac{g^2}{(4\pi)^2} \left[\not{p} \xi - (3 + \xi)m \right] C_2(R) \log \frac{\tilde{\Lambda}^2}{p^2} + \not{p} \Delta_{Z_\psi} - \Delta_m + \text{finite}. \quad (5.72)$$

The cancellation of the divergence at a renormalization scale μ implies then that the counter-terms are given by:

$$\begin{aligned} \Delta_{Z_\psi} &= \frac{g^2}{(4\pi)^2} \left[-\xi C_2(R) \right] \log \frac{\tilde{\Lambda}^2}{\mu^2} + \text{finite}. \\ \Delta_m &= \frac{g^2}{(4\pi)^2} \left[-(3 + \xi)C_2(R) \right] m \log \frac{\tilde{\Lambda}^2}{\mu^2} + \text{finite}. \end{aligned} \quad (5.73)$$

Finally, let us consider the vertex counter-term Δ_g^ψ . This is extracted from the divergent part of the 1-loop correction to the 3-point function between 2 matter fermions and 1 gauge boson. There are 2 loop diagrams plus 1 counter-term diagram. The result can be written in the following form:

$$\begin{array}{c} \mu \quad a \\ \text{---}\text{---}\text{---} \\ \text{---}\text{---}\text{---} \end{array} + \begin{array}{c} \mu \quad a \\ \text{---}\text{---}\text{---} \\ \text{---}\text{---}\text{---} \end{array} + \begin{array}{c} \mu \quad a \\ \text{---}\text{---}\text{---} \\ \text{---}\text{---}\text{---} \end{array} \overset{\Delta_g^\psi}{\bullet} = i g t^a \Gamma^\mu(p_i). \quad (5.74)$$

One finds, after a rather easy computation:

$$\Gamma^\mu(p_i) = \frac{g^2}{(4\pi)^2} \left[\xi C_2(R) + \frac{3 + \xi}{4} C_2(G) \right] \log \frac{\tilde{\Lambda}^2}{p^2} + \Delta_g^\psi + \text{finite}. \quad (5.75)$$

The cancellation of the divergence at a renormalization scale μ implies then that the counter-term is given by:

$$\Delta_g^\psi = \frac{g^2}{(4\pi)^2} \left[-\xi C_2(R) - \frac{3 + \xi}{4} C_2(G) \right] \log \frac{\tilde{\Lambda}^2}{\mu^2} + \text{finite}. \quad (5.76)$$

From the logarithmic μ -dependence of the counter-terms that we have computed, we can finally deduce the β function for the gauge coupling g . Putting all the contributions together, the ξ -dependence cancels out, and one is left with the following result:

$$\beta(g) = \frac{g^3}{(4\pi)^2} \left[-\frac{11}{3} C_2(G) + \frac{4}{3} C(R) \right]. \quad (5.77)$$

The crucial feature of this formula is that since $C_2(G)$ and $C(R)$ are by definition semi-positive and positive definite, the fermions always give a positive contribution, whereas the gauge bosons give a vanishing contribution for Abelian groups and a negative contribution for non-Abelian groups. As a result, for non-Abelian groups with sufficiently many gauge bosons relative to matter fermions, the β -function becomes negative.

The running coupling at an arbitrary energy scale μ' is defined by the differential equation

$$\mu' \frac{d\bar{g}(\mu')}{d\mu'} = \beta(\bar{g}(\mu')), \quad (5.78)$$

with the boundary condition:

$$\bar{g}(\mu) = g. \quad (5.79)$$

The solution takes the form

$$\bar{g}^{-2}(\mu') - \bar{g}^{-2}(\mu) = -\frac{b}{(4\pi)^2} \log \frac{\mu'^2}{\mu^2}, \quad (5.80)$$

or

$$g^2(\mu') = \frac{g^2}{1 - \frac{b g^2}{(4\pi)^2} \log \frac{\mu'^2}{\mu^2}}, \quad (5.81)$$

where:

$$b = -\frac{11}{3}C_2(G) + \frac{4}{3}C(R). \quad (5.82)$$

Notice finally that in a generic mass-dependent renormalization scheme, the finite parts of the counter-terms computed above may depend on μ/m . It is then more complicated to solve the renormalization group equations, because the running of the mass is coupled to the running of the gauge coupling. These effects are negligible at energy scales sufficiently far away from the mass scale. But around the mass scale they are responsible for a transition in the value of the β -function, matching the values with and without the fermion. This threshold effect is associated to the decoupling of the fermion at energies below its mass, and represents an effect that is actually also showing up, though in a different way, in mass-independent renormalization schemes, like minimal subtraction. We will discuss this point more in detail in the context of effective theories.

5.4 Effective action

Let us now see how one can compute the quantum effective action for the gauge fields in Yang-Mills theories, focusing for simplicity on the case of massless fermions. It is convenient to start by rescaling the gauge fields by the coupling constant g_0 to rewrite the theory as

$$\mathcal{L} = -\frac{1}{4g_0^2} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi} i \not{D} \psi \quad (5.83)$$

where now:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c, \quad (5.84)$$

$$D_\mu A_\nu^a = \partial_\mu A_\nu^a + f^{abc} A_\mu^b A_\nu^c, \quad (5.85)$$

$$D_\mu \psi = \partial_\mu \psi - i A_\mu^a t^a \psi. \quad (5.86)$$

The infinitesimal gauge transformations become then:

$$\delta A_\mu^a = D_\mu \alpha^a, \quad (5.87)$$

$$\delta \psi = i \alpha^a t^a \psi. \quad (5.88)$$

We then split the fields into a classical background part plus a quantum fluctuation part, over which we then integrate in the path integral defining the effective action. We consider a non-zero background only for gauge fields, and decompose:

$$A_\mu^a = \bar{A}_\mu^a + \eta_\mu^a. \quad (5.89)$$

One finds correspondingly that:

$$F_{\mu\nu}^a = \bar{F}_{\mu\nu}^a + \bar{D}_\mu \eta_\nu^a - \bar{D}_\nu \eta_\mu^a + f^{abc} \eta_\mu^b \eta_\nu^c, \quad (5.90)$$

$$D_\mu \psi = \bar{D}_\mu \psi - i \eta_\mu^a t^a \psi, \quad (5.91)$$

where the covariant derivative \bar{D}_μ with respect to the background gauge field is defined by the following expressions:

$$\bar{D}_\mu \eta_\nu^a = \partial_\mu \eta_\nu^a + f^{abc} \bar{A}_\mu^b \eta_\nu^c, \quad (5.92)$$

$$\bar{D}_\mu \psi = \partial_\mu \psi - i \bar{A}_\mu^a t^a \psi. \quad (5.93)$$

Regarding the background field as fixed, the theory has now a gauge symmetry which is implemented by transformations of η_μ :

$$\delta \eta_\mu^a = \bar{D}_\mu \alpha^a + f^{abc} \eta_\mu^b \alpha^c, \quad (5.94)$$

$$\delta \psi = i \alpha^a t^a \psi. \quad (5.95)$$

We can gauge-fix this symmetry by using the same Faddeev-Popov procedure as before. In this case, we take as gauge-fixing function:

$$G^a(\eta) = \bar{D}^\mu \eta_\mu^a. \quad (5.96)$$

The corresponding kinetic operator for the ghost action is:

$$\frac{\delta G^a(\eta^\alpha)}{\delta \alpha^b} = \bar{D}^\mu \frac{\delta \eta_\mu^{a\alpha}}{\delta \alpha^b} = \bar{D}^\mu (\bar{D}_\mu \delta^{ab} + f^{abc} \eta_\mu^c). \quad (5.97)$$

Going through the same steps as before to introduce the ghosts, and choosing here $\xi = 1$ for simplicity, one finally arrives at the following Lagrangian:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4g_0^2} (\bar{F}_{\mu\nu}^a + \bar{D}_\mu \eta_\nu^a - \bar{D}_\nu \eta_\mu^a + f^{abc} \eta_\mu^b \eta_\nu^c) (\bar{F}^{a\mu\nu} + \bar{D}^\mu \eta^{a\nu} - \bar{D}^\nu \eta^{a\mu} + f^{abc} \eta^{b\mu} \eta^{c\nu}) \\ & -\frac{1}{2g_0^2} (\bar{D}^\mu \eta_\mu^a)^2 + \bar{\psi} (i \bar{\mathcal{D}} + \eta^a t_a) \psi - \bar{c}^a \bar{D}^\mu (\bar{D}_\mu c^a + f^{abc} \eta_\mu^b c^c). \end{aligned} \quad (5.98)$$

In this way of performing the gauge fixing, the gauge-fixed action automatically inherits a new local symmetry mixing background and fluctuation fields, in which the background fields \bar{A}_μ^a transform as gauge fields and the fluctuation fields η_μ^a as matter fields in the adjoint representation. More precisely, the infinitesimal form of these new transformations is given by:

$$\delta \bar{A}_\mu^a = \bar{D}_\mu \beta^a, \quad (5.99)$$

$$\delta \eta_\mu^a = -f^{abc} \eta_\mu^b \beta^c, \quad (5.100)$$

$$\delta c^a = -f^{abc} c^b \beta^c, \quad (5.101)$$

$$\delta \psi = i \beta^a t^a \psi. \quad (5.102)$$

The integration measure of the path integral over η , ψ and c defining the effective action is invariant under these transformations, and as a result the quantum effective action $\Gamma[\bar{A}]$ for the background field is by construction invariant under the above background gauge transformations $\delta \bar{A}_\mu^a = \bar{D}_\mu \beta^a$. This manifest gauge invariance is very helpful, and greatly simplifies the computation of the effective action.

In the 1-loop approximation, the effective action $\Gamma[\bar{A}]$ is determined by the terms in the above Lagrangian which are quadratic in the fluctuation fields η_μ^a , c^a and ψ . These take the following form:

$$\mathcal{L}_{\text{quad}} = -\frac{1}{2g^2} \eta^{a\mu} (\Delta_\eta)_{\mu\nu}^{ab} \eta^{b\nu} + \bar{c}^a (\Delta_c)^{ab} c^b + \bar{\psi} \sqrt{\Delta_\psi} \psi + \text{counter-terms}, \quad (5.103)$$

where:

$$(\Delta_\eta)_{\mu\nu}^{ab} = -\bar{D}^{2ab} \eta_{\mu\nu} - 2f^{abc} \bar{F}_{\mu\nu}^c, \quad (5.104)$$

$$(\Delta_c)^{ab} = -\bar{D}^{2ab}, \quad (5.105)$$

$$\Delta_\psi = -\bar{\mathcal{D}}^2. \quad (5.106)$$

Recalling that the structure constants give the matrix elements of the generators in the adjoint representation, $f^{abc} = i(t_G^a)^{bc}$, and working out the square of the Dirac operator, these kinetic operators can be rewritten as follows, after suppressing the gauge indices:

$$\Delta_{\eta\mu\nu} = -\bar{D}^2 \eta_{\mu\nu} + \bar{F}_{\alpha\beta}^a \Sigma_{\eta\mu\nu}^{\alpha\beta} t_G^a, \quad \text{with} \quad \Sigma_{\eta\mu\nu}^{\alpha\beta} = i(\delta_\mu^\alpha \delta_\nu^\beta - \delta_\nu^\alpha \delta_\mu^\beta), \quad (5.107)$$

$$\Delta_\psi = -\bar{D}^2 + \bar{F}_{\alpha\beta}^a \Sigma_\psi^{\alpha\beta} t_R^a, \quad \text{with} \quad \Sigma_\psi^{\alpha\beta} = \frac{i}{4} [\gamma^\alpha, \gamma^\beta], \quad (5.108)$$

$$\Delta_c = -\bar{D}^2 + \bar{F}_{\alpha\beta}^a \Sigma_c^{\alpha\beta} t_G^a, \quad \text{with} \quad \Sigma_c^{\alpha\beta} = 0. \quad (5.109)$$

We recognize these as the minimal gauge-invariant d'Alembertian operators with an additional magnetic moment interaction with gyromagnetic ratio $g = 2$, in the appropriate representation of the gauge and Lorentz groups for each of the fields η , ψ and c :

$$\Delta_\phi = -\bar{D}^2 + \bar{F}_{\alpha\beta}^a \Sigma_\phi^{\alpha\beta} t_\phi^a \quad (5.110)$$

The 1-loop correction to the effective action is now given by the determinants of these

kinetic operators, or more precisely by:

$$\begin{aligned}\Gamma^1[\bar{A}] &= \text{const.} + \frac{i}{2} \text{tr} \log \Delta_\eta - i \text{tr} \log \Delta_c - i \text{tr} \log \sqrt{\Delta_\psi} + \Delta^1 S \\ &= \text{const.} + \frac{i}{2} \text{tr} \log \Delta_\eta - i \text{tr} \log \Delta_c - \frac{i}{2} \text{tr} \log \Delta_\psi + \Delta^1 S.\end{aligned}\quad (5.111)$$

This effective action can be organized in a derivative expansion. No zero-derivative effective potential is allowed by gauge invariance, and thus the first leading term in this low-energy expansion is a correction to the kinetic term, with an extra mild logarithmic dependence on derivatives:

$$\Gamma^1[\bar{A}] = \int d^4x \left\{ -\frac{1}{4} \left[\frac{a}{(4\pi)^2} + \frac{b}{(4\pi)^2} \log \left(\frac{\Lambda^2}{-\square} \right) \right] \bar{F}_{\mu\nu}^a \bar{F}^{a\mu\nu} + \Delta^1 \mathcal{L} + \dots \right\}. \quad (5.112)$$

Finally, imposing a renormalization condition defining the coupling g at scale μ from the 1PI 2-point function amounts to require that this correction vanishes at the scale μ , i.e. when $-\square \bar{A}_\mu^a = \mu^2 A_\mu^a$. This fixes the counter-term to:

$$\Delta^1 \mathcal{L} = \frac{1}{4} \left[\frac{a}{(4\pi)^2} + \frac{b}{(4\pi)^2} \log \left(\frac{\Lambda^2}{\mu^2} \right) \right] \bar{F}_{\mu\nu}^a \bar{F}^{a\mu\nu}. \quad (5.113)$$

The renormalized effective action at 1-loop order and leading order in the derivative expansion takes then the following form:

$$\Gamma[\bar{A}] = \int d^4x \left\{ -\frac{1}{4} \left[g^{-2} - \frac{b}{(4\pi)^2} \log \left(\frac{-\square}{\mu^2} \right) \right] \bar{F}_{\mu\nu}^a \bar{F}^{a\mu\nu} + \dots \right\}. \quad (5.114)$$

Switching now to momentum space, we can identify the above wave-function correction with the inverse running coupling at the scale k . More precisely, we have

$$\Gamma[\bar{A}] = \int \frac{d^4k}{(2\pi)^4} \left[-\frac{1}{2} \bar{g}^{-2}(k) \tilde{A}_\mu^a(-k) \left(k^2 \eta^{\mu\nu} - k^\mu k^\nu \right) \tilde{A}_\nu^a(k) + \dots \right], \quad (5.115)$$

where

$$\bar{g}^{-2}(k) = g^{-2} - \frac{b}{(4\pi)^2} \log \left(\frac{k^2}{\mu^2} \right). \quad (5.116)$$

Comparing with the 1-loop running coupling, we see that the coefficient b coincides with the numerical coefficient defining the β function:

$$\beta = \frac{b g^3}{(4\pi)^2}. \quad (5.117)$$

To compute the coefficient b , we can consider slowly varying external fields and expand the determinants defining the 1-loop contribution to the effective action in powers of the external field. Each of the 3 fluctuation fields η , ψ and c contribute effectively as if it had a kinetic operator Δ_ϕ with the appropriate spin matrix $\Sigma_\phi^{\alpha\beta}$, of the form:

$$\Delta_\phi = -\square + V_{\phi^2 A} + V_{\phi^2 F} + V_{\phi^2 A^2}. \quad (5.118)$$

where

$$V_{\phi^2 A} = i\left(\bar{A}^{a\mu}\partial_\mu + \partial_\mu\bar{A}^{a\mu}\right)t_\phi^a, \quad (5.119)$$

$$V_{\phi^2 A^2} = \bar{A}^{a\mu}\bar{A}_\mu^b t_\phi^a t_\phi^b, \quad (5.120)$$

$$V_{\phi^2 F} = \bar{F}_{\alpha\beta}^a \Sigma_\phi^{\alpha\beta} t_\phi^a. \quad (5.121)$$

The inverse of the operator \square defines a free effective propagator, whereas the quantities $V_{\phi^2 A}$, $V_{\phi^2 A^2}$ and $V_{\phi^2 F}$ correspond to three kinds of effective interaction vertices with the background fields. Expanding the determinants, one gets then a diagrammatic expansion based on these propagators and vertices. For each term, one has:

$$\begin{aligned} \text{tr log } \Delta_\phi &= \text{tr log } (-\square) + \text{tr log } \left[1 - \frac{1}{\square} \left(V_{\phi^2 A} + V_{\phi^2 A^2} + V_{\phi^2 F}\right)\right] \\ &= \text{const.} - \sum_{n=1}^{\infty} \frac{1}{n} \text{tr} \left[\frac{1}{\square} \left(V_{\phi^2 A} + V_{\phi^2 A^2} + V_{\phi^2 F}\right)\right]^n. \end{aligned} \quad (5.122)$$

All the terms that are linear in the background field \bar{A} trivially vanish, as required by gauge invariance. At the quadratic level, instead, there is one contribution with 1 vertex $V_{\phi^2 A^2}$ and two contributions from 2 vertices $V_{\phi^2 A}$ and $V_{\phi^2 F}$, which cannot interfere. One finds then:

$$\begin{aligned} \text{tr log } \Delta_\phi &= -\frac{1}{2} \text{tr} \left[\frac{1}{\square} V_{\phi^2 A} \frac{1}{\square} V_{\phi^2 A}\right] - \text{tr} \left[\frac{1}{\square} V_{\phi^2 A^2}\right] - \frac{1}{2} \text{tr} \left[\frac{1}{\square} V_{\phi^2 F} \frac{1}{\square} V_{\phi^2 F}\right] + \dots \\ &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \\ &= \int \frac{d^4 k}{(2\pi)^2} \tilde{A}_\mu^a(-k) \left(\pi_{\phi ab}^{1\mu\nu}(k) + \pi_{\phi ab}^{2\mu\nu}(k) + \pi_{\phi ab}^{3\mu\nu}(k)\right) \tilde{A}_\nu^b(k). \end{aligned} \quad (5.123)$$

The three diagrams are easily computed by introducing a Feynman parameter x , shifting the integration momentum to $q = p + kx$, going to the Euclidean space and using dimensional regularization. They involve the quantity $\Delta = -x(1-x)k^2$. The trace over gauge indices always produces:

$$\text{tr} [t_\phi^a t_\phi^b] = C(R_\phi) \delta^{ab}. \quad (5.124)$$

The trace over the spin components can instead be of the following two types:

$$\text{tr} [\mathbb{1}_\phi] = d(j_\phi), \quad \text{tr} [\Sigma_\phi^{\rho\sigma} \Sigma_\phi^{\alpha\beta}] = C(j_\phi) (\eta^{\rho\alpha} \eta^{\sigma\beta} - \eta^{\rho\beta} \eta^{\sigma\alpha}), \quad (5.125)$$

where for the spin $j = 0, 1/2, 1$ fluctuations c, ψ, η one finds:

$$d(0) = 1, \quad d(1/2) = 4, \quad d(1) = 4, \quad (5.126)$$

$$C(0) = 0, \quad C(1/2) = 1, \quad C(1) = 2. \quad (5.127)$$

For the first diagram, one finds:

$$\begin{aligned}
\pi_{\phi ab}^{1\mu\nu} &= -\frac{1}{2} \text{tr}[t_\phi^a t_\phi^b] \text{tr}[\mathbb{1}_\phi] \int \frac{d^4 p}{(2\pi)^4} \frac{(2p+k)^\mu (2p+k)^\nu}{p^2(p+k)^2} \\
&= -\frac{i}{2} d(j_\phi) C(R_\phi) \delta^{ab} \int_0^1 dx \tilde{\mu}^{4-d} \int \frac{d^d q_E}{(2\pi)^4} \frac{-q_E^2 \eta^{\mu\nu} + (1-2x)^2 k^\mu k^\nu}{(q_E^2 + \Delta)^2} \\
&= \frac{i}{(4\pi)^2} (k^2 \eta^{\mu\nu} - k^\mu k^\nu) \delta^{ab} \left(\frac{1}{6} d(j_\phi) C(R_\phi) \right) \left(\log \frac{\tilde{\Lambda}^2}{k^2} + \text{finite} \right). \tag{5.128}
\end{aligned}$$

For the second diagram, one finds a vanishing result:

$$\begin{aligned}
\pi_{\phi ab}^{2\mu\nu} &= \text{tr}[t_\phi^a t_\phi^b] \text{tr}[\mathbb{1}_\phi] \int \frac{d^4 p}{(2\pi)^4} \frac{\eta^{\mu\nu}}{p^2} \\
&= -i d(j_\phi) C(R_\phi) \delta^{ab} \tilde{\mu}^{4-d} \int \frac{d^d q_E}{(2\pi)^4} \frac{1}{q_E^2} \\
&= 0. \tag{5.129}
\end{aligned}$$

Finally, for the third diagram one obtains:

$$\begin{aligned}
\pi_{\phi ab}^{3\mu\nu} &= -2 \text{tr}[t_\phi^a t_\phi^b] \text{tr}[\Sigma_\phi^{\rho\sigma} \Sigma_\phi^{\alpha\beta}] \int \frac{d^4 p}{(2\pi)^4} \frac{(k_\rho \eta_{\mu\sigma})(k_\alpha \eta_{\nu\beta})}{p^2(p+k)^2} \\
&= -2i C(j_\phi) C(R_\phi) \delta^{ab} \int_0^1 dx \tilde{\mu}^{4-d} \int \frac{d^d q_E}{(2\pi)^4} \frac{k^2 \eta^{\mu\nu} - k^\mu k^\nu}{(q_E^2 + \Delta)^2} \\
&= \frac{i}{(4\pi)^2} (k^2 \eta^{\mu\nu} - k^\mu k^\nu) \delta^{ab} \left(-2C(j_\phi) C(R_\phi) \right) \left(\log \frac{\tilde{\Lambda}^2}{k^2} + \text{finite} \right). \tag{5.130}
\end{aligned}$$

Putting everything together, we find then:

$$\text{tr} \log \Delta_\phi = \frac{i}{2} \int \frac{d^4 k}{(2\pi)^2} \frac{c_\phi}{(4\pi)^2} \log \frac{\tilde{\Lambda}^2}{k^2} \tilde{A}_\mu^a(-k) (k^2 \eta^{\mu\nu} - k^\mu k^\nu) \tilde{A}_\nu^a(k) + \text{finite}, \tag{5.131}$$

where

$$c_\phi = \left(\frac{1}{3} d(j_\phi) - 4C(j_\phi) \right) C(R_\phi). \tag{5.132}$$

For the three different fields, this gives:

$$c_c = \frac{1}{3} C(G), \quad c_\psi = -\frac{8}{3} C(R), \quad c_\eta = -\frac{20}{3} C(G). \tag{5.133}$$

The coefficient b of the β -function is then given by:

$$\begin{aligned}
b &= \frac{1}{2} c_\eta - c_c - \frac{1}{2} c_\psi \\
&= -\frac{11}{3} C(G) + \frac{4}{3} C(R). \tag{5.134}
\end{aligned}$$

6 Effective theories

6.1 Low-energy effective theories

It is natural to believe that in order to describe a physical phenomenon occurring at some typical length scale, it should be sufficient, in first approximation, to use a theory involving degrees of freedom with a wave-length comparable to or larger than that scale. Possible microscopic details concerning much shorter wave-lengths should induce only mild indirect effects, and it should be possible to account for these through suitable parameters in the theory describing the relevant macroscopic degrees of freedom. In other words, if one is interested in describing phenomena limited to some minimal length scale, one can use a simpler macroscopic theory instead of the full microscopic one, with parameters that can either be computed from the microscopic theory if this is known or measured from experiments. This type of effective theories proves to be very useful every time a large separation of scales arises.

An effective quantum field theory is a quantum field theory emerging as an approximate description of some theory valid below a certain maximal energy scale M . The typical situation in which such an effective field theory naturally emerges and becomes relevant is when particles with very different masses arise in a renormalizable quantum field theory. Suppose for instance that there is a first field φ with a small mass m and a second field ϕ with a large mass M , such that $m \ll M$. Physical amplitudes computed in such a theory will have a complicated non-analytic dependence on the external momenta, coming from the fact that virtual particles contribute poles or cuts whenever they go close to mass-shell. Suppose however that we want to use such a theory only to describe scattering of light particles associated to φ at energies E much smaller than M , without any real heavy particle associated to ϕ as external state. We expect then that the effects related to the exchange of virtual heavy particles should be small and accountable in simple terms. This is indeed true, due to the fact that such virtual states can now have only momenta which are much smaller than the scale M . In tree diagrams, this is just implied by momentum conservation, whereas in loop diagrams it becomes manifest only with certain choices of regularization. This property implies that virtual states can never become real, and that the non-analytic part of their propagator never becomes relevant. One can then think of expanding the propagator of such virtual heavy particles in power series of p/M :

$$\frac{i}{p^2 - M^2 + i\epsilon} \rightarrow \frac{-i}{M^2} \frac{1}{1 - p^2/M^2} \simeq \frac{-i}{M^2} \left[1 + \frac{p^2}{M^2} + \left(\frac{p^2}{M^2}\right)^2 + \dots \right]. \quad (6.1)$$

The effect of each term in the expansion can then be mimicked by some new local effective interaction among the light fields, suppressed by some inverse power of M . Clearly, such an expansion will break down as soon as $p \sim M$. Physically, this is because at such scale the heavy particles ϕ can be produced on-shell as real states, due to the imaginary part of the full propagator, and it becomes then clearly impossible to account indirectly for their effects without including them as genuine degrees of freedom in the theory. But if p/M is small, one can reach an arbitrary accuracy by retaining sufficiently many orders in the series.

From the point of view of the quantum effective action, the above situation looks as follows. If one wants to use the theory at arbitrary energies E , then the whole quantum effective action is needed. However, for $E \ll M$, only the effective action for the light fields is relevant, and the heavy fields can be integrated out in the functional integral. Moreover, the contribution to the effective action produced by the process of integrating out the heavy fields is effectively local, in the sense that it can be expanded as an infinite series of new local operators. This reflects the fact that for $E \ll M$ the virtual heavy particles can propagate only over a small finite distance. All these new operators will have dimension greater than 4 and will have a coefficient that is suppressed by some inverse power of M . As a result, the effective theory is not renormalizable, even if the original theory was so, in the sense that the subtraction of divergences will require the renormalization of all the coefficients parametrizing this infinite number of new local operators. This is however not a problem. On the contrary, it reflects that this theory has by construction a more limited predictive power than the original one, limited to $E < M$. The important observation is that whenever E/M is small, then any arbitrary finite accuracy can be reached by retaining a sufficient but finite number of orders in the expansion in such parameter, and thus a finite number of operators. The theory will then become predictive, to some extent, once the coefficients of this limited set of operators has been fixed from experiments. This has to be contrasted to the situation in the original exact renormalizable theory, where it is possible to predict any observable with infinite accuracy after fitting only a finite set of couplings. The difference is thus that in the effective theory the accuracy at fixed E can only be increased if the number of measured parameters is increased. On the other hand, the theory is much simpler.

The above reasoning implies that doing experiments with limited maximal energy E and finite accuracy, it is only possible to assess a theory describing the involved physics as an effective theory below some energy scale M above E . This is because adding any higher-dimensional operator suppressed by inverse powers of M will produce corrections that are suppressed by some power of E/M , which becomes unobservable if M is sufficiently large. For example, the Standard Model has been established to be indeed the correct description of particle interactions only up to the Electroweak scale, but it may be that the true theory differs from it at higher energies. The fact that the standard model is renormalizable only tells that this is not necessary, and that in principle it could be the exact theory describing particle interactions. Actually, it is believed that even in this extreme hypothesis, the Standard Model should eventually break down at the Planck scale, where gravitational interactions become important also for elementary particles and can no longer be ignored. This is because General relativity is non-renormalizable as a quantum field theory, due to the fact that its coupling is dimensionfull, and is thus forced to be an effective theory with a validity range which is at most the Planck scale.

Summarizing, an effective theory is an approximate theory reproducing the IR behavior of some more fundamental theory below a certain energy scale, with the virtue of being much simpler and the limitation of losing its predictivity in the UV, beyond the defining energy scale. The classical Lagrangian of such a theory consists in general of a finite set of operators with dimension 4, which are called marginal because they are equally relevant

at all the scales, a finite number of operators of dimension smaller than 4, which are called relevant because they become important in the IR, and finally an infinite set of operators of dimension greater than 4, which are called irrelevant because they become unimportant in the IR. If the full theory is known, the coefficients of these operators can be computed. If on the other hand the full theory is not known, one can write down all the possible operators compatible with the assumed symmetries and think of their coefficients as parametrizing our ignorance about the behavior of the theory beyond a certain scale. This effective theory can then also be used at the quantum level, provided some care is used to renormalize it. In fact, one can define the effective theory order by order in \hbar by a matching procedure, consisting in fixing the parameters of the effective theory in such a way that its 1PI amplitudes reproduce the amplitudes of the exact theory that are 1PI with respect to light fields, at some given order in the low-energy expansion in inverse powers of M .

6.2 Matching of parameters in a simple example

Let us now illustrate the matching procedure that must be followed to properly define an effective theory in a simple toy model, involving two scalar fields φ and ϕ with masses m and M and a cubic interaction parametrized by a coupling κ with dimensions of a mass. The Lagrangian of the full theory is then:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 - \frac{1}{2}\kappa\varphi^2\phi. \quad (6.2)$$

Let us now suppose that $\kappa \sim m$ and $M \gg m$. One can then integrate out the heavy fields ϕ and define an effective theory for the light fields φ below the scale M .

At the tree level, i.e. leading order in \hbar , the situation is particularly simple. The only effective 1PI interaction vertex among φ 's that can be induced by exchanging virtual heavy ϕ 's is a quartic vertex, obtained by gluing two cubic vertices with a heavy propagator. At leading order in p_i^2/M^2 , this has no momentum dependence and corresponds thus to an operator without derivatives. The effective theory will then have the form:

$$\mathcal{L}_{\text{eff}}^0 = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}C_0^{(4,0)}\varphi^4 + \dots \quad (6.3)$$

In order to fix the coefficient $C_0^{(4,0)}$, we simply require that the above effective Lagrangian should reproduce the same answer as the full theory for the 1PI 4-point function of φ , at leading order in p_i^2/M^2 . This means that the matching condition is:

$$\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \end{array} \cdot C_0^{(4,0)}. \quad (6.4)$$

For the 3 diagrams in the exact theory, one simply finds $-i\kappa^2/(\Delta - M^2)$, where $\Delta = s, t, u$. The effective vertex in the low-energy theory gives instead $-iC_0^{(4,0)}$. Neglecting terms of order Δ/M^2 , the matching condition is then:

$$3i\frac{\kappa^2}{M^2} = -iC_0^{(4,0)}. \quad (6.5)$$

This fixes:

$$C_0^{(4,0)} = a_0^{(4,0)} \frac{\kappa^2}{M^2}, \quad a_0^{(4,0)} = -3. \quad (6.6)$$

At next-to-leading order in p_i^2/M^2 , 2 powers of momentum are involved and the matching requires the introduction of a new operator with 2 derivatives and 4 fields, with a coefficient of the form $C_0^{(4,2)} = a_0^{(4,2)} \kappa^2/M^4$. Similarly, all the higher-order terms in p_i^2/M^2 are matched to operators involving $2n$ derivatives and 4 fields, with coefficients of the form $C_0^{(4,2n)} = a_0^{(4,2n)} \kappa^2/M^{2n+2}$. The structure of the full tree-level effective Lagrangian can finally be parametrized in terms of numerical coefficients $a_0^{(4,2n)} \sim \mathcal{O}(1)$ in the following way:

$$\mathcal{L}_{\text{eff}}^0 = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{1}{4!} a_0^{(4,0)} \frac{\kappa^2}{M^2} \varphi^4 - \frac{1}{2} a_0^{(4,2)} \frac{\kappa^2}{M^4} \varphi^2 \partial_\mu \varphi \partial^\mu \varphi + \dots \quad (6.7)$$

Note that since $a_0^{(4,0)} < 0$, the potential is not bounded from below. But this is just an accidental feature of this toy model, and it does not affect the general discussion of this section.

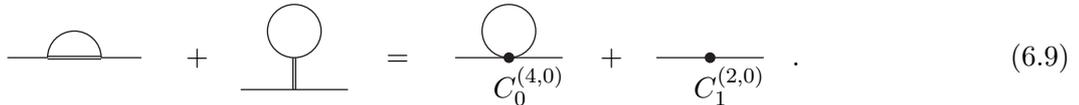
The same result for the tree-level effective Lagrangian can be obtained from the path-integral representation of the effective action, as a result of completing the squares in the quadratic Lagrangian for the heavy fields and redefining a new shifted heavy fields. This amounts indeed to using the equations of motion of the heavy field, using the propagator and expanding it as an infinite series of local differential operators.

At the 1-loop level, i.e. next-to-leading order in \hbar , there are two types of 1PI effective interaction vertices among φ 's that get corrected by the exchange of virtual heavy ϕ 's: quadratic and a quartic vertices. At leading order in p_i^2/M^2 , these have no momentum dependence and correspond thus to operators without derivatives. The effective theory will then have the form:

$$\mathcal{L}_{\text{eff}}^1 = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} \left(m^2 + C_1^{(2,0)} \right) \varphi^2 - \frac{1}{4!} \left(C_0^{(4,0)} + C_1^{(4,0)} \right) \varphi^4 + \dots \quad (6.8)$$

In order to fix the coefficients $C_1^{(2,0)}$ and $C_1^{(4,0)}$, we require that the above effective Lagrangian should reproduce the same answer as the full theory for the 1PI 2- and 4-point functions, at leading order in p_i^2/M^2 .

The matching condition for the 2-point function is encoded in the following equation between diagrams in the exact theory and in the effective theory:



$$\text{[Diagrammatic equation (6.9)]} \quad (6.9)$$

Evaluating the diagrams in the $\overline{\text{MS}}$ scheme at zero-momentum, one obtains:

$$\begin{aligned} & i \frac{\kappa^2}{(4\pi)^2} \left(1 - \frac{M^2}{M^2 - m^2} \log \frac{M^2}{\mu^2} + \frac{m^2}{M^2 - m^2} \log \frac{m^2}{\mu^2} \right) - \frac{i}{2} \frac{\kappa^2}{(4\pi)^2} \frac{m^2}{M^2} \left(1 - \log \frac{m^2}{\mu^2} \right) \\ & = \frac{i}{2} \frac{C_0^{(4,0)}}{(4\pi)^2} m^2 \left(1 - \log \frac{m^2}{\mu^2} \right) - i C_1^{(2,0)}. \end{aligned} \quad (6.10)$$

In order to avoid large logarithms, the renormalization scale must be chosen close to the scale defining the effective theory. Taking then $\mu = M$ and keeping only the leading and next-to-leading order terms in m^2/M^2 , one finds:

$$i \frac{\kappa^2}{(4\pi)^2} \left(1 - \frac{1}{2} \frac{m^2}{M^2} + \frac{3}{2} \frac{m^2}{M^2} \log \frac{m^2}{M^2} \right) = \frac{i}{2} \frac{C_0^{(4,0)}}{(4\pi)^2} m^2 \left(1 - \log \frac{m^2}{M^2} \right) - i C_1^{(2,0)}. \quad (6.11)$$

Notice now that on both sides of this expression there are still terms involving the quantity $\log(m^2/M^2)$, which becomes large when m is much smaller than M . However, using the value $C_0^{(4,0)} = -3\kappa^2/M^2$ determined by the tree-level matching, we see that their form is exactly the same in the exact and the effective theory result appearing in the two sides, and they thus drop out of the matching condition. This is a consequence of the fact that the tree-level matched low-energy effective theory must reproduce the correct IR singularities for $m \rightarrow 0$. Finally, dropping the subleading term in m^2/M^2 , we find:

$$C_1^{(2,0)} = a_1^{(2,0)} \frac{\kappa^2}{(4\pi)^2}, \quad a_1^{(2,0)} = -1. \quad (6.12)$$

Subleading orders in p_i^2/M^2 are matched with corrections to the coefficients of the higher-derivative operators. The correction to the coefficient of the operator involving $2n$ derivatives and 2 fields is found to behave as $C_1^{(2,2n)} = a_1^{(2,2n)} \kappa^2 / [(4\pi)^2 M^{2n}]$.

The matching condition for the 4-point function can be worked out in a similar way. It consists of the following equation between diagrams:

$$= \begin{array}{c} C_0^{(4,0)} C_0^{(4,0)} \\ \text{diagram} \end{array} + \begin{array}{c} C_0^{(4,0)} \\ \text{diagram} \end{array} + \begin{array}{c} C_0^{(4,0)} \\ \text{diagram} \end{array} + \begin{array}{c} C_1^{(4,0)} \\ \text{diagram} \end{array}. \quad (6.13)$$

Choosing as before $\mu = M$, to minimize logarithms, one finds again that the residual logarithms of the type $\log(m^2/M^2)$ are identical in the full theory and the effective theory computations, and drop therefore out of the matching, as a consequence of the value of $C_0^{(4,0)}$ fixed by the tree-level matching. Finally, the matching condition fixes:

$$C_1^{(4,0)} = a_1^{(4,0)} \frac{\kappa^4}{(4\pi)^2 M^4}, \quad a_1^{(4,0)} \sim \mathcal{O}(1). \quad (6.14)$$

Again, subleading orders in p_i^2/M^2 are matched with corrections to the coefficients of the higher-derivative operators. The correction for the operator involving $2n$ derivatives and 4 fields is of the form $C_1^{(4,2n)} = a_1^{(4,2n)} \kappa^4 / [(4\pi)^2 M^{2n+4}]$.

Finally, the effective Lagrangian at the 1-loop order can be parametrized with the numerical coefficients $a_0^{(4,2n)} \sim \mathcal{O}(1)$ representing the leading tree-level corrections plus some numerical coefficients $a_1^{(4,2n)} \sim \mathcal{O}(1)$ parametrizing the 1-loop corrections, and takes

the following form:

$$\begin{aligned}
\mathcal{L}_{\text{eff}}^1 &= \frac{1}{2} \left(1 + a_1^{(2,2)} \frac{\kappa^2}{(4\pi)^2 M^2} \right) \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} \left(m^2 + a_1^{(2,0)} \frac{\kappa^2}{(4\pi)^2} \right) \varphi^2 \\
&\quad - \frac{1}{4!} \left(a_0^{(4,0)} + a_1^{(4,0)} \frac{\kappa^2}{(4\pi)^2 M^2} \right) \frac{\kappa^2}{M^2} \varphi^4 \\
&\quad - \frac{1}{2} \left(a_0^{(4,2)} + a_1^{(4,2)} \frac{\kappa^2}{(4\pi)^2 M^2} \right) \frac{\kappa^2}{M^4} \varphi^2 \partial_\mu \varphi \partial^\mu \varphi + \dots .
\end{aligned} \tag{6.15}$$

It is then clear that in this simple example the dimensionless loop expansion parameter is given by:

$$l = \frac{\kappa^2}{(4\pi)^2 M^2} . \tag{6.16}$$

Note that in this example based on scalar fields, the 1-loop correction to the mass parameter is not necessarily small compared to the tree-level value m^2 . Indeed, since no symmetry is restored when $m^2 \rightarrow 0$, the correction is additive rather than multiplicative. This is different from what happens for fermions, where the correction is multiplicative because for $m \rightarrow 0$ a chiral symmetry is restored. In fact, we see that in the above toy example involving only a super-renormalizable coupling κ , the natural value of the mass is fixed by the 1-loop correction to be close to the scale defined by that coupling, namely $m^2 \sim \kappa^2/(4\pi)^2$. In more general examples involving a light scalar field coupled through a dimensionless coupling λ , for example to heavy fermions of mass M , one would instead find that the natural value of the mass is close to the cut off, namely $m^2 \sim \lambda^2 M^2/(4\pi)^2$.

It should be emphasized that in the above matching at the 1-loop level, we have neglected loop diagrams in the effective theory which are built with higher-derivative vertices. This is because these diagrams give contributions which are further suppressed by the small parameter:

$$\epsilon = \frac{m^2}{M^2} . \tag{6.17}$$

In dimensional regularization, this is manifestly true for each such diagram. Indeed, in such schemes power divergences are absent and automatically replaced by a suitable power of the light mass m , which combines with the inverse power of the large scale M multiplying the higher-derivative operator used as vertex to give some power of the above dimensionless parameter. In regularizations with a momentum cut-off at the scale M , on the other hand, each such diagram gives a sizeable unsuppressed contribution, because power divergences provide positive powers of the cut-off M which cancel the negative powers of M coming with the operator. One would then have to resum all the diagrams of a given type involving vertices with arbitrary high derivatives. Since the physical predictions of the theory should not depend on the choice of renormalization scheme, the resummation of these diagrams should eventually give back a suppression factor. From these considerations, it clearly emerges that mass-independent renormalization schemes are much more convenient to use than mass-dependent schemes in effective theories, because they allow to automatically disentangle power divergences and to manifestly organize the theory as a $1/M$ expansion.

In this scheme, the charged fermion therefore automatically decouples at energies much lower than its mass. Note moreover that $\Pi(k^2, \mu)$ remains small for k and μ much smaller than m , so that this scheme provides indeed a reliable description of the theory below the fermion mass threshold.

In order to avoid solving the differential equation for the running coupling with the exact μ -dependent β -function, one can use a matching procedure to take care of the transition across the threshold. For $\mu \gg m$, the coupling runs with the approximately constant $\beta \simeq 4/3 e^2/(4\pi)^2$, whereas for $\mu \ll m$ it does not run any further since $\beta \simeq 0$:

$$\bar{e}(\mu') \simeq \begin{cases} \bar{e}(\mu) \left(1 - \frac{4}{3} \frac{\bar{e}(\mu)^2}{(4\pi)^2} \log \frac{\mu'^2}{\mu^2}\right)^{-1/2}, & \mu, \mu' \gg m, \\ \bar{e}(\mu) & \mu, \mu' \ll m. \end{cases} \quad (6.25)$$

One can then match these two constant and logarithmically running behaviors around the scale m . If one does that brutally, one finds:

$$\bar{e}(\mu) \simeq \bar{e}(0) \left(1 - \frac{4}{3} \frac{\bar{e}(0)^2}{(4\pi)^2} \theta(\mu - m) \log \frac{\mu^2}{m^2}\right)^{-1/2}. \quad (6.26)$$

Proceeding in this way, however, the approximate asymptotic value of the coupling that is obtained at very low scales relative to its values at high scales suffers a sizable mismatch compared to the true value obtained from the exact running. This is due to the fact that below its mass threshold the decoupling fermion still contributes a small tail to the β -function, which when accumulated gives a finite effect. One can however correct for this mismatch and take into account this effect by adding a suitable threshold correction to the running of the coupling. This threshold correction can be determined by using the renormalized function $\Pi(0, \mu)$ at zero momentum to relate the coupling at very low energies $\bar{e}(0)$ to the coupling $\bar{e}(\mu)$ at energies μ well above the mass threshold but not too much beyond. Indeed, as long as μ is not too big compared to m , $\Pi(0, \mu)$ is small and the running coupling approximately coincides with the renormalized coupling. Comparing the definition of the sliding scale renormalization scheme with the ordinary physical definition of the low-energy coupling, we see that the residue at the photon pole is not 1 but rather $1 - \Pi(0, \mu)$. This implies that the effective coupling is actually rescaled by $(1 - \Pi(0, \mu))^{-1/2}$. One finds then, taking formally $\mu \gg m$:

$$\begin{aligned} \bar{e}(\mu) &\simeq \bar{e}(0) \left(1 - \Pi(0, \mu)\right)^{-1/2} \simeq \bar{e}(0) \left(1 + \frac{1}{2} \Pi(0, \mu)\right) \\ &\simeq \bar{e}(0) + \frac{4 \bar{e}(0)^3}{(4\pi)^2} \int_0^1 dx x(1-x) \log \left[\frac{\mu^2 x(1-x)}{m^2} \right] \\ &\simeq \bar{e}(0) + \frac{2 \bar{e}(0)^3}{3(4\pi)^2} \left(\log \frac{\mu^2}{m^2} - \frac{5}{3} \right). \end{aligned} \quad (6.27)$$

Matching this with the logarithmic running for $\mu' > \mu \gg m$, one finally finds:

$$\bar{e}(\mu) \simeq \bar{e}(0) \left(1 - \frac{4}{3} \frac{\bar{e}(0)^2}{(4\pi)^2} \theta(\mu - m) \left(\log \frac{\mu^2}{m^2} - \frac{5}{3} \right)\right)^{-1/2}. \quad (6.28)$$

This expression is similar to the one derived with the brutal matching, except for a discontinuous shift at $\mu = m$ that accounts for threshold effects.

In the $\overline{\text{MS}}$ scheme, the renormalized $\Pi(k^2, \mu)$ is obtained simply by substituting the cut-off $\tilde{\Lambda}$ with the renormalization scale μ . One finds then:

$$\Pi(k^2, \mu) = -\frac{8e^2}{(4\pi)^2} \int_0^1 dx x(1-x) \log \left[\frac{m^2 - k^2 x(1-x)}{\mu^2} \right]. \quad (6.29)$$

This gives now a constant β function:

$$\beta = \frac{4}{3} \frac{e^3}{(4\pi)^2}. \quad (6.30)$$

In this scheme, the charged fermion does therefore not seem to automatically decouple at energies much lower than its mass. However, we see that $\Pi(k^2, \mu)$ grows large for k and μ much smaller than m , due to a large logarithm of the type $\log(m^2/\mu^2)$. This signals that such a scheme is actually not reliable to describe the theory below the fermion mass threshold. One is then forced to switch to an effective theory below the mass scale m , where the fermion is integrated out. In such an effective theory, the β function vanishes. Matching the full theory and the effective theory at the scale m , one finds then the same situation as in the sliding scale scheme. One can at this point incorporate by hand the threshold correction computed in that scheme to improve the brutal matching.

In theories with several well separated mass scales m_i , one can generalize this matching procedure and define a succession of effective theories with gradually less degrees of freedom. At each threshold m_i , the corresponding field is integrated out, and the effective theory above the threshold is matched to the new simpler effective theory below the threshold. In order to get a quantitatively accurate result for the running of dimensionless couplings, at each threshold one has to include the finite threshold correction accounting for the residual effects of the decoupling state.

6.4 Effective action for scalar electrodynamics

As a nice example where the effective field theory concept can be very efficiently used, let us now consider the theory of a spin-0 point-particle with charge e and mass m in an electromagnetic field. The Lagrangian is given by:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - m^2 |\phi|^2, \quad (6.31)$$

where:

$$D_\mu = \partial_\mu - ieA_\mu. \quad (6.32)$$

The effective action at energies much smaller than the mass m can be obtained by integrating out the charged particles and computing the resulting effective action for the gauge fields. Due to the nature of the interaction vertices, there is no tree-level effect and

the leading new operators arise only at the 1-loop level. The relevant kinetic operator for the scalar field fluctuations is given by:

$$-\frac{\delta^2 S}{\delta\phi\delta\phi^*} = D^2 + m^2. \quad (6.33)$$

The 1-loop correction to the effective action, normalized in such a way that for vanishing fields it vanishes and defined after analytic continuation to Euclidean space, is given by:

$$\begin{aligned} \Gamma^1(A) &= -\text{tr} \log \frac{-D_E^2 + m^2}{-\square_E + m^2} + \Delta S^1 \\ &= \int_0^\infty \frac{d\tau}{\tau} \left(\text{tr} [e^{-\tau(-D_E^2 + m^2)}] - \text{tr} [e^{-\tau(-\square_E + m^2)}] \right) + \Delta S^1. \end{aligned} \quad (6.34)$$

where

$$D_{E\mu} = \partial_{E\mu} - ieA_{E\mu}. \quad (6.35)$$

The basic object to compute is now the first-quantized partition function:

$$Z(\tau, A_E) = \text{tr} [e^{-\tau\hat{H}}], \quad (6.36)$$

for a point-particle in four-dimensional Euclidean space with Hamiltonian given by:

$$\hat{H} = -D_E^2 + m^2. \quad (6.37)$$

Recall now that the Hamiltonian for a non-relativistic point-particle of mass \hat{m} and charge e interacting with an ordinary scalar potential \hat{V} and an electromagnetic vector potential \hat{A} is given by $\hat{H} = -D_E^2/(2\hat{m}) + \hat{V}$. Comparing with the above expression, we can identify $\hat{m} = 1/2$, $\hat{V} = m^2$ and $\hat{A} = A_E(q_E)$. The corresponding Lagrangian is then given by $\hat{L} = (\hat{m}/2)\dot{q}_E^2 - \hat{V} - e\dot{q}_E \cdot \hat{A}$, that is:

$$\hat{L} = \frac{1}{4}\dot{q}_E^2 - m^2 - e\dot{q}_E \cdot A(q_E). \quad (6.38)$$

Using this expression, and analytically continuing to Euclidean time $t \rightarrow -i\tau$, we finally arrive to the following path-integral representation of the partition function:

$$Z(\tau, A_E) = \int_{\text{p.b.c}} \mathcal{D}q_E \exp \left\{ - \int_0^\tau d\tau' \hat{L}_E[q_E(\tau')] \right\}. \quad (6.39)$$

where:

$$\hat{L}_E = \frac{1}{4}\dot{q}_E^2 + m^2 - ie\dot{q}_E \cdot A_E(q_E). \quad (6.40)$$

For generic electromagnetic backgrounds, the Lagrangian is not quadratic in q_E and one can at best evaluate the path-integral perturbatively in powers of the couplings. However, for slowly varying electromagnetic fields $F_{\mu\nu}$ one can consider a derivative expansion and try to compute the leading order dependence on $F_{\mu\nu}$, neglecting derivatives of it. In

this approximation of constant field strength, one can choose a gauge where the electromagnetic potential is given simply by $A_\mu(x) = F_{\mu\nu}x^\nu/2$. In Euclidean space, this gives:

$$A_E^\mu(q_E) = \frac{1}{2}F_E^{\mu\nu}q_E^\nu, \quad (6.41)$$

where:

$$F_E^{\mu\nu} = \begin{pmatrix} 0 & E_E^x & E_E^y & E_E^z \\ -E_E^x & 0 & B_E^z & -B_E^y \\ -E_E^y & -B_E^z & 0 & B_E^x \\ -E_E^z & B_E^y & -B_E^x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -iE^x & -iE^y & -iE^z \\ iE^x & 0 & B^z & -B^y \\ iE^y & -B^z & 0 & B^x \\ iE^z & B^y & -B^x & 0 \end{pmatrix}. \quad (6.42)$$

The Lagrangian defining the partition function becomes then quadratic:

$$\hat{L}_E = \frac{1}{4}\dot{q}_E^\mu\dot{q}_E^\mu + m^2 - \frac{ie}{2}F_E^{\mu\nu}\dot{q}_E^\mu q_E^\nu. \quad (6.43)$$

This means that the path-integral is now Gaussian and can therefore be evaluated in an exact way. To do so, we expand the integration field in a complete basis of periodic functions in the interval $[0, \tau]$:

$$q_E^\mu(\tau') = q_0^\mu + \sum_{n=1}^{\infty} \left(q_n^\mu \phi_n(\tau') + q_n^{*\mu} \phi_n^*(\tau') \right). \quad (6.44)$$

The functions ϕ_n are orthonormal plane-waves with discretized frequencies:

$$\phi_n(\tau') = \frac{1}{\sqrt{\tau}} e^{i\omega_n \tau'}, \quad \omega_n = \frac{2\pi n}{\tau}. \quad (6.45)$$

Plugging this decomposition into the Euclidean action, and performing the time integration with the help of the orthonormality condition, one finds:

$$\begin{aligned} \hat{S}_E &= m^2\tau + \sum_{n=1}^{\infty} \left[\frac{1}{2}\omega_n^2 q_n^{*\mu} q_n^\mu - e\omega_n F_E^{\mu\nu} q_n^{*\mu} q_n^\nu \right] \\ &= m^2\tau + \frac{1}{2} \sum_{n=1}^{\infty} \omega_n^2 q_n^{*\mu} M_n^{\mu\nu}(F) q_n^\nu, \end{aligned} \quad (6.46)$$

where

$$\begin{aligned} M_n^{\mu\nu}(F) &= \delta^{\mu\nu} - \frac{2e}{\omega_n} F_E^{\mu\nu} \\ &= \begin{pmatrix} 1 & i\frac{2e}{\omega_n}E^x & i\frac{2e}{\omega_n}E^y & i\frac{2e}{\omega_n}E^z \\ -i\frac{2e}{\omega_n}E^x & 1 & -\frac{2e}{\omega_n}B^z & \frac{2e}{\omega_n}B^y \\ -i\frac{2e}{\omega_n}E^y & \frac{2e}{\omega_n}B^z & 1 & -\frac{2e}{\omega_n}B^x \\ -i\frac{2e}{\omega_n}E^z & -\frac{2e}{\omega_n}B^y & \frac{2e}{\omega_n}B^x & 1 \end{pmatrix}. \end{aligned} \quad (6.47)$$

The path-integral representation for the partition function gives then:

$$\begin{aligned}
Z(\tau, F) &= \int_{\text{p.b.c.}} \mathcal{D}q_E \exp \left\{ -S_E[q_E] \right\} \\
&= \mathcal{N} \int dq_0^\mu e^{-m^2\tau} \prod_{n=1}^{\infty} \int dq_n^\mu \int dq_n^{*\mu} \exp \left\{ -\frac{1}{2} \omega_n^2 q_n^{*\mu} M_n^{\mu\nu}(F) q_n^\mu \right\} \\
&= \mathcal{N}' (TV) e^{-m^2\tau} \prod_{n=1}^{\infty} \det^{-1} \left[M_n^{\mu\nu}(F) \right].
\end{aligned} \tag{6.48}$$

The normalization factor can now be fixed by requiring that for vanishing electromagnetic fields, $F = 0$, one should obtain the result for a free particle. Keeping an arbitrary dimension $d = 4 - 2\epsilon$ for later use, this is:

$$Z(\tau, 0) = \frac{TV}{(4\pi\tau)^{d/2}} e^{-m^2\tau}. \tag{6.49}$$

Since $M_n^{\mu\nu}(0) = \delta^{\mu\nu}$, one has $\det M_n^{\mu\nu}(0) = 1$, and therefore the properly normalized result for $F \neq 0$ is simply:

$$Z(\tau, F) = \frac{TV}{(4\pi\tau)^{d/2}} e^{-m^2\tau} \prod_{n=1}^{\infty} \det^{-1} M_n^{\mu\nu}(F). \tag{6.50}$$

Now, starting from the definition of the 4 by 4 matrix $M_n^{\mu\nu}(F)$, one computes:

$$\det \left[M_n^{\mu\nu}(F) \right] = 1 - \left(\frac{2e}{\omega_n} \right)^2 (\vec{E}^2 - \vec{B}^2) - \left(\frac{2e}{\omega_n} \right)^4 (\vec{E} \cdot \vec{B})^2 \tag{6.51}$$

This result depends, as it should, only on the two possible combinations of electric and magnetic fields that are Lorentz invariant, namely:

$$\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} (\vec{E}^2 - \vec{B}^2), \tag{6.52}$$

$$\mathcal{G} = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = -\vec{E} \cdot \vec{B}. \tag{6.53}$$

One can thus finally rewrite the determinant as:

$$\begin{aligned}
\det M_n^{\mu\nu}(F) &= 1 + 2\mathcal{F} \left(\frac{2e}{\omega_n} \right)^2 - \mathcal{G}^2 \left(\frac{2e}{\omega_n} \right)^4 \\
&= \left[1 + \left(\frac{2ea_+}{\omega_n} \right)^2 \right] \left[1 + \left(\frac{2ea_-}{\omega_n} \right)^2 \right].
\end{aligned} \tag{6.54}$$

where the quantities a_+ and a_- are defined by:

$$a_{\pm} = \sqrt{\mathcal{F} \pm \sqrt{\mathcal{F}^2 + \mathcal{G}^2}}. \tag{6.55}$$

Finally, the infinite product over modes can be evaluated by using the infinite product representation of the sinh function:

$$\prod_{n=1}^{\infty} \left[1 + \left(\frac{2ea_{\pm}}{\omega_n} \right)^2 \right] = \prod_{n=1}^{\infty} \left[1 + \frac{(e\tau a_{\pm})^2}{\pi^2 n^2} \right] = \frac{\sinh(e\tau a_{\pm})}{e\tau a_{\pm}}. \tag{6.56}$$

The partition function for $F \neq 0$ is then given by the following compact expression:

$$Z(\tau, F) = \frac{TV}{(4\pi\tau)^{d/2}} e^{-m^2\tau} \frac{e\tau a_+}{\sinh(e\tau a_+)} \frac{e\tau a_-}{\sinh(e\tau a_-)}. \quad (6.57)$$

Putting everything together, and adding an appropriate power of an arbitrary scale $\tilde{\mu}$ to restore ordinary dimensions, the 1-loop correction to the effective action in the approximation of slowly varying fields is given by:

$$\begin{aligned} \Gamma^1[F] &= \int_0^\infty \frac{d\tau}{\tau} \left(Z(\tau, F) - Z(\tau, 0) \right) + \Delta S^1 \\ &= \frac{TV \tilde{\mu}^{d-4}}{(4\pi)^{d/2}} \int_0^\infty \frac{d\tau}{\tau^{1+d/2}} e^{-m^2\tau} \left(\frac{e\tau a_+}{\sinh(e\tau a_+)} \frac{e\tau a_-}{\sinh(e\tau a_-)} - 1 \right) + \Delta S^1. \end{aligned} \quad (6.58)$$

This expression contains a divergent correction to the free quadratic Lagrangian of electromagnetism, which is interpreted as the wave-function renormalization, plus an infinite series of finite higher-order corrections, inducing non-linearities in the effective equations of motion. These terms can be obtained by expanding the above expression in powers of e as:

$$\frac{e\tau a_+}{\sinh(e\tau a_+)} \frac{e\tau a_-}{\sinh(e\tau a_-)} - 1 = \sum_{k=1}^{\infty} (e\tau)^{2k} f_k(F). \quad (6.59)$$

The first few coefficients are:

$$f_1(F) = -\frac{1}{3}\mathcal{F} = -\frac{1}{12}F_{\mu\nu}F^{\mu\nu}, \quad (6.60)$$

$$f_2(F) = \frac{1}{90} \left[7\mathcal{F}^2 + \mathcal{G}^2 \right] = \frac{1}{1440} \left[7(F_{\mu\nu}F^{\mu\nu})^2 + (F_{\mu\nu}\tilde{F}^{\mu\nu})^2 \right], \quad (6.61)$$

$$\dots \quad (6.62)$$

Using this expansion, rewriting the factor TV as an integral over space-time and performing the τ integral explicitly for each term, one finds then:

$$\Gamma^1[F] = \int d^4x \left(\sum_{k=1}^{\infty} c_k e^{2k} f_k(F) + \Delta\mathcal{L}^1 \right), \quad (6.63)$$

where:

$$\begin{aligned} c_k &= \frac{\tilde{\mu}^{d-4}}{(4\pi)^{d/2}} \int_0^\infty d\tau \tau^{2k-1-d/2} e^{-m^2\tau} \\ &= \frac{\Gamma(2k-d/2)}{(4\pi)^{d/2}} \tilde{\mu}^{d-4} (m^2)^{d/2-2k}. \end{aligned} \quad (6.64)$$

The integral is divergent only for the first term $k = 1$, which has the same structure as the tree-level Lagrangian. Using $d = 4 - 2\epsilon$, sending $\epsilon \rightarrow 0$, and defining as usual $\tilde{\Lambda} = \sqrt{4\pi} e^{-\gamma/2} e^{1/(2\epsilon)} \tilde{\mu}$, one finds:

$$c_1 = \frac{1}{(4\pi)^2} \log \frac{\tilde{\Lambda}^2}{m^2}, \quad (6.65)$$

$$c_k = \frac{\Gamma(2(k-1))}{(4\pi)^2} m^{-4(k-1)}, \quad k = 2, 3, \dots \quad (6.66)$$

Finally, adding up the tree-level and 1-loop contributions, one finds the following result for the effective action:

$$\Gamma[F] = \int d^4x \left[-\frac{1}{4} \left(1 + \frac{e^2}{3(4\pi)^2} \log \frac{\tilde{\Lambda}^2}{m^2} + \Delta_Z \right) F_{\mu\nu} F^{\mu\nu} + \frac{e^4}{1440(4\pi)^2 m^4} \left[7(F_{\mu\nu} F^{\mu\nu})^2 + (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right] + \dots \right]. \quad (6.67)$$

Imposing the renormalization condition that the correction to the 1PI 2-point function vanishes at a scale μ fixes the counter-term to be of the form:

$$\Delta_Z = -\frac{e^2}{3(4\pi)^2} \log \frac{\mu^2}{m^2}. \quad (6.68)$$

This leaves:

$$\Gamma[F] = \int d^4x \left[-\frac{1}{4} \left(1 + \frac{e^2}{3(4\pi)^2} \log \frac{\mu^2}{m^2} \right) F_{\mu\nu} F^{\mu\nu} + \frac{e^4}{1440(4\pi)^2 m^4} \left[7(F_{\mu\nu} F^{\mu\nu})^2 + (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right] + \dots \right]. \quad (6.69)$$

As expected, the effective action takes the form of a series in inverse powers of m . More precisely, besides the operator F^2 of dimension 4, which is relevant at all energies, there is an infinite series of operators of the form F^{2+2n} with dimension $4 + 4n$ and coefficient m^{-4n} , which are less and less relevant at energies below m . The dimensionless expansion parameter is therefore $(eF)/m^2$.

Note finally that by rescaling the gauge fields by the gauge coupling e , one can also write the above result for the effective action in the following form:

$$\Gamma[F] = \int d^4x \left[-\frac{1}{4} \bar{e}^{-2}(m) F_{\mu\nu} F^{\mu\nu} + \frac{1}{1440(4\pi)^2 m^4} \left[7(F_{\mu\nu} F^{\mu\nu})^2 + (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right] + \dots \right], \quad (6.70)$$

in terms of an effective coupling at the scale m given by:

$$\bar{e}^{-2}(m) = e^{-2} - \frac{1}{3(4\pi)^2} \log \frac{m^2}{\mu^2}. \quad (6.71)$$

Comparing with the general form of the running coupling in a gauge theory, we can deduce that the β function for this Abelian theory with a charged scalar field is given by:

$$\beta(e) = \frac{e^3}{3(4\pi)^2}. \quad (6.72)$$

6.5 Effective action for spinor electrodynamics

The computation of the previous section can be easily generalized to the more relevant case of a spin-1/2 point-particle with charge e and mass m in an electromagnetic field. The Lagrangian describing the theory is given by:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i\not{D} \pm m) \psi, \quad (6.73)$$

where as before:

$$D_\mu = \partial_\mu - ieA_\mu. \quad (6.74)$$

The relevant kinetic operator for the spinor field fluctuations, which enters the determinants of the effective action, is given by:

$$-\frac{\delta^2 S}{\delta\psi \delta\bar{\psi}} = -i\mathcal{D} \mp m. \quad (6.75)$$

The sign of the mass is irrelevant. Indeed, as a consequence of the properties of γ -matrices one has:

$$\text{tr} \log(-i\mathcal{D} + m) = \text{tr} \log(i\mathcal{D} + m) = \frac{1}{2} \text{tr} \log(\mathcal{D}^2 + m^2). \quad (6.76)$$

Moreover, one easily verifies that:

$$\mathcal{D}^2 = D^2 - \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu}. \quad (6.77)$$

The 1-loop correction to the effective action, normalized in such a way that for vanishing fields it vanishes and defined after analytic continuation to Euclidean space, is given by:

$$\begin{aligned} \Gamma^1(A) &= \frac{1}{2} \text{tr} \log \frac{-D_E^2 - e/2 \sigma_E \cdot F_E + m^2}{-\square_E + m^2} + \Delta S^1 \\ &= -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \left(\text{tr} [e^{-\tau(-D_E^2 - e/2 \sigma_E \cdot F_E + m^2)}] - \text{tr} [e^{-\tau(-\square_E + m^2)}] \right) + \Delta S^1, \end{aligned} \quad (6.78)$$

where

$$D_{E\mu} = \partial_{E\mu} - ieA_{E\mu}. \quad (6.79)$$

The basic object to compute is in this case the first-quantized partition function

$$Z(\tau, A_E) = \text{tr} [e^{-\tau \hat{H}}], \quad (6.80)$$

for a point-particle in four-dimensional Euclidean space with Hamiltonian given by:

$$\hat{H} = -D_E^2 - \frac{e}{2} \sigma_E \cdot F_E + m^2, \quad (6.81)$$

and Lagrangian:

$$\hat{L} = \frac{1}{4} \dot{q}_E^2 - m^2 - e \dot{q}_E \cdot A(q_E) + \frac{e}{2} \sigma_E \cdot F_E(q_E). \quad (6.82)$$

Going to Euclidean space one finally obtains the following Lagrangian:

$$\hat{L}_E = \frac{1}{4} \dot{q}_E^2 + m^2 - ie \dot{q}_E \cdot A_E(q_E) + \frac{e}{2} \sigma_E \cdot F_E(q_E). \quad (6.83)$$

In this case, the trace defining the partition function factorizes into a trace over spinor indices of the exponential of the spin-interaction times a trace over modes that is identical

to the one arising for a spin-0 particle. Since $\sigma_E \cdot F_E = \sigma \cdot F$, the result for constant fields is:

$$Z(\tau, F) = \frac{TV}{(4\pi\tau)^{d/2}} e^{-m^2\tau} \text{tr} [e^{e\tau/2\sigma \cdot F}] \frac{e\tau a_+}{\sinh(e\tau a_+)} \frac{e\tau a_-}{\sinh(e\tau a_-)}. \quad (6.84)$$

The trace over spinor indices encodes the effect due to the interactions between the spin of the charged particle and the electromagnetic field. It can be computed by recalling that:

$$\{\sigma^{\mu\nu}, \sigma^{\rho\sigma}\} = 2(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - 2i\gamma_5\epsilon^{\mu\nu\rho\sigma}. \quad (6.85)$$

It follows that:

$$\begin{aligned} (\sigma \cdot F)^2 &= \frac{1}{2} \{\sigma_{\mu\nu}, \sigma_{\rho\sigma}\} F^{\mu\nu} F^{\rho\sigma} \\ &= 2F_{\mu\nu} F^{\mu\nu} + 2iF_{\mu\nu} \tilde{F}^{\mu\nu} \\ &= 8(\mathcal{F} + i\gamma_5\mathcal{G}). \end{aligned} \quad (6.86)$$

Using the now the definitions of a_{\pm} , one has:

$$\mathcal{F} = \frac{1}{2}(a_+^2 + a_-^2), \quad \mathcal{G} = -i a_+ a_-. \quad (6.87)$$

It follows that:

$$(\sigma \cdot F)^2 = 4[(a_+ + a_-)^2 P_+ + (a_+ - a_-)^2 P_-]. \quad (6.88)$$

in terms of the chiral projectors

$$P_{\pm} = \frac{1}{2}(1 \pm \gamma_5). \quad (6.89)$$

Finally, this means that:

$$(\sigma \cdot F)^{2n} = 2^{2n} [(a_+ + a_-)^{2n} P_+ + (a_+ - a_-)^{2n} P_-], \quad (6.90)$$

$$(\sigma \cdot F)^{2n+1} = 2^{2n} F_{\mu\nu} [(a_+ + a_-)^{2n} \sigma^{\mu\nu} P_+ + (a_+ - a_-)^{2n} \sigma^{\mu\nu} P_-]. \quad (6.91)$$

Recalling then that $\text{tr}[P_{\pm}] = 2$ and $\text{tr}[\sigma^{\mu\nu} P_{\pm}] = 0$, one finds:

$$\begin{aligned} \text{tr}[e^{e\tau/2\sigma \cdot F}] &= 2 \sum_{n=0}^{\infty} \frac{1}{(2n)!} (e\tau)^{2n} [(a_+ + a_-)^{2n} + (a_+ - a_-)^{2n}] \\ &= 2 \cosh[e\tau(a_+ + a_-)] + 2 \cosh[e\tau(a_+ - a_-)] \\ &= 4 \cosh(e\tau a_+) \cosh(e\tau a_-). \end{aligned} \quad (6.92)$$

This gives then the following result for the partition function:

$$Z(\tau, F) = 4 \frac{TV}{(4\pi\tau)^{d/2}} e^{-m^2\tau} \frac{e\tau a_+}{\tanh(e\tau a_+)} \frac{e\tau a_-}{\tanh(e\tau a_-)}. \quad (6.93)$$

The 1-loop correction to the effective action in the approximation of slowly varying fields is now given by:

$$\begin{aligned}\Gamma^1[F] &= -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \left(Z(\tau, F) - Z(\tau, 0) \right) + \Delta S^1 \\ &= -2 \frac{TV \tilde{\mu}^{d-4}}{(4\pi)^{d/2}} \int_0^\infty \frac{d\tau}{\tau^{1+d/2}} e^{-m^2\tau} \left(\frac{e\tau a_+}{\tanh(e\tau a_+)} \frac{e\tau a_-}{\tanh(e\tau a_-)} - 1 \right) + \Delta S^1.\end{aligned}\quad (6.94)$$

As before, this expression contains a divergent correction to the free quadratic Lagrangian plus an infinite series of finite higher-order corrections, which can be obtained by expanding the above expression in powers of e as:

$$-2 \left(\frac{e\tau a_+}{\tanh(e\tau a_+)} \frac{e\tau a_-}{\tanh(e\tau a_-)} - 1 \right) = \sum_{k=1}^{\infty} (e\tau)^{2k} f_k(F). \quad (6.95)$$

The first few coefficients are:

$$f_1(F) = -\frac{4}{3} \mathcal{F} = -\frac{1}{3} F_{\mu\nu} F^{\mu\nu}, \quad (6.96)$$

$$f_2(F) = \frac{2}{45} \left[4 \mathcal{F}^2 + 7 \mathcal{G}^2 \right] = \frac{1}{360} \left[4 (F_{\mu\nu} F^{\mu\nu})^2 + 7 (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right], \quad (6.97)$$

$$\dots \quad (6.98)$$

Using this expansion, rewriting the factor TV as an integral over space-time and performing the τ integral explicitly for each term, one finds then:

$$\Gamma^1[F] = \int d^4x \left(\sum_{k=1}^{\infty} c_k e^{2k} f_k(F) + \Delta \mathcal{L}^1 \right), \quad (6.99)$$

where the coefficients c_k have exactly the same values as for scalars:

$$c_1 = \frac{1}{(4\pi)^2} \log \frac{\tilde{\Lambda}^2}{m^2}, \quad (6.100)$$

$$c_k = \frac{\Gamma(2(k-1))}{(4\pi)^2} m^{-4(k-1)}, \quad k = 2, 3, \dots \quad (6.101)$$

Finally, adding up the tree-level and 1-loop contributions, one finds the following result for the effective action:

$$\begin{aligned}\Gamma[F] &= \int d^4x \left[-\frac{1}{4} \left(1 + \frac{4e^2}{3(4\pi)^2} \log \frac{\tilde{\Lambda}^2}{m^2} + \Delta_Z \right) F_{\mu\nu} F^{\mu\nu} \right. \\ &\quad \left. + \frac{e^4}{360(4\pi)^2 m^4} \left[4 (F_{\mu\nu} F^{\mu\nu})^2 + 7 (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right] + \dots \right].\end{aligned}\quad (6.102)$$

Imposing the renormalization condition that the correction to the 1PI 2-point function vanishes at a scale μ fixes the counter-term to be of the form:

$$\Delta_Z = -\frac{4e^2}{3(4\pi)^2} \log \frac{\mu^2}{m^2}. \quad (6.103)$$

This leaves:

$$\Gamma[F] = \int d^4x \left[-\frac{1}{4} \left(1 + \frac{4e^2}{3(4\pi)^2} \log \frac{\mu^2}{m^2} \right) F_{\mu\nu} F^{\mu\nu} + \frac{e^4}{360(4\pi)^2 m^4} \left[4(F_{\mu\nu} F^{\mu\nu})^2 + 7(F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right] + \dots \right]. \quad (6.104)$$

As expected, the effective action takes again the form of a series in inverse powers of m , as in the case of scalar particles but with different numerical coefficients, and the dimensionless expansion parameter is as before $(eF)/m^2$.

Note finally that by rescaling the gauge fields by gauge coupling e , one can also write the above result for the effective action in the following form:

$$\Gamma[F] = \int d^4x \left[-\frac{1}{4} \bar{e}^{-2}(m) F_{\mu\nu} F^{\mu\nu} + \frac{1}{360(4\pi)^2 m^4} \left[4(F_{\mu\nu} F^{\mu\nu})^2 + 7(F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right] + \dots \right], \quad (6.105)$$

in terms of an effective coupling at the scale m given by:

$$\bar{e}^{-2}(m) = e^{-2} - \frac{4}{3(4\pi)^2} \log \frac{m^2}{\mu^2}. \quad (6.106)$$

Comparing with the general form of the running coupling in a gauge theory, we can deduce that the β function for this Abelian theory with a charged fermion field is given by:

$$\beta(e) = \frac{4e^3}{3(4\pi)^2}. \quad (6.107)$$

6.6 Pair-production by constant electromagnetic fields

The parameters a_+ and a_- on which the 1-loop correction to the quantum effective action for scalar and spinor electrodynamics depends are in general complex. Indeed, their explicit expressions in terms of the electric and magnetic fields \vec{E} and \vec{B} are given by:

$$a_{\pm} = \frac{1}{\sqrt{2}} \sqrt{\pm \sqrt{(\vec{E}^2 - \vec{B}^2)^2 + 4(\vec{E} \cdot \vec{B})^2} - (\vec{E}^2 - \vec{B}^2)}. \quad (6.108)$$

We see that a_+ is always real, but a_- can become imaginary. As a result, Γ has not only a real dispersive part, but also an imaginary absorptive part. Since the real part of Γ is identified with the vacuum energy times the total time, its imaginary part can be identified with the decay width of the vacuum. More precisely:

$$\Gamma = -T \left(E_{\Omega} - i \frac{\Gamma_{\Omega}}{2} \right). \quad (6.109)$$

The probability per unit time and volume that the vacuum decays is then given by $P = \Gamma_{\Omega}/V$, that is:

$$P = 2 \frac{\text{Im}\Gamma}{TV}. \quad (6.110)$$

This effect is interpreted as due to the possibility for the electromagnetic field to decay through the creation of pairs of charged particle/antiparticle pairs out of the vacuum.

In the simplest situation where $\vec{E} \perp \vec{B}$, a_- becomes imaginary when $|\vec{E}| > |\vec{B}|$. Let us then focus on the simplest situation where $E \neq 0$ and $B = 0$, as the prototype situation where the phenomenon of pair creation can occur. In this situation, we have simply:

$$a_+ = 0, \quad a_- = iE. \quad (6.111)$$

The effective action is then given by:

$$\Gamma[F] = \frac{TV}{(4\pi)^2} \int_0^\infty \frac{d\tau}{\tau^3} e^{-m^2\tau} \left[n_B \left(\frac{e\tau E}{\sin(e\tau E)} - 1 \right) - 2n_F \left(\frac{e\tau E}{\tan(e\tau E)} - 1 \right) \right] + S + \Delta S. \quad (6.112)$$

We see that the integrand has now poles for $\tau = n\pi/(eE)$. The way to treat this poles is dictated by the $i\epsilon$ prescription on the free propagator of the charged particles, which amounts to take:

$$m^2 \rightarrow m^2 - i\epsilon. \quad (6.113)$$

This is equivalent to saying that the integration contour in τ should be slightly deformed above the real axis, with an infinitesimal constant and positive imaginary part. This implies that the integral in τ will consist in a real contribution coming from the principal part of the integral plus an imaginary part coming from the half residues at the poles. More precisely, one has:

$$\frac{e\tau E}{\sin(e\tau E)} = \text{PP} \frac{e\tau E}{\sin(e\tau E)} + i\pi \sum_{n=1}^{\infty} (-1)^n \delta\left(\tau - \frac{n\pi}{|eE|}\right), \quad (6.114)$$

$$\frac{e\tau E}{\tan(e\tau E)} = \text{PP} \frac{e\tau E}{\tan(e\tau E)} + i\pi \sum_{n=1}^{\infty} \delta\left(\tau - \frac{n\pi}{|eE|}\right). \quad (6.115)$$

It follows that:

$$P = 2 \frac{\text{Im} \Gamma[F]}{TV} = \frac{(eE)^2}{8\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^2} \left((-1)^{n+1} n_B + 2n_F \right) e^{-n\pi m^2/|eE|}. \quad (6.116)$$

For typical particle masses and realizable electric fields, the exponent is tiny and the terms with $n > 1$ can certainly be neglected. One finds then:

$$P \simeq \frac{(eE)^2}{8\pi^4} \left(n_B + 2n_F \right) e^{-\pi m^2/|eE|}. \quad (6.117)$$

The above effect is a non-perturbative effect. Indeed, in the limit in which the coupling e is small, the decay rate has an essential singularity and vanishes faster than any power of e . This means that this effect cannot be realized at any finite order in perturbation theory. This fact is actually easy to understand in a physical way. Imagine for this to compute the amplitude for n photons will equal frequency ω to produce a pair of charged particle

and antiparticle. This is proportional to e^n , and has a kinematical threshold given by the condition $n\omega \geq 2m$. Now, in order to mimic a constant background electromagnetic field, representing a coherent state of very soft photons, we have to take the frequency of these photons to zero and their number to infinity: $\omega \rightarrow 0$, $n \rightarrow \infty$. This shows that the effect is infinitely suppressed in perturbation theory.

The fact that the probability has the form of an exponential suggests that it may have a semiclassical interpretation in terms of tunneling through a potential barrier. This is indeed the case, and one can understand at least qualitatively the result by thinking of the particle/antiparticle pair as a system that is confined at any given point hidden in the vacuum by a very localized potential barrier of height $2m$. Applying an electric field, one creates however a new contribution to the potential that decreases linearly with the distance, whose slope is $-|eE|$. The creation of the pair out of the vacuum corresponds then essentially to the process for a particle of mass $M \sim m$ through a barrier of height $V \sim 2m$ and size $\Delta \sim 2m/|eE|$. The semiclassical probability for this tunneling is given by the formula:

$$\log P \simeq -2 \int_{x_1}^{x_2} dx \sqrt{2MV(x)}. \quad (6.118)$$

The order of magnitude of the integral can now be estimated even without knowing the precise form of the barrier, which affects only the $\mathcal{O}(1)$ normalization, and gives a result in qualitative agreement with the precise quantum field theory computation:

$$\log P \sim -\Delta \sqrt{MV} \sim -\frac{m^2}{|eE|}. \quad (6.119)$$

Unfortunately, the above effect is too small to be observable, even for the lightest charged particles and the strongest electric fields that can presently be realized experimentally.

7 Symmetries and anomalies

7.1 Classical symmetries and Nöther's theorem

Consider first a classical field theory involving some set of fields collectively denoted by ϕ . In the canonical formulation, the dynamics is specified by an action functional depending on the fields ϕ and their first derivatives:

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi). \quad (7.1)$$

The equations of motion are then obtained as the second order Euler-Lagrange equations following from the least action principle:

$$\frac{\delta S}{\delta \phi} \equiv \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0. \quad (7.2)$$

The dynamics can also be rewritten in Hamiltonian form, by performing a Legendre transform with respect to the canonical momentum $\pi = \delta S / \delta \partial_0 \phi$. The Hamiltonian functional has the form:

$$H = \int d^3\vec{x} \mathcal{H}(\phi, \partial_i \phi, \pi). \quad (7.3)$$

The equations of motions are then decomposed into the system of coupled first order Hamilton equations:

$$\dot{\phi} = \frac{\delta H}{\delta \pi} \equiv \frac{\partial \mathcal{H}}{\partial \pi}, \quad \dot{\pi} = -\frac{\delta H}{\delta \phi} \equiv -\frac{\partial \mathcal{H}}{\partial \phi} + \partial_i \frac{\partial \mathcal{H}}{\partial \partial_i \phi}, \quad (7.4)$$

In terms of Poisson brackets, these equations become

$$\dot{\phi} = \{\phi, H\}_P, \quad \dot{\pi} = \{\pi, H\}_P. \quad (7.5)$$

For a generic functional F on phase space, one finds then:

$$\dot{F} = \{F, H\}_P. \quad (7.6)$$

Conservation laws take the form of Lorentz-invariant continuity equations for a current J^μ , which is then said to be conserved:

$$\partial_\mu J^\mu = 0. \quad (7.7)$$

The integral of the time component of such a current defines a conserved charge:

$$Q = \int d^3\vec{x} J^0. \quad (7.8)$$

Assuming that Q does not depend explicitly on time, one has then $\dot{Q} = 0$ and thus:

$$\{Q, H\}_P = 0. \quad (7.9)$$

The crucial general result about conserved quantities is Nöther's theorem. This states that conserved quantities are associated to continuous global symmetries of the action S . Let us briefly review how this comes about. For simplicity, let us focus on internal symmetries, which do not affect the space-time coordinates. We consider then an infinitesimal transformation depending on a small real parameter $\delta\alpha$, of the form:

$$\delta_\alpha\phi = \delta\alpha \frac{\delta\phi}{\delta\alpha}. \quad (7.10)$$

Using the Euler-Lagrange equations of motion, one finds that the variation of the action under such a transformation takes the form:

$$\delta_\alpha S = \int d^4x \partial_\mu (\delta\alpha J_\alpha^\mu), \quad (7.11)$$

where

$$J_\alpha^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} \frac{\delta\phi}{\delta\alpha}. \quad (7.12)$$

The invariance of S under transformations with constant $\delta\alpha$, namely $\delta_\alpha S = 0$, implies then that the above current is conserved:

$$\partial_\mu J_\alpha^\mu = 0. \quad (7.13)$$

The corresponding conserved charge is given by:

$$Q_\alpha = \int d^3\vec{x} \pi \frac{\delta\phi}{\delta\alpha}. \quad (7.14)$$

It is straightforward to verify that the charge Q_α generates the infinitesimal symmetry transformations in phase space, in the sense that:

$$\frac{\delta\phi}{\delta\alpha} = \{\phi, Q_\alpha\}_P, \quad \frac{\delta\pi}{\delta\alpha} = \{\pi, Q_\alpha\}_P. \quad (7.15)$$

For a generic functional F on phase space, one finds then:

$$\frac{\delta F}{\delta\alpha} = \{F, Q_\alpha\}_P. \quad (7.16)$$

Moreover, if there are several symmetries forming some Lie group with structure constants f^{abc} , the corresponding charges Q^a realize the group algebra through Poisson brackets:

$$\{Q^a, Q^b\}_P = f^{abc} Q^c. \quad (7.17)$$

Note finally that generalized transformations with non-constant $\delta\alpha$ do not represent an invariance of the action, but induce a simple variation of S :

$$\delta_\alpha S = \int d^4x \partial_\mu \delta\alpha J_\alpha^\mu. \quad (7.18)$$

A theory with fields ϕ possessing a global symmetry can be promoted to a new theory involving also a gauge field $A_{\alpha\mu}$ with suitable couplings and transformation laws, in which

the symmetry becomes local. To perform this gauging, one can proceed as follows. One first assumes that the full action splits into two separately gauge-invariant parts S_G and S_M as follows:

$$S = \int d^4x \mathcal{L}_G(A_{\alpha\nu}, \partial_\mu A_{\alpha\nu}) + \int d^4x \mathcal{L}_M(\phi, \partial_\mu \phi, A_{\alpha\nu}). \quad (7.19)$$

The new field $A_{\alpha\mu}$ is then taken to transform as a connection:

$$\delta_\alpha A_{\alpha\mu} = g^{-1} \partial_\mu \delta\alpha. \quad (7.20)$$

The current of the interacting theory is defined as the source term coming from S_M in the equation of motion for $A_{\alpha\mu}$:

$$J_{\alpha\mu} = -g^{-1} \frac{\delta S_M}{\delta A_{\alpha\mu}}. \quad (7.21)$$

The equations of motion for the gauge field take then the form:

$$\frac{\delta S_G}{\delta A_{\alpha\mu}} = g J_{\alpha\mu}. \quad (7.22)$$

The current J_α^μ is conserved, as a consequence of the invariance of S_M under local transformations with $\delta\alpha$ vanishing at infinity. Indeed, the only source of non-stationarity of S_M comes from $\delta_\alpha A_{\alpha\mu}$, and thus:

$$\delta_\alpha S_M = \int d^4x \frac{\delta S_M}{\delta A_{\alpha\mu}} \delta_\alpha A_{\alpha\mu} = - \int d^4x J_\alpha^\mu \partial_\mu \delta\alpha. \quad (7.23)$$

Integrating by parts, we see that the invariance of S_M implies:

$$\partial_\mu J_\alpha^\mu = 0. \quad (7.24)$$

This is the same conservation law as in the global case, except that J_α^μ may now depend also on $A_{\alpha\mu}$.

The standard example of this gauging procedure is quantum electrodynamics with charged fermions. The starting point is the Lagrangian for a free fermion:

$$\mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi. \quad (7.25)$$

This has the global invariance $\delta_\alpha \psi = i \delta\alpha \psi$, leading to the conserved current

$$J^\mu = \bar{\psi} \gamma^\mu \psi. \quad (7.26)$$

The gauged version of this theory is obtained by minimally coupling the gauge field to the fermion current through the covariant derivative $D_\mu = \partial_\mu - i g A_\mu$, and is described by the following Lagrangian:

$$\mathcal{L} = i \bar{\psi} \gamma^\mu D_\mu \psi - m \bar{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (7.27)$$

This has now the local invariance $\delta_\alpha \psi = i \delta\alpha \psi$, $\delta_\alpha A_\mu = g^{-1} \partial_\mu \delta\alpha$, and the conserved current is again:

$$J^\mu = \bar{\psi} \gamma^\mu \psi. \quad (7.28)$$

7.2 Quantum symmetries and Ward identities

In the operatorial formulation of a quantum field theory, the fields become operators ϕ acting on a Hilbert space of particle states $|N\rangle$, and Poisson brackets become commutators or anticommutators:

$$\{\dots, \dots\}_{\text{P}} \rightarrow -i[\dots, \dots]. \quad (7.29)$$

The field operators satisfy the Euler-Lagrange differential equations of motion $\delta S/\delta\phi = 0$, as functions of the coordinates:

$$\frac{\delta S}{\delta\phi} \equiv \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = 0. \quad (7.30)$$

The time evolution is dictated by the Heisenberg equations of motion, and for any operator O constructed from the fields one has

$$\dot{O} = -i[O, H]. \quad (7.31)$$

The basic objects that one considers at the quantum level are however correlation functions, defined by the vacuum expectation value of time-ordered products of fields:

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \langle\Omega|T\phi(x_1)\cdots\phi(x_n)|\Omega\rangle. \quad (7.32)$$

It turns out that such correlation functions satisfy a slightly different type of equations of motion, compared to the field operators. Indeed, when the term involving $\partial_0\pi$ in the equation of motion hits the time step-functions defining the T -product, one gets contact terms:

$$\partial_0 T\pi(x)\phi(y) = [\pi(x), \phi(y)]\delta(t_x - t_y) = -i\delta(x - y). \quad (7.33)$$

One finds then that the equations of motion from the fields translate into the following relations between correlation functions, called Schwinger-Dyson equations:

$$\langle\frac{\delta S}{\delta\phi}(x)\phi(x_1)\cdots\phi(x_n)\rangle = i\sum_k\langle\phi(x_1)\cdots\phi(x_{k-1})\phi(x_{k+1})\cdots\phi(x_n)\rangle\delta(x - x_k). \quad (7.34)$$

In the alternative path-integral formulation of a quantum field theory, the correlation functions can also be computed through a functional integral over all possible paths, weighted by a phase given by the action:

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \frac{\int\mathcal{D}\phi\phi(x_1)\cdots\phi(x_n)e^{iS}}{\int\mathcal{D}\phi e^{iS}}. \quad (7.35)$$

In this formulation, the equations of motion satisfied by correlation functions can be derived more directly, in a way that is moreover very similar to the classical case. One uses a variational approach and considers an arbitrary infinitesimal field variation $\delta\phi$ vanishing at infinity. One looks then at the path-integral with some field insertions and

views this transformation as a shift in the dummy integration variables. This should leave the result unchanged, implying that:

$$\delta \left(\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) e^{iS} \right) = 0. \quad (7.36)$$

Since the Jacobian of the field transformation is 1, so that $\delta \mathcal{D}\phi = 0$, this equation implies:

$$\int \mathcal{D}\phi \left(i \delta\phi(x) \frac{\delta S}{\delta\phi}(x) \phi(x_1) \cdots \phi(x_n) + \sum_k \phi(x_1) \cdots \delta\phi(x_k) \cdots \phi(x_n) \right) e^{iS} = 0. \quad (7.37)$$

Requiring that this should hold for any $\delta\phi$ and dividing by the path-integral without field insertions, one finally recovers the Schwinger-Dyson equations:

$$\left\langle \frac{\delta S}{\delta\phi}(x) \phi(x_1) \cdots \phi(x_n) \right\rangle = i \sum_k \langle \phi(x_1) \cdots \phi(x_{k-1}) \phi(x_{k+1}) \cdots \phi(x_n) \rangle \delta(x - x_k). \quad (7.38)$$

The Schwinger-Dyson equations for the correlation functions of the theory also imply some relations on S -matrix elements. Indeed, recall that the S -matrix elements can be obtained through the LSZ reduction formula, as amputated correlation functions. They depend therefore only on the most singular part of the correlation functions, with one pole for each external particle. As a result, they satisfy then the classical equations of motion, because the contact terms in the Schwinger-Dyson equations always miss at least one of the poles and are not enough singular to contribute.

Concerning conservations laws, it is naively expected that conserved quantities of the classical theory should lead to corresponding conserved quantities in the quantum theory. More precisely, in the operatorial formalism any classical conservation law should turn into an equation for the corresponding current operator:

$$\partial_\mu J_\alpha^\mu = 0. \quad (7.39)$$

This leads to a conserved charge operator Q_α defined in the same way as in the classical theory. The symmetry transformations associated to the conserved charge Q_α are then realized through operatorial transformations induced by the operator $U(\alpha) = e^{i\alpha Q_\alpha}$. For an infinitesimal transformation, one finds in particular:

$$\frac{\delta O}{\delta\alpha} = i [O, Q_\alpha]. \quad (7.40)$$

Moreover, conserved charges must commute with H if they do not depend explicitly on time:

$$[Q_\alpha, H] = 0. \quad (7.41)$$

As for the equation of motion, which implies some identities for correlation functions involving the fields, the conservation equation $\partial_\mu J_\alpha^\mu = 0$ implies some identities for correlation functions, involving now the current. Again, the difference with respect to the

operatorial equations consists in some contact terms, which are relevant off-shell but not on-shell. In the operatorial formulation, these contact terms arise from the T -product, as before. Using the form of the current and the canonical commutation relations, one finds then the following Ward identity:

$$\langle \partial_\mu J_\alpha^\mu(x) \phi(x_1) \cdots \phi(x_n) \rangle = i \sum_k \langle \phi(x_1) \cdots \frac{\delta \phi}{\delta \alpha}(x_k) \cdots \phi(x_n) \rangle \delta(x - x_k). \quad (7.42)$$

In the path-integral formulation, these Ward identities can be derived by proceeding as for the equations of motion, but considering an infinitesimal symmetry transformation of the type:

$$\delta_\alpha \phi = \delta \alpha \frac{\delta \phi}{\delta \alpha}. \quad (7.43)$$

If this corresponds to a classical global symmetry, $\delta_\alpha S = 0$ for constant $\delta \alpha$. However, for non-constant $\delta \alpha$ vanishing at infinity, one finds:

$$\delta_\alpha S = \int d^4x \partial_\mu \delta \alpha J_\alpha^\mu = - \int d^4x \delta \alpha \partial_\mu J_\alpha^\mu. \quad (7.44)$$

Consider then the path integral with some field insertions, and view this transformation as a change of the dummy integration variables. This should leave the result unchanged:

$$\delta_\alpha \left(\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) e^{iS} \right) = 0. \quad (7.45)$$

Assuming that the Jacobian of the transformation is 1, as it turns out to be for most of the relevant symmetries, so that $\delta_\alpha \mathcal{D}\phi = 0$, this relation implies:

$$\int \mathcal{D}\phi \left(-i \int d^4x \delta \alpha(x) \partial_\mu J_\alpha^\mu(x) \phi(x_1) \cdots \phi(x_n) + \sum_k \phi(x_1) \cdots \delta \alpha(x_k) \frac{\delta \phi}{\delta \alpha}(x_k) \cdots \phi(x_n) \right) e^{iS} = 0. \quad (7.46)$$

Requiring this to hold for any $\delta \alpha$ and dividing by the path-integral without insertions, one recovers finally the same Ward identity as in the operator formalism:

$$\langle \partial_\mu J_\alpha^\mu(x) \phi(x_1) \cdots \phi(x_n) \rangle = i \sum_k \langle \phi(x_1) \cdots \frac{\delta \phi}{\delta \alpha}(x_k) \cdots \phi(x_n) \rangle \delta(x - x_k). \quad (7.47)$$

The Ward identities satisfied by correlation functions imply again relations for S -matrix elements, as was the case for the equations of motion. Again, the extra terms modifying the operatorial conservation law do however not contribute to the S -matrix elements, because they are not enough singular. The S -matrix elements satisfy therefore the classical conservation laws.

To illustrate the role and the importance of Ward identities, let us consider the standard example of quantum electrodynamics. Consider in particular the special correlation function $\langle J^\mu(x) \psi(y_1) \bar{\psi}(y_2) \rangle$. The Ward identity gives in this case:

$$\langle \partial_\mu J^\mu(x) \psi(y_1) \bar{\psi}(y_2) \rangle = -\langle \psi(y_1) \bar{\psi}(y_2) \rangle \delta(x - y_1) + \langle \psi(y_1) \bar{\psi}(y_2) \rangle \delta(x - y_2). \quad (7.48)$$

The left hand side of this relation corresponds to the coupling M^μ between one unphysical gauge boson and two fermions, whereas the right hand side contains the fermion propagator S . More precisely, taking the Fourier transform of this relation, with momenta k , p_1 and p_2 which satisfy $k = p_2 - p_1$ by translational invariance, one finds:

$$k_\mu M^\mu(p_1, p_2) = S(\not{p}_2) - S(\not{p}_1). \quad (7.49)$$

Diagrammatically, this implies the following relation between the connected 3-point and 2-point correlation functions:

$$k_\mu \times \text{diagram} = \text{diagram} - \text{diagram} . \quad (7.50)$$

In order to get the corresponding relation between S -matrix elements, one needs to amputate the two external fermion lines, by writing

$$M^\mu(p_1, p_2) = S(\not{p}_1) (-i \Gamma^\mu(p_1, p_2)) S(\not{p}_2). \quad (7.51)$$

The Ward identity becomes then

$$-i k_\mu \Gamma^\mu(p_1, p_2) = S^{-1}(\not{p}_1) - S^{-1}(\not{p}_2). \quad (7.52)$$

Finally, one can use the 1PI decomposition of the propagator and the vertex

$$S(\not{p}) = \frac{i}{\not{p} - m - \Sigma(\not{p})}, \quad \Gamma^\mu(p_1, p_2) = \gamma^\mu + \Lambda^\mu(p_1, p_2), \quad (7.53)$$

to rewrite the Ward identity in the following form:

$$(p_2 - p_1)_\mu \Lambda^\mu(p_1, p_2) = \Sigma(\not{p}_2) - \Sigma(\not{p}_1). \quad (7.54)$$

This implies that unphysical longitudinal photons are decoupled on-shell, since then $\Sigma(\not{p}_i) = \Sigma(m)$, and have a simple and rigidly determined effect off-shell. This property is crucial for the consistency of the quantum theory, and in particular for the correct decoupling of unphysical negative norm photon states.

7.3 Regularization and anomalies

In quantum field theory, there are UV divergences. One needs therefore to regularize the theory with some finite cut-off, renormalize it, and finally remove the cut-off. Due to this complication, the formal derivation of the Ward identities can happen to be invalidated, with the appearance of so-called anomalies. The classical symmetry is then broken by quantum effects.

Quantum anomalies in a classical symmetry can appear only if there does not exist any UV regularization of the theory which manifestly preserves that symmetry. However, anomalies are actually finite IR effects. They do not depend on the regularization method, but only on which symmetries this respects. They represent thus genuine physical effects.

The precise way anomalies show up varies with the formalism, but always rests on some subtlety related to regularization. In the operatorial formalism, the subtlety is that the current J_α^μ associated to a classical symmetry is a composite field, involving products of fields at the same point, which gives a singular behavior. In the path integral formalism, the subtlety is that the measure $\mathcal{D}\phi$ is a formal infinite-dimensional product which needs to be properly defined, and which can give rise to unexpected Jacobians under transformations. In a perturbative diagrammatic expansion, the subtlety is that certain loop diagrams are linearly divergent, and shifting the momentum integration variables is not trivially allowed.

The physical effect of an anomaly in a symmetry existing in the classical limit is that the symmetry disappears at the quantum level. More precisely, it is violated by specific and computable effects. For global symmetries, which correspond to true restrictions on the theory, this anomalous effect is perfectly consistent. It may happen and means that the classical selection rules are violated at the quantum level in a specific way. The predicted violation of selection rules associated to these effects has been verified experimentally in several situations. For local symmetries, which correspond to fake redundancies of the theory, the appearance of an anomaly is inconsistent, because unphysical states do then no longer decouple and ruin unitarity. It must therefore be excluded. The consistency requirements implied by the absence of such catastrophic effects are satisfied in a non-trivial way by relevant physical models, like for instance the Standard Model.

7.4 Anomalies in global chiral symmetries

The prototypical situation where a quantum anomaly arises is the chiral symmetry for fermions. Consider for instance the theory of a massless Dirac fermion interacting with an Abelian gauge field in the standard minimal way through $D_\mu = \partial_\mu - igA_\mu$:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (7.55)$$

This has a $U(1)$ local gauge symmetry associated to group elements of the form $e^{i\alpha}$ and acting as:

$$\begin{aligned} \delta\psi &= i\delta\alpha\psi, & \delta\bar{\psi} &= -i\delta\alpha\bar{\psi} \\ \delta A_\mu &= g^{-1}\partial_\mu\delta\alpha. \end{aligned} \quad (7.56)$$

The corresponding conserved current is:

$$J^\mu = \bar{\psi}\gamma^\mu\psi. \quad (7.57)$$

It also has a $U(1)_5$ global chiral symmetry associated to group elements of the form $e^{i\alpha_5\gamma_5}$ and acting as:

$$\begin{aligned}\delta_5\psi &= i\delta\alpha_5\gamma_5\psi, & \delta_5\bar{\psi} &= i\delta\alpha_5\bar{\psi}\gamma_5 \\ \delta_5A_\mu &= 0.\end{aligned}\tag{7.58}$$

The corresponding conserved current is:

$$J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi.\tag{7.59}$$

At the classical level, both symmetries are present and:

$$\partial_\mu J^\mu = 0, \quad \partial_\mu J_5^\mu = 0.\tag{7.60}$$

At the quantum level, however, it is impossible to regularize the theory while preserving both of these symmetries, and one of the two is broken. Using a regularization preserving $U(1)$ but not $U(1)_5$, one finds for instance:

$$\langle\partial_\mu J^\mu\rangle = 0, \quad \langle\partial_\mu J_5^\mu\rangle \neq 0.\tag{7.61}$$

The anomaly can be computed in many different ways. In order to illustrate the fact that it is a genuine physical effect which does not depend on the details of the regularization, but only on its symmetry properties, it is useful to study how it appears in the operatorial formulation, in the path-integral formulation, and also in the perturbative evaluation of amplitudes in terms of Feynman diagrams.

In the operatorial formalism, we need to regularize the current operator in a gauge invariant way. This can be done by the so-called point-splitting method, defining a regularized version of the current composite operator by splitting apart the two fermions fields that it involves of some small distance ϵ^μ :

$$J_5^\mu(x, \epsilon) = \bar{\psi}\left(x + \frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\psi\left(x - \frac{\epsilon}{2}\right) \exp\left\{ig\int_{x-\epsilon/2}^{x+\epsilon/2} dy^\mu A_\mu(y)\right\}.\tag{7.62}$$

The last Wilson line factor is needed in order for the regularized current to be invariant under local gauge transformations. Using the equations of motion $\gamma^\mu\partial_\mu\psi = ig\gamma^\mu\psi A_\mu$, one computes then the divergence of the current:

$$\partial_\mu J_5^\mu(x, \epsilon) = -igJ_5^\mu(x, \epsilon)\left[A_\mu(y)\Big|_{x-\epsilon/2}^{x+\epsilon/2} - \partial_\mu\int_{x-\epsilon/2}^{x+\epsilon/2} dy^\nu A_\nu(y)\right].\tag{7.63}$$

At leading order in ϵ this yields:

$$\partial_\mu J_5^\mu(x, \epsilon) = igJ_5^\mu(x, \epsilon)F_{\mu\nu}(x)\epsilon^\nu.\tag{7.64}$$

Taking the vacuum expectation value of this object, and treating $A_\mu(x)$ as an external field, one finds:

$$\langle\partial_\mu J_5^\mu(x, \epsilon)\rangle = ig\langle J_5^\mu(x, \epsilon)\rangle F_{\mu\nu}(x)\epsilon^\nu.\tag{7.65}$$

The quantity appearing on the right hand side is essentially the fermion propagator in the background field $A_\mu(x)$, taken between the two points $x^\mu \pm \epsilon^\mu/2$. For small ϵ , one finds indeed:

$$\langle J_5^\mu(x, \epsilon) \rangle = i \operatorname{tr} \left[\gamma_5 \gamma^\mu S(A(x), \epsilon) \right]. \quad (7.66)$$

This expression has a singularity for $\epsilon \rightarrow 0$, which can be computed by expanding the fermion propagator in powers of the external gauge field $A_\mu(x)$. Diagrammatically, this corresponds to the following series:

$$\begin{array}{c} \text{====>} \\ \text{====>} \end{array} = \text{--->} + \text{--->} \begin{array}{c} \text{wavy} \\ \text{line} \end{array} \text{--->} + \dots \quad (7.67)$$

Each photon insertion comes with an additional inverse power of the momentum in Fourier space, and makes therefore the propagator less singular in the UV, which corresponds to short distances in configuration space. On the other hand, photon insertions also bring some further γ matrices, which are needed to make the trace over spinor indices non zero. Indeed, a trace involving γ_5 is non-vanishing only if it involves at least 4 other γ^μ matrices, and $\operatorname{tr}[\gamma_5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta] = -4i\epsilon^{\mu\nu\alpha\beta}$. One finds then the following situation. The term without any insertion is singular, but does not contribute to the trace. The term with just one insertion is still singular and contributes to the trace. Finally the terms with several insertions are regular and although they contribute to the trace they can be discarded. Computing then the relevant diagram with one insertion, one finds:

$$\begin{aligned} \langle J_5^\mu(x, \epsilon) \rangle &= ig \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \int \frac{d^4 p}{(2\pi)^4} e^{ip\epsilon} \operatorname{tr} \left[\gamma_5 \gamma^\mu \frac{1}{\not{p}} A(k) \frac{1}{\not{p} - \not{k}} \right] \\ &= -4g \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \int \frac{d^4 p}{(2\pi)^4} e^{ip\epsilon} \frac{\epsilon^{\mu\alpha\beta\gamma} p_\alpha A_\beta(k) (p-k)_\gamma}{p^2 (p-k)^2} \\ &= -4g \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \epsilon^{\mu\alpha\beta\gamma} k_\beta A_\gamma(k) \int \frac{d^4 p}{(2\pi)^4} e^{ip\epsilon} \frac{p_\alpha}{p^2 (p-k)^2}. \end{aligned} \quad (7.68)$$

In the limit of small ϵ , the integral in p becomes linearly divergent and can be easily computed. One finds:

$$\int \frac{d^4 p}{(2\pi)^4} e^{ip\epsilon} \frac{p_\alpha}{p^4} = -\frac{i}{8\pi^2} \frac{\epsilon_\alpha}{\epsilon^2}. \quad (7.69)$$

This gives then:

$$\langle J_5^\mu(x, \epsilon) \rangle = \frac{ig}{4\pi^2} \frac{\epsilon_\alpha}{\epsilon^2} \epsilon^{\mu\alpha\beta\gamma} F_{\beta\gamma}(x). \quad (7.70)$$

For the divergence of the current, this implies:

$$\langle \partial_\mu J_5^\mu(x, \epsilon) \rangle = -\frac{g^2}{4\pi^2} \frac{\epsilon_\alpha \epsilon^\nu}{\epsilon^2} \epsilon^{\mu\alpha\beta\gamma} F_{\beta\gamma}(x) F_{\mu\nu}(x). \quad (7.71)$$

Finally, one can take the limit $\epsilon \rightarrow 0$ in a symmetric way, with

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon^\alpha \epsilon^\beta}{\epsilon^2} = \frac{1}{4} \eta^{\alpha\beta}. \quad (7.72)$$

This yields a non-trivial finite result:

$$\langle \partial_\mu J_5^\mu(x) \rangle = -\frac{g^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}(x) F_{\alpha\beta}(x). \quad (7.73)$$

On the other hand, proceeding in the same way for the other current one finds

$$\langle \partial_\mu J^\mu(x) \rangle = 0. \quad (7.74)$$

In the path-integral formalism, we need to suitably define and regularize the functional integration measure. This can be done by expanding the fermion fields in a basis of eigenmodes of the kinetic operator $i\mathcal{D}$:

$$i\mathcal{D}\psi_n(x) = \lambda_n \psi_n(x). \quad (7.75)$$

It is useful to temporarily make an analytic continuation to Euclidean space. The operator $i\mathcal{D}$ is then Hermitian, and the basis is guaranteed to be orthonormal and complete:

$$\int d^4x \psi_m^\dagger(x) \psi_n(x) = \delta_{mn}, \quad \sum_n \psi_n(x) \psi_n^\dagger(y) = \delta(x-y). \quad (7.76)$$

The fields ψ and $\bar{\psi}$, which must be treated as independent, can now be expanded as follows:

$$\psi(x) = \sum_n a_n \psi_n(x), \quad \bar{\psi}(x) = \sum_n \hat{a}_n \psi_n(x). \quad (7.77)$$

The path integral measure is then naturally defined as:

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} = \prod_n da_n d\hat{a}_n. \quad (7.78)$$

Consider now the local version of the chiral transformation, namely $\delta_5 \psi = i\delta\alpha_5 \gamma_5 \psi$ and $\delta_5 \bar{\psi} = i\delta\alpha_5 \bar{\psi} \gamma_5$ with non-constant $\delta\alpha_5$, to derive the Ward identity for the chiral current. These transformation act as follows on the mode coefficients a_n and \hat{a}_n :

$$\delta a_m = \sum_n (\delta_{mn} + \delta C_{mn}) a_n, \quad \delta \hat{a}_m = \sum_n (\delta_{mn} + \delta C_{mn}) \hat{a}_n, \quad (7.79)$$

where:

$$\delta C_{mn} = i \int d^4x \delta\alpha_5(x) \psi_m^\dagger(x) \gamma_5 \psi_n(x). \quad (7.80)$$

The Jacobian associated to each of these transformations has the form:

$$J = \det(1 + \delta C) = \exp \left\{ \text{tr} \log(1 + \delta C) \right\} = \exp \left\{ \text{tr} \delta C \right\}. \quad (7.81)$$

This gives a result of the form:

$$J = \exp \left\{ i \int d^4x \delta\alpha_5(x) \mathcal{A}(x) \right\}, \quad (7.82)$$

where:

$$\mathcal{A}(x) = \sum_n \psi_n^\dagger(x) \gamma_5 \psi_n(x). \quad (7.83)$$

If $\mathcal{A}(x) \neq 0$, then $J \neq 1$ and an additional term appears in the derivation of the Ward identity. Indeed, viewing the transformation as a change of variables, one should have:

$$\delta_5 \left(\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS} \right) = 0 \quad (7.84)$$

The action transforms as before, but the measure gets rescaled by J^{-2} :

$$\begin{aligned} e^{iS} &\rightarrow e^{iS} \exp \left\{ i \int d^4x \delta\alpha_5(x) \partial_\mu J_5^\mu(x) \right\}, \\ \mathcal{D}\psi \mathcal{D}\bar{\psi} &\rightarrow \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ -2i \int d^4x \delta\alpha_5(x) \mathcal{A}(x) \right\}. \end{aligned} \quad (7.85)$$

We conclude then that the Ward identity becomes in this case:

$$\langle \partial_\mu J_5^\mu(x) \rangle = 2 \mathcal{A}(x). \quad (7.86)$$

The anomalous exponent \mathcal{A} is ambiguous and needs to be regularized, since by using the completeness property of the $\psi_n(x)$ one formally finds $\mathcal{A}(x) = \text{tr}[\gamma_5] \delta(0) = 0 \cdot \infty$. One can introduce for this a cut-off Λ to damp the contribution of modes with large eigenvalues and define:

$$\mathcal{A}(x, \Lambda) = \sum_n \psi_n^\dagger(x) \gamma_5 e^{-\lambda_n^2/\Lambda^2} \psi_n(x). \quad (7.87)$$

This corresponds to the following trace over the spectrum of states, with the zero-mode omitted:

$$\mathcal{A}(x, \Lambda) = \text{Tr}' \left[\gamma_5 e^{-(i\mathcal{D})^2/\Lambda^2} \right]. \quad (7.88)$$

To evaluate this, we first use the identity:

$$\begin{aligned} (i\mathcal{D})^2 &= -\frac{1}{4} \{ \gamma^\mu, \gamma^\nu \} \{ D_\mu, D_\nu \} - \frac{1}{4} [\gamma^\mu, \gamma^\nu] [D_\mu, D_\nu] \\ &= -D^2 + \frac{g}{2} \sigma^{\mu\nu} F_{\mu\nu}. \end{aligned} \quad (7.89)$$

We next expand the argument of the trace in powers of A_μ . Each power of A_μ comes with a factor Λ^{-1} or Λ^{-2} , and at most two γ matrices. The leading contribution to $\mathcal{A}(x, \Lambda)$ for large Λ comes then from the term involving the minimal number of powers of A_μ required to make the trace over spinor indices non-vanishing. This gives:

$$\mathcal{A}(x, \Lambda) = \frac{g^2}{8\Lambda^4} \text{Tr}' \left[e^{-\square/\Lambda^2} \right] \text{tr} \left[\gamma_5 \sigma^{\mu\nu} \sigma^{\alpha\beta} \right] F_{\mu\nu}(x) F_{\alpha\beta}(x). \quad (7.90)$$

The trace over spinor indices gives then $\text{tr}[\gamma_5 \sigma^{\mu\nu} \sigma^{\alpha\beta}] = 4i\epsilon^{\mu\nu\alpha\beta}$, whereas the remaining trace over free states gives:

$$\text{Tr}'\left[e^{-\square/\Lambda^2}\right] = \int \frac{d^4 p}{(2\pi)^4} e^{p^2/\Lambda^2} = \frac{i}{16\pi^2} \Lambda^4. \quad (7.91)$$

We are finally left with a finite result when $\Lambda \rightarrow \infty$:

$$\mathcal{A}(x) = -\frac{g^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}(x) F_{\alpha\beta}(x). \quad (7.92)$$

This reproduces the same result as before for the anomalous Ward identity:

$$\langle \partial_\mu J_5^\mu(x) \rangle = -\frac{g^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}(x) F_{\alpha\beta}(x). \quad (7.93)$$

One the other hand, proceeding similarly one finds:

$$\langle \partial_\mu J^\mu(x) \rangle = 0. \quad (7.94)$$

It is instructive to study also how the anomaly emerges within a perturbative approach in terms of Feynman diagrams. It turns out that it entirely comes from a linearly divergent one-loop triangle diagram. To see this, consider the matrix element of J_5^μ between the vacuum and a 2-photon state, which should be directly sensitive to the anomaly. At the one-loop level and in momentum space, this receives contributions from two similar diagrams, which are related by the crossing $(\alpha, k_1) \leftrightarrow (\beta, k_2)$:

$$\begin{aligned} T^{\mu\alpha\beta}(k_1, k_2) &= \begin{array}{c} \mu \\ q \end{array} \begin{array}{c} l+k_2 \\ \nearrow \\ \bullet \\ \searrow \\ l-k_1 \end{array} \begin{array}{c} \text{wavy} \\ \nearrow \\ \text{wavy} \\ \searrow \end{array} \begin{array}{c} \alpha \\ k_1 \\ \beta \\ k_2 \end{array} + \begin{array}{c} \mu \\ q \end{array} \begin{array}{c} l+k_1 \\ \nearrow \\ \bullet \\ \searrow \\ l-k_2 \end{array} \begin{array}{c} \text{wavy} \\ \nearrow \\ \text{wavy} \\ \searrow \end{array} \begin{array}{c} \alpha \\ k_1 \\ \beta \\ k_2 \end{array} \\ &= ig^2 \int \frac{d^4 l}{(2\pi)^4} \left\{ \text{tr} \left[\gamma^\mu \gamma_5 \frac{1}{\not{l}-\not{k}_1} \gamma^\beta \frac{1}{\not{l}-\not{k}_1+\not{k}_2} \gamma^\alpha \frac{1}{\not{l}+\not{k}_2} \right] \right. \\ &\quad \left. + \text{tr} \left[\gamma^\mu \gamma_5 \frac{1}{\not{l}-\not{k}_2} \gamma^\alpha \frac{1}{\not{l}-\not{k}_2+\not{k}_1} \gamma^\beta \frac{1}{\not{l}+\not{k}_1} \right] \right\}. \quad (7.95) \end{aligned}$$

Taking the divergence of the current corresponds to contracting this result with the momentum flowing in at the vertex, namely $q_\mu = (k_1 + k_2)_\mu$. One finds then in each of the two diagrams a factor $q_\mu \gamma^\mu \gamma_5$, which can be decomposed as follows:

$$q_\mu \gamma^\mu \gamma_5 = (\not{l} + \not{k}_2) \gamma_5 + \gamma_5 (\not{l} - \not{k}_1) = (\not{l} + \not{k}_1) \gamma_5 + \gamma_5 (\not{l} - \not{k}_2). \quad (7.96)$$

In each term of $q_\mu T^{\mu\alpha\beta}$ there is then one propagator denominator that cancels, and finally one finds:

$$\begin{aligned} q_\mu T^{\mu\alpha\beta}(k_1, k_2) &= ig^2 \int \frac{d^4 l}{(2\pi)^4} \left\{ \text{tr} \left[\gamma_5 \frac{1}{\not{l}-\not{k}_1} \gamma^\beta \frac{1}{\not{l}-\not{k}_1+\not{k}_2} \gamma^\alpha - \gamma_5 \frac{1}{\not{l}-\not{k}_1+\not{k}_2} \gamma^\alpha \frac{1}{\not{l}+\not{k}_2} \gamma^\beta \right] \right. \\ &\quad \left. + (\alpha, k_1) \leftrightarrow (\beta, k_2) \right\}. \quad (7.97) \end{aligned}$$

If one could now freely shift l , each diagram would be antisymmetric under the crossing $(\alpha, k_1) \leftrightarrow (\beta, k_2)$ and the result would cancel. But the integral is linearly divergent and must be regularized. A shift in the integration variable leaves then a finite surface term:

$$\begin{aligned}\Delta(a) &= \int \frac{d^4 l}{(2\pi)^4} [f(l+a) - f(l)] = \int \frac{d^4 l}{(2\pi)^4} [a^\mu \partial_\mu f(l) + \dots] \\ &= \frac{i}{8\pi^2} \lim_{l \rightarrow \infty} a^\mu l_\mu l^2 f(l).\end{aligned}\tag{7.98}$$

Applying this to the expression for $q_\mu T^{\mu\alpha\beta}$ with the appropriate shifts ($a^\mu = (2k_1 - k_2)^\mu$ in the first term of the first row and $a^\mu = (2k_2 - k_1)^\mu$ in the first term of the second row) and evaluating the spinorial traces, one finds two identical terms which add up and finally give:

$$q_\mu T^{\mu\alpha\beta}(k_1, k_2) = \frac{g^2}{2\pi^2} \epsilon^{\alpha\beta\rho\sigma} k_{1\rho} k_{2\sigma}.\tag{7.99}$$

This result finally implies the same anomalous Ward identity in configuration space that was derived before, namely:

$$\langle \partial_\mu J_5^\mu(x) \rangle = -\frac{g^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}(x) F_{\alpha\beta}(x).\tag{7.100}$$

A similar computation for the gauge current yields instead:

$$\langle \partial_\mu J^\mu(x) \rangle = 0.\tag{7.101}$$

7.5 Generalizations

The analysis of chiral anomalies can be extended to theories with several fermions and more general symmetries, forming a group G with generators satisfying $[T^a, T^b] = i f^{abc} T^c$. Consider for instance a theory with massless Dirac fermions interacting with non-Abelian gauge fields with $D_\mu = \partial_\mu - ig A_\mu^a T^a$:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}.\tag{7.102}$$

This has a group G of local gauge symmetries associated to group elements of the form $e^{i\alpha^a T^a}$ and acting as:

$$\begin{aligned}\delta\psi &= i\delta\alpha^a T^a \psi, & \delta\bar{\psi} &= -i\delta\alpha^a T^a \bar{\psi} \\ \delta A_\mu^a &= g^{-1} \partial_\mu \delta\alpha^a + f^{abc} A_\mu^b \delta\alpha^c.\end{aligned}\tag{7.103}$$

The corresponding covariantly conserved currents are given by:

$$J^{a\mu} = \bar{\psi}\gamma^\mu T^a \psi.\tag{7.104}$$

It also has a group G_5 of global chiral symmetries associated to group elements of the form $e^{i\alpha_5^a T^a \gamma_5}$ and acting as:

$$\begin{aligned}\delta_5\psi &= i\delta\alpha_5^a T^a \gamma_5 \psi, & \delta_5\bar{\psi} &= i\delta\alpha_5^a \bar{\psi} \gamma_5 T^a \\ \delta_5 A_\mu^a &= f^{abc} A_\mu^b \delta\alpha_5^c.\end{aligned}\tag{7.105}$$

The corresponding covariantly conserved currents are given by:

$$J_5^{a\mu} = \bar{\psi}\gamma^\mu\gamma_5 T^a\psi \quad (7.106)$$

At the classical level, both symmetries are present and:

$$D_\mu J^{a\mu} = 0, \quad D_\mu J_5^{a\mu} = 0. \quad (7.107)$$

At the quantum level, however, one finds:

$$\langle D_\mu J^{a\mu} \rangle = 0, \quad \langle D_\mu J_5^{a\mu} \rangle = -\frac{g^2}{16\pi^2} d^{abc} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^b F_{\alpha\beta}^c, \quad (7.108)$$

in terms of the symmetric constants:

$$d^{abc} = \text{symtr}[T^a T^b T^c]. \quad (7.109)$$

In the computations done so far, we have used regularizations preserving manifestly the local gauge symmetry, and found an anomaly in the global chiral symmetry. One can however generalize these computations by using families of regularizations that depend on a continuous parameter ξ , and which preserve the gauge symmetry only for $\xi = 0$. The deformation concerns respectively the phase of the Wilson line factor, the gauge field dependence of the operator used to regulate the Jacobian and a shift in the loop momentum in the three methods that have been used. It turns out that such regularizations preserve then the chiral symmetry for some other non-zero value of the parameter, which we can conventionally take to be $\xi = 1$. One finds then that for a generic value of ξ the two Ward identities become:

$$\langle D_\mu J^{a\mu} \rangle = 2\xi \mathcal{A}^a, \quad \langle D_\mu J_5^{a\mu} \rangle = 2(1 - \xi) \mathcal{A}^a, \quad (7.110)$$

where

$$\mathcal{A}^a = -\frac{g^2}{32\pi^2} d^{abc} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^b F_{\alpha\beta}^c. \quad (7.111)$$

This shows that it is possible to preserve either the gauge symmetry, for $\xi = 0$, or the chiral symmetry, for $\xi = 1$, but not both simultaneously. Since the gauge symmetry is local and the chiral symmetry is global, we are forced by consistency to choose the option where gauge invariance is preserved and chiral symmetry is sacrificed.

7.6 Anomalies in local gauge symmetries

Much as an ordinary gauge symmetry is local thanks to a vector gauge field A_μ , a global chiral symmetry can be made local by introducing an additional axial gauge field $A_{5\mu}$. Consider for instance a theory with massless Dirac fermions interacting vectorially with some gauge fields A_μ^a and axially with some other gauge fields $A_{5\mu}^a$, with couplings determined by the covariant derivative $D_\mu = \partial_\mu - igA_\mu^a T^a - ig\gamma_5 A_{5\mu}^a T^a$:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4}F_{5\mu\nu}^a F_5^{a\mu\nu}. \quad (7.112)$$

This theory has two independent groups G and G_5 of local symmetries, associated to the currents:

$$\begin{aligned} J^{a\mu} &= \bar{\psi}\gamma^\mu T^a \psi, \\ J_5^{a\mu} &= \bar{\psi}\gamma^\mu \gamma_5 T^a \psi. \end{aligned} \quad (7.113)$$

The occurrence of an anomaly implies now a breakdown of gauge-invariance for the quantum effective action of the gauge fields, obtained by integrating over the quantum fluctuations of the fermion fields:

$$\Gamma[A, A_5] = -i \log \langle e^{iS[\psi, A, A_5]} \rangle. \quad (7.114)$$

Indeed, under infinitesimal gauge transformation one gets:

$$\begin{aligned} \delta\Gamma[A, A_5] &= \int d^4x \delta\alpha^a(x) \langle D_\mu J^{a\mu}(x) \rangle, \\ \delta_5\Gamma[A, A_5] &= \int d^4x \delta\alpha_5^a(x) \langle D_\mu J_5^{a\mu}(x) \rangle. \end{aligned} \quad (7.115)$$

In the presence of a non-trivial anomaly, these cannot be made both zero, and the theory becomes therefore unavoidably inconsistent. A change in regularization allows to shift but not to eliminate the anomaly. It corresponds to adding to the action a local non-invariant counter-term:

$$\Delta\Gamma[\xi, A, A_5] = \text{local counter-term}. \quad (7.116)$$

The anomalous correlation functions are those with an odd number of γ_5 : $\langle J_5 J J \rangle$ and $\langle J_5 J_5 J_5 \rangle$. The standard choice is to preserve G and give up G_5 , but in this situation it is just an arbitrary choice.

To uncover the nature of the anomaly occurring in this theory, it is convenient to reformulate it in a more symmetric way in terms of two chiral sectors, defined by taking the following linear combinations of fields:

$$\psi_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\psi, \quad A_{L,R}^{a\mu} = A^{a\mu} \pm A_5^{a\mu}. \quad (7.117)$$

The symmetries get then reshuffled to G_L and G_R , with currents

$$J_{L,R}^{a\mu} = \frac{1}{2}(J^{a\mu} \pm J_5^{a\mu}). \quad (7.118)$$

In this language, the anomalous correlation functions are those with only L or R fields: $\langle J_L J_L J_L \rangle$ and $\langle J_R J_R J_R \rangle$. Notice now that $\psi_{L,R}$ is equivalent to $\psi_{R,L}^c$, and one can thus reinterpret this theory as a chiral fermion interacting with a gauge field with group $G_L \times G_R$ in a representation of the type $(R, 1) \oplus (1, R^c)$. In this formulation, it becomes clear that this theory can be generalized to less symmetric situations, with an arbitrary gauge group and chiral fields in arbitrary representations. Moreover, the occurrence of an anomaly is reinterpreted in this formulation as a consequence of the fact that the involved fermion is a chiral Weyl fermion, rather than a Dirac fermion.

As the simplest example of theory where a local gauge symmetry gets an anomaly, consider now the theory of a massless Weyl fermion interacting with an Abelian gauge field in the standard way with $D_\mu = \partial_\mu - igA_\mu$:

$$\mathcal{L} = i\bar{\chi}\gamma^\mu D_\mu\chi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (7.119)$$

This has a $U(1)$ local gauge symmetry associated to group elements of the form $e^{i\alpha}$ and acting as:

$$\begin{aligned} \delta\chi &= i\delta\alpha\chi, & \delta\bar{\chi} &= -i\delta\alpha\bar{\chi}, \\ \delta A_\mu &= g^{-1}\partial_\mu\delta\alpha. \end{aligned} \quad (7.120)$$

The corresponding conserved current is:

$$J^\mu = \bar{\chi}\gamma^\mu\chi. \quad (7.121)$$

There are no extra global symmetries in this case. At the classical level, the symmetry is present and:

$$\partial_\mu J^\mu = 0. \quad (7.122)$$

At the quantum level, however, it is impossible to regularize the theory while preserving this symmetry, because of the chiral nature of the fermion field. One finds then:

$$\langle\partial_\mu J^\mu\rangle \neq 0. \quad (7.123)$$

The precise form of the gauge anomaly can be computed in the same way as for the chiral anomaly, in all the three different approaches that we have seen. The fact that χ is a Weyl fermion of some definite chirality $\eta = \pm 1$ implies that one can rewrite it as a projection of a Dirac fermion ψ :

$$\chi = \frac{1}{2}(1 + \eta\gamma_5)\psi. \quad (7.124)$$

One can then rewrite the current as:

$$J^\mu = \frac{1}{2}(\bar{\psi}\gamma^\mu\psi + \eta\bar{\psi}\gamma^\mu\gamma_5\psi). \quad (7.125)$$

From the form of this expression, we see that the computation is almost identical to that of the chiral anomaly, except for a factor $\eta/2$, and the result is:

$$\langle\partial_\mu J^\mu\rangle = -\frac{\eta g^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x). \quad (7.126)$$

As in the case of global symmetries, this computation can be generalized to more complicated situations. Consider for instance a more general theory with massless Weyl fermions interacting with a non-Abelian gauge field with $D_\mu = \partial_\mu - igA_\mu^a T^a$:

$$\mathcal{L} = i\bar{\chi}\gamma^\mu D_\mu\chi - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}. \quad (7.127)$$

This possesses a local gauge symmetry associated to group elements of the form $e^{i\alpha^a T^a}$ and acting as:

$$\begin{aligned}\delta\chi &= i\delta\alpha^a T^a \chi, & \delta\bar{\chi} &= -i\delta\alpha^a T^a \bar{\chi}, \\ \delta A_\mu^a &= g^{-1}\partial_\mu\delta\alpha^a + f^{abc}A_\mu^b\delta\alpha^c.\end{aligned}\tag{7.128}$$

The corresponding covariantly conserved currents are given by:

$$J^{a\mu} = \bar{\chi}\gamma^\mu T^a \chi.\tag{7.129}$$

As before, the theory does not possess any additional and independent global symmetries. At the classical level, the symmetry is present and:

$$D_\mu J^{a\mu} = 0.\tag{7.130}$$

At the quantum level, however, one finds:

$$\langle D_\mu J^{a\mu} \rangle = -\frac{\eta g^2}{32\pi^2} d^{abc} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^b F_{\alpha\beta}^c.\tag{7.131}$$

in terms of the symmetric constants:

$$d^{abc} = \text{symtr} [T^a T^b T^c].\tag{7.132}$$

As already emphasized, gauge anomalies must be avoided, since they ruin the consistency of the theory. Chiral gauge theories can thus be consistent only if the various contributions to gauge anomalies cancel out. Using the general result for the contribution to the anomaly from one fermion in a given representation, one can derive the condition for anomalies to cancel in a general chiral gauge theory. Notice for this that a fermion of chirality η in a representation R is equivalent to a fermion of opposite chirality $-\eta$ in the conjugate representation R^c , and these give thus the same contribution to the anomaly, since $d^{abc}(R) = -d^{abc}(R^c)$. Notice also that a chiral fermion in a real representation does not contribute to the anomaly, because in this case $d^{abc} = 0$. As a consequence, only massless fermions can contribute to the anomaly, since chiral fermions can only have Majorana mass terms and these are gauge invariant only if the representation is real. Finally, anomaly cancellation implies therefore a non-trivial restriction on the allowed spectrum of massless chiral fermions, which concerns only their representations and which does not depend on any continuous parameter:

$$\sum_R \eta_R d^{abc}(R) = 0.\tag{7.133}$$

7.7 General structure of anomalies

In order to describe the general mathematical structure of anomalies, it is convenient to use the formulation of gauge theories in terms of differential forms. A generic Yang-Mills

theory with group G can indeed be described, after rescaling the coupling g into the gauge fields, in terms of the following differential forms:

$$\begin{aligned} A &= A_\mu^a T^a dx^\mu, \\ F &= \frac{1}{2} F_{\mu\nu}^a T^a dx^\mu dx^\nu, \\ J &= J_\mu^a T^a dx^\mu. \end{aligned} \tag{7.134}$$

The covariant derivative can be represented with the help of the exterior derivative d , which when applied to a p -form produces a $(p + 1)$ -form:

$$D = d + [A, \dots], \quad D^2 = [F, \dots]. \tag{7.135}$$

The relation between $F_{\mu\nu}^a$ and A_μ^a implies that the corresponding forms F and A are related by:

$$F = dA + A^2. \tag{7.136}$$

Using the Hodge dual operation $*$, which converts any p -form into a dual $(4 - p)$ -form, the Bianchi identities, the equations of motion and the classical conservation laws can be written as:

$$DF = 0, \quad D^*F = -*J, \quad D^*J = 0. \tag{7.137}$$

For the particular case of an Abelian theory with group $U(1)$, the above formulae simplify, because:

$$D = d, \quad D^2 = 0. \tag{7.138}$$

The field strength is then just:

$$F = dA, \tag{7.139}$$

and the Bianchi identities, equations of motion and classical conservation laws become:

$$dF = 0, \quad d^*F = -*J, \quad d^*J = 0. \tag{7.140}$$

Consider now the theory of a massless Dirac fermion with a generic group G of local gauge symmetries and a minimal $U(1)_5$ global chiral symmetry. This is the prototypical example where anomalies arise. The anomaly is encoded in the following gauge-invariant 4-form constructed out of the 2-form F :

$$\mathcal{A} = -\frac{1}{8\pi^2} \text{tr } F^2. \tag{7.141}$$

The local version of the conservation law takes the form of a deformed continuity equation ruling the flow of charge and can be written as:

$$d^*J_5 = 2\mathcal{A}. \tag{7.142}$$

The integrated version of this conservation law defines instead the total variation of charge between asymptotic past and future:

$$\Delta Q_5 = 2 \int \mathcal{A}. \quad (7.143)$$

Now, it turns out that the anomaly is a closed form, as a consequence of the Bianchi identity satisfied by F :

$$d\mathcal{A} = 0. \quad (7.144)$$

It is also locally exact, in the sense that if F can be expressed in terms of A , then the anomaly can be rewritten as

$$\mathcal{A} = -\frac{1}{8\pi^2} dC, \quad (7.145)$$

in terms of a Chern-Simons 3-form depending on the 1-form A :

$$C = A dA + \frac{2}{3} A^3. \quad (7.146)$$

The local conservation law is violated by any non-zero F . Note that one can define a new current $\tilde{J}_5 = J_5 - 2^*C$ which is conserved: $d^*\tilde{J}_5 = 0$, but this is not gauge invariant. The global conservation law is on the other hand preserved for any F that can be written in terms of an A . But, it is violated by an integer for topologically non-trivial fiber bundles, where F cannot be described by a globally defined A . More precisely, the Atiyah-Singer index theorem states that the integral of \mathcal{A} is the index of the Dirac operator, which counts the difference between the numbers n_L and n_R of its L -handed and R -handed zero-modes:

$$\begin{aligned} \int A &= \dim \ker(i\mathcal{D}P_L) - \dim \ker(i\mathcal{D}P_R) \\ &= n_L - n_R. \end{aligned} \quad (7.147)$$

This difference can be non-zero only in a topologically non-trivial background F , which somehow distinguishes the two chiralities. In fact, it is straightforward to show that the eigenmodes of $i\mathcal{D}$ with non-zero eigenvalues occur in pairs of opposite chiralities and eigenvalues. The index can then be written also as:

$$\begin{aligned} \int A &= \int d^4x \sum_n \psi_n^\dagger(x) \gamma_5 \psi_n(x) \\ &= \text{Tr} [\gamma_5]. \end{aligned} \quad (7.148)$$

From this writing, we see that the path-integral evaluation of the anomaly represents a physicist proof of the Atiyah-Singer theorem.

7.8 Anomalies in the standard model

Global-symmetry anomalies turn out to play a prominent role in the low-energy effective description of the Standard Model. Consider indeed the $SU(3) \times U(1)$ gauge theory of

strong and electromagnetic interactions, in the limit where the $SU(2)$ weak interactions are neglected. This is a vectorial theory, with no axial couplings distinguishing chiralities. Neglecting the masses and the electromagnetic coupling of the first family of quarks $q = (u, d)$, the model has an approximate $SU(2)_L \times SU(2)_R$ global symmetry, rotating independently $q_L = (u_L, d_L)$ and $q_R = (u_R, d_R)$. In the axial-vector nomenclature, we have then the following symmetries and currents:

$$\begin{aligned} SU(2) : J^{a\mu} &= \bar{q}\gamma^\mu\tau^a q && \text{(isospin symmetry)}, \\ SU(2)_5 : J^{a\mu} &= \bar{q}\gamma^\mu\tau^a\gamma_5 q && \text{(chiral symmetry)}. \end{aligned} \quad (7.149)$$

The selection rules associated to the isospin symmetry are observed to indeed approximately hold true, whereas those predicted by the chiral symmetry are not at all. The interpretation of this fact is that the vacuum contains a quark condensate $\langle\bar{q}q\rangle$ and spontaneously breaks this symmetry. The Goldstone bosons are $\bar{q}q'$ bound states identified with the pion triplet $\pi = (\pi^+, \pi^-, \pi^0)$. Since this symmetry is chiral, the approximate Ward identity ruling its consequences is potentially anomalous.

The chiral symmetry currents J_{5a}^μ have non-zero matrix elements between the vacuum and the pion states, which are parametrized as:

$$\langle\Omega|J_5^{a\mu}(0)|\pi^b(p)\rangle = -if_\pi p^\mu\delta^{ab}. \quad (7.150)$$

This implies:

$$\langle\Omega|\partial_\mu J_5^{a\mu}(0)|\pi^b(p)\rangle = f_\pi m_\pi^2\delta^{ab}. \quad (7.151)$$

The approximate operatorial conservation law for the chiral symmetry must therefore be of the type:

$$\partial_\mu J_5^{a\mu} = m_\pi^2 f_\pi \pi^a(x). \quad (7.152)$$

At the quantum level, this conservation law induces an anomalous Ward identity for correlation functions involving the current which is schematically of the following type:

$$\langle\partial_\mu J_5^{a\mu}\rangle = m_\pi^2 f_\pi \pi^a + 2\mathcal{A}^a. \quad (7.153)$$

The first term proportional to m_π encodes the small explicit breaking of the symmetry. The second is a possible anomaly.

To compute the anomalous contribution to the Ward identity, we just need to apply the general results for anomalies, for the case of an $SU(2)_5$ global symmetry interfering with an $SU(3) \times U(1)$ local gauge symmetry. The result is:

$$\mathcal{A}^a(x) = - \sum_{\text{groups}} \frac{g^2}{32\pi^2} d^{aBC} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^B F_{\alpha\beta}^C, \quad (7.154)$$

where in this case:

$$d^{aBC} = \text{tr} [\tau^a T^B T^C]. \quad (7.155)$$

For the $SU(3)$ part, the matrices T^A act on a different space than the matrices τ^a ; the trace factorizes then into two pieces and trivially vanishes: $d^{aAB} = 0$. For the $U(1)$ part, one has $T = \sqrt{2} \text{diag}(2/3, -1/3)$ for each color; this gives a non-zero trace with $\tau^3 = \text{diag}(1/2, -1/2)$: $d^{aQQ} = \delta^{a3}$. Finally, one finds:

$$\mathcal{A}^a(x) = -\delta^{a3} \frac{\alpha_{\text{em}}}{8\pi} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}. \quad (7.156)$$

This means that the conservation law of $J_5^{3\mu}$ corresponding to the π^0 is anomalous, whereas those of $J_5^{1,2\mu}$ corresponding to the π^\pm are regular.

A remarkable implication of this anomaly is that it is the dominant reason for the observed decay $\pi^0 \rightarrow \gamma\gamma$. To see this, we can consider the low-energy effective theory for the pions π^a and the photon γ . According to the derived Ward identities, the effective action $\Gamma[\pi, A]$ must behave under infinitesimal $SU(2)$ and $SU(2)_5$ transformations as:

$$\begin{aligned} \delta\Gamma[\pi, A] &= 0 \\ \delta_5\Gamma[\pi, A] &= -\int d^4x \delta\alpha_5^a(x) \left[m_\pi^2 f_\pi \pi^a(x) + 2\mathcal{A}^a(x) \right]. \end{aligned} \quad (7.157)$$

At the linearized level, the π^a and A_μ fields transforms as follows:

$$\begin{aligned} \delta\pi^a &= \epsilon^{abc} \delta\alpha^b \pi^c, & \delta A_\mu &= 0, \\ \delta_5\pi^a &= \delta\alpha^a f_\pi, & \delta_5 A_\mu &= 0. \end{aligned} \quad (7.158)$$

In order to reproduce the violations of the chiral symmetry by the mass and anomaly terms, $\Gamma[\pi, A]$ must contain the following two breaking terms:

$$\Gamma[\pi, A] \supset \int d^4x \left[-\frac{1}{2} m_\pi^2 \vec{\pi} \cdot \vec{\pi} + \frac{\alpha_{\text{em}}}{4\pi} \frac{\pi^3}{f_\pi} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \right]. \quad (7.159)$$

The second term gives a contribution to the $\pi^0 \rightarrow \gamma\gamma$ decay rate, which is equal to:

$$\Gamma = \frac{\alpha_{\text{em}}^2}{64\pi^3} \frac{m_\pi^3}{f_\pi^2}. \quad (7.160)$$

This turns out to be in very good agreement with experiment, within a few percent of error. This was also one of the first pieces of evidence for the fact that there are 3 colors of quarks.

Local-symmetry anomalies also show up in the Standard Model. Consider now the full $SU(3) \times SU(2) \times U(1)$ theory of strong, weak and electromagnetic interactions, with the 3 generations of all the known quarks and leptons. This is a chiral gauge theory, and it therefore potentially suffers from gauge anomalies. However, it turns out that the contributions from the various matter fermions cancel, within each family. The quantum numbers of a family of quarks and leptons are the following:

$$\begin{aligned} \begin{pmatrix} u_1 & u_2 & u_3 \\ d_1 & d_2 & d_3 \end{pmatrix}_L &: (3, 2)_{1/6}, & \begin{pmatrix} \nu_l \\ l \end{pmatrix}_L &: (1, 2)_{-1/2}, \\ \begin{pmatrix} u_1 & u_2 & u_3 \\ d_1 & d_2 & d_3 \end{pmatrix}_R^c &: (\bar{3}, 1)_{-2/3}, & \nu_R^c &: (1, 1)_0 \\ \begin{pmatrix} u_1 & u_2 & u_3 \\ d_1 & d_2 & d_3 \end{pmatrix}_R^c &: (\bar{3}, 1)_{1/3}, & l_R^c &: (1, 1)_1 \end{aligned} \quad (7.161)$$

The coefficients of the various anomalies are given by:

$$C^{abc} = \sum_R d^{abc}(R) = \sum_R \text{symtr}[T^a T^b T^c]_R. \quad (7.162)$$

A non-trivial anomaly can arise only when there are 0, 2 or 3 non-Abelian generators, and any number of Abelian generators. This leaves 5 potential types of anomalies, which all cancel.

- $SU(3) - SU(3) - SU(3)$
Only triplets contribute: $C \sim 2 - 2 = 0$.
- $SU(2) - SU(2) - SU(2)$
Even doublets do not contribute, because they are real: $C = 0$.
- $U(1) - U(1) - U(1)$
All fields contribute proportionally to the cube of their hypercharge:
 $C \sim 6 \cdot (1/6)^3 + 3 \cdot (-2/3)^3 + 3 \cdot (1/3)^3 + 2 \cdot (-1/2)^3 + (1)^3 = 0$.
- $SU(3) - SU(3) - U(1)$
Only triplets contribute proportionally to their hypercharge, so that
 $C \sim 2 \cdot (1/6) + (-2/3) + (1/3) = 0$.
- $SU(2) - SU(2) - U(1)$
Only doublets contribute proportionally to their hypercharge, so that
 $C \sim 3 \cdot (1/6) + (-1/2) = 0$.

Therefore, all the dangerous gauge anomalies cancel in the standard model. The need for this cancellation led to the prediction of the existence of the t quark, before its discovery. Another remarkable fact is that the gauge group $SU(3) \times SU(2) \times U(1)$ of the Standard Model can be unified in the larger simple group $SO(10)$, with all the $15 + 1$ fermions of a family becoming a single 16 representation. The absence of anomaly cancellation would then follow simply from the fact that the 16 spinorial representation is real.

7.9 Scale anomaly and renormalization group functions

Another important symmetry that is plagued by quantum anomalies is scale invariance. This is a space-time symmetry, which arises for instance in theories possessing only dimensionless couplings, so that the Lagrangian does not involve any characteristic energy or length scale. It is then expected that such a theory is invariant under changes of the absolute definition of energy or length scales. This can be formalized by associating to every field ϕ of the theory a scaling dimension Δ_ϕ , and defining the scale transformation as follows:

$$x^\mu \rightarrow e^{-t} x^\mu, \quad \phi(x) \rightarrow e^{-\Delta_\phi t} \phi(e^{-t} x). \quad (7.163)$$

At the classical level, the Lagrangian of a theory without dimensionful parameters is automatically invariant under such scaling transformations, provided one chooses the scaling

dimension of each field to be equal to its canonical dimension: $\Delta_\phi = d_\phi$. At the infinitesimal level, the above scaling transformations read:

$$\delta x^\mu = -\delta t x^\mu, \quad \delta \phi = -\delta t (\Delta_\phi + x^\mu \partial_\mu) \phi. \quad (7.164)$$

By Nöther's theorem, the presence of such a continuous global symmetry implies the existence of a conserved current S^μ , satisfying $\partial_\mu S^\mu = 0$. This current can be derived by applying Nöther's theorem. Since it is a space-time transformation, the resulting current can be related to the energy momentum tensor, after suitably improving it by adding terms which are automatically conserved and which do not contribute to the integrated charges. More precisely, recall that the canonical energy momentum tensor can be improved to a physical energy-momentum tensor $\theta^{\mu\nu}$, which encodes in an equivalent way the conservation laws associated to translational invariance but has the additional property of being symmetric and gauge invariant. Using this tensor, one can then derive a similar improved version of the angular momentum tensor, which is written simply as $J^{\mu\nu\rho} = \theta^{\mu\nu} x^\rho - \theta^{\mu\rho} x^\nu$ and is automatically conserved as a consequence of the fact that $\theta^{\mu\nu}$ is conserved and symmetric. Similarly, we will see below that the dilatation current associated to scale invariance is given by $S^\mu = \theta^{\mu\nu} x_\nu$, and that it is automatically conserved whenever $\theta^{\mu\nu}$ happens to be traceless.

To derive more precisely the conservation laws related to dilatations, it is convenient to gauge the Poincaré group of global space-time symmetries, with the introduction of the metric field $g_{\mu\nu}$. The Lagrangian is built in such a way to be now invariant under the local version of space-time translations, that is diffeomorphisms:

$$\delta x^\mu = \delta \xi^\mu, \quad \delta g_{\mu\nu} = -\partial_\mu \delta \xi_\nu - \partial_\nu \delta \xi_\mu. \quad (7.165)$$

As already argued in general, one can now derive essentially the same conserved currents that were implied by Nöther's theorem for the global version of the symmetries as the physical currents entering the equations of motions of the new gauging fields as source terms. In this case, this physical currents are identified with the physical symmetric energy-momentum tensor, which is defined by the functional derivative of the action for the original matter fields ϕ with respect to the metric field:

$$\theta^{\mu\nu} = 2 \frac{\delta S_M}{\delta g_{\mu\nu}}. \quad (7.166)$$

With this definition, the energy-momentum tensor $\theta^{\mu\nu}$ is automatically symmetric and invariant under any symmetry of the original non-gauged theory. Moreover, the variation of the action S_M under diffeomorphisms takes the form:

$$\begin{aligned} \delta S_M &= \int d^4x \frac{\delta S_M}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = -\frac{1}{2} \int d^4x \theta^{\mu\nu} (\partial_\mu \delta \xi_\nu + \partial_\nu \delta \xi_\mu) \\ &= \int d^4x \partial_\mu \theta^{\mu\nu} \delta \xi_\nu. \end{aligned} \quad (7.167)$$

The invariance of S_M under diffeomorphisms implies thus that $\theta^{\mu\nu}$ is indeed conserved:

$$\partial_\mu \theta^{\mu\nu} = 0. \quad (7.168)$$

Now, within this gauged theory, the original global scale transformations can be represented as a scale transformation of the metric and the fields, leaving the coordinates unchanged:

$$g_{\mu\nu}(x) \rightarrow e^{2t} g_{\mu\nu}(x), \quad \phi(x) \rightarrow e^{-\Delta_\phi t} \phi(x). \quad (7.169)$$

At the infinitesimal level, this means:

$$\delta g_{\mu\nu} = 2 \delta t g_{\mu\nu}, \quad \delta \phi \rightarrow -\delta t \Delta_\phi \phi. \quad (7.170)$$

The transformation $\delta g_{\mu\nu} = 2 \delta t g_{\mu\nu}$ on the metric is similar to the one that would be associated to a diffeomorphism with $\xi^\mu = -\delta t x^\mu$. However, only the metric transforms and the coordinates remain constant, so that distances are truly changed by the scale transformation, contrarily to what would happen for a diffeomorphism. Note also that one should eventually set $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ to recover the original situation without gravity. As before, the variation of S_M comes entirely from the variation of $g_{\mu\nu}$, since it is stationary with respect to any variation of the fields ϕ due to their equation of motion. One finds then simply:

$$\begin{aligned} \delta S_M &= \int d^4 x \frac{\delta S_M}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \\ &= \int d^4 x \theta_\mu^\mu \delta t. \end{aligned} \quad (7.171)$$

The invariance of S_M under scale transformations implies thus that:

$$\theta_\mu^\mu = 0. \quad (7.172)$$

This can finally be rewritten as an ordinary continuity equation,

$$\partial_\mu S^\mu = 0, \quad (7.173)$$

for a dilatation current which can be identified with:

$$S^\mu = \theta^{\mu\nu} x_\nu. \quad (7.174)$$

As an interesting example of scale invariant theory, let us consider for instance a generic gauge theory with massless Dirac fermions. The Lagrangian is given by:

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} + i\bar{\psi} \not{D} \psi. \quad (7.175)$$

In this case, the physical energy-momentum tensor is found to be:

$$\theta^{\mu\nu} = F^{a\mu\rho} F_\rho^{a\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma}^a F^{a\rho\sigma} + \frac{i}{2} \bar{\psi} \left(\gamma^\mu D^\nu + \gamma^\nu D^\mu - 2\eta^{\mu\nu} \not{D} \right) \psi. \quad (7.176)$$

Using the Dirac equation of motion, one can easily verify that this is indeed traceless:

$$\theta_\mu^\mu = 0. \quad (7.177)$$

Let us now see in this example what happens to scale invariance at the quantum level. As already anticipated, the need for regularizing the theory unavoidably implies the introduction of a finite energy scale. As a result, the regularization needed to renormalize the theory cannot manifestly preserve the scaling symmetry, and there is in general an anomaly that emerges:

$$\langle \theta_\mu^\mu \rangle = 2\mathcal{A}. \quad (7.178)$$

The precise way in which the anomaly arises depends again on the formalism and on the regularization that are used. For simplicity, we will use here a perturbative diagrammatic approach and dimensional regularization. The subtlety is then coming from the fact that in $d \neq 4$ the Lagrangian is no longer scale invariant, because the gauge coupling becomes dimensionful, and this can be cured only by introducing some reference energy scale $\tilde{\mu}$. Correspondingly, when computing θ_μ^μ one finds a non-zero result. Indeed, using again the Dirac equation one obtains $\theta_\mu^\mu = (d-4)/4 F_{\mu\nu}^a F^{a\mu\nu}$. The anomaly is then given by the 1-loop matrix element of this quantity:

$$\mathcal{A} = \frac{d-4}{8} \langle F_{\mu\nu}^a F^{a\mu\nu} \rangle. \quad (7.179)$$

When $d = 4 - 2\epsilon$ and ϵ is taken to be small, the numerical prefactor in this expression vanishes like ϵ , but the correlation function has a UV divergence which translates into a $1/\epsilon$ pole, and a finite result is left in the limit $\epsilon \rightarrow 0$:

$$\mathcal{A} = -\frac{\epsilon}{4} \langle F_{\mu\nu}^a F^{a\mu\nu} \rangle. \quad (7.180)$$

The required matrix element can be computed in a straightforward way by recalling the background gauge computation of the leading quadratic term in the quantum effective action for gauge theories that was done in Chapter 5. Indeed, it is given by

$$\langle F_{\mu\nu}^a F^{a\mu\nu} \rangle = \frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} A_\mu^a(-k) \pi^{ab\mu\nu}(k^2) A_\nu^b(k), \quad (7.181)$$

in terms of the 1-loop 2-point function of gauge fields in the background gauge, which has the form:

$$\pi^{ab\mu\nu}(k^2) = i(k^2 \eta^{\mu\nu} - k^\mu k^\nu) \delta^{ab} \left(\frac{b}{(4\pi)^2} \frac{1}{\epsilon} + \text{finite} \right). \quad (7.182)$$

Plugging this result back into the expression for the anomaly, one finds a finite term for $\epsilon \rightarrow 0$, which is given by:

$$\mathcal{A} = \frac{b}{4(4\pi)^2} F_{\mu\nu}^a F^{a\mu\nu}. \quad (7.183)$$

The numerical coefficient b , which controls the logarithmic divergence giving rise to the $1/\epsilon$ pole in the 2-point function, is given by:

$$b = -\frac{11}{3} C(G) + \frac{4}{3} C(R). \quad (7.184)$$

This same coefficient also controls the β function of the gauge coupling, which as we have seen in Chapter 5 is given by:

$$\beta(g) = \frac{b g^3}{(4\pi)^2}. \quad (7.185)$$

Finally, the anomalous Ward identity for scale invariance can therefore be written in terms of the beta function of the gauge coupling as:

$$\langle \theta_\mu^\mu \rangle = \frac{\beta(g)}{2g^3} F_{\mu\nu}^a F^{a\mu\nu}. \quad (7.186)$$

The above result shows that the coefficient of the quantum anomaly plaguing scale invariance is controlled by the β function of the gauge coupling. This shows that the origin of this effect is simply the fact that at the quantum level the gauge coupling is not a dimensionless constant, but a function of the renormalization scale μ , with a scaling dimension that is controlled by the β function. Indeed, the β function encodes by definition the rate of change of the coupling under a rescaling $\mu \rightarrow \mu + \delta\mu$ of the renormalization scale:

$$\delta g = \beta(g) \frac{\delta\mu}{\mu}. \quad (7.187)$$

Now, according to our previous definitions the scale symmetry acts as $x^\mu \rightarrow e^{-t} x^\mu$ on length scales, and must therefore act as $\mu \rightarrow e^t \mu$ on energy scales. This means that $\delta\mu = \mu \delta t$ and finally one deduces that:

$$\delta g = \beta(g) \delta t. \quad (7.188)$$

This effect implies a new term in the variation of the quantum effective action Γ_M , coming from the variation of the coupling in the gauge kinetic term. As a result, the scale invariance of the renormalized effective action does not imply that $\langle \theta_\mu^\mu \rangle$ vanishes, but rather that it compensates the variation coming from the running of the coupling. A straightforward computation shows that indeed this reproduces the result that we have just computed:

$$\begin{aligned} \langle \theta_\mu^\mu \rangle &= \frac{\delta \Gamma_M}{\delta g} \frac{\delta g}{\delta t} = \frac{\partial}{\partial g} \left(-\frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} \right) \beta(g) \\ &= \frac{\beta(g)}{2g^3} F_{\mu\nu}^a F^{a\mu\nu}. \end{aligned} \quad (7.189)$$

Since the β function gets corrected order by order in the perturbation expansion for weak coupling g , the scale anomaly receives corrections at all loop orders. This is in contrast with what happens for chiral and gauge anomalies, where the anomaly receives contributions only at the 1-loop level.

From the study of the above example of gauge theories, it is now clear how to write down the basic Ward identity for scale invariance at the quantum level also for more general

theories involving any kind of dimensionless coupling λ , in terms of the corresponding β function β_λ . It is just:

$$\langle \theta_\mu^\mu \rangle = \beta_\lambda \frac{\delta \Gamma_M}{\delta \lambda}. \quad (7.190)$$

This equation expresses the fact that the quantum coupling acquires, besides its vanishing classical scaling dimension, a non-vanishing anomalous scaling dimension given by its β function divided by itself:

$$\Delta_\lambda = d_\lambda + \frac{d \log \lambda}{dt} = 0 + \frac{\beta_\lambda}{\lambda}. \quad (7.191)$$

Similarly, it is now natural to guess that at the quantum level also the fields ϕ acquire, besides their classical scaling dimension, an anomalous scaling dimension given by the associated γ function γ_ϕ :

$$\Delta_\phi = d_\phi + \frac{d \log \sqrt{Z_\phi}}{dt} = d_\phi + \gamma_\phi. \quad (7.192)$$

The fact that these expectations are correct can now be confirmed by deriving the Callan-Symanzik equations satisfied by correlation functions as Ward-identities with field insertions for this quantum scale invariance concerning correlation functions involving additional fields, which involve the anomalous dimensions of both the couplings and the fields. To show this, let us start from the Ward identity associated to scale invariance, for correlation functions involving n fields ϕ and integrated over the position of the current. Applying the general formula with $\theta_\mu^\mu = \partial_\mu S^\mu$, this reads:

$$\sum_k \langle \phi(x_1) \cdots \frac{\delta \phi}{\delta t}(x_k) \cdots \phi(x_n) \rangle = i \int d^4 x \langle \theta_\mu^\mu(x) \phi(x_1) \cdots \phi(x_n) \rangle. \quad (7.193)$$

The left hand side of this relation can be rewritten more explicitly by using the form of the scale transformation law on the fields: $\delta \phi / \delta t = -(\Delta_\phi + x_k^\mu \partial / \partial x_k^\mu) \phi$. The right-hand side, on the other hand, is a bit more subtle to evaluate precisely. In there were no divergences, it would vanish. But as in the simplest case without field insertions studied above, one finds actually a finite result after regularization, due to the fact that there are logarithmic divergences. In fact, one can argue as before that the source of this anomaly is due to the fact that the couplings acquire a non-trivial scale dependence after renormalization. This suggests that the insertion of $i \int d^4 x \theta_\mu^\mu$ in the correlation function should give the variation of the remaining correlation function induced by the quantum scale transformation law of the couplings, which is extracted by acting with the operator $\delta \lambda / \delta t \partial / \partial \lambda = \beta_\lambda \partial / \partial \lambda$. One is then led to the following equation:

$$- \sum_k \left(\Delta_\phi + x_k^\mu \frac{\partial}{\partial x_k^\mu} \right) \langle \phi(x_1) \cdots \phi(x_n) \rangle = \beta_\lambda \frac{\partial}{\partial \lambda} \langle \phi(x_1) \cdots \phi(x_n) \rangle. \quad (7.194)$$

One can then rewrite the Ward identity as a differential equation for the n -point correlation function:

$$\left(\sum_k x_k^\mu \frac{\partial}{\partial x_k^\mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + n \Delta_\phi \right) G^{(n)}(x_1, \dots, x_n; \lambda, \mu) = 0. \quad (7.195)$$

Finally, we can use usual dimensional analysis to rewrite this equation in a more familiar form. Indeed, the canonical dimension of $G^{(n)}$ is n times the canonical dimension d_ϕ of the involved fields ϕ , and the relative dependence on the dimensionful parameters x_k and μ is therefore constrained as follows:

$$\left(\sum_k x_k^\mu \frac{\partial}{\partial x_k^\mu} - \mu \frac{\partial}{\partial \mu} + n d_\phi \right) G^{(n)}(x_1, \dots, x_n; \lambda, \mu) = 0. \quad (7.196)$$

Using this relation we can then trade $\sum_k x_k^\mu \partial/\partial x_k^\mu$ with $\mu \partial/\partial \mu - n d_\phi$, and rewrite the Ward identity as:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + n(\Delta_\phi - d_\phi) \right) G^{(n)}(x_1, \dots, x_n; \lambda, \mu) = 0. \quad (7.197)$$

We see then that by identifying the anomalous dimension as $\gamma_\phi = \Delta_\phi - d_\phi$ one indeed reproduces the Callan-Symanzik equation:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + n\gamma_\phi \right) G^{(n)}(x_1, \dots, x_n; \lambda, \mu) = 0. \quad (7.198)$$

Summarizing, scale invariance is generically realized in an anomalous way in quantum field theories, and this corresponds to the running of couplings and wave-function factors with the energy scale. More precisely, the β functions for the couplings represent genuine quantum anomalies, which modify the structure of the conservation law, whereas the γ functions for the fields represent a deviation of their scaling dimensions with respect to the canonical dimensions. Theories with vanishing β are therefore scale invariant also at the quantum level, in the same sense as in the classical limit. Theories with a non-vanishing β , on the other hand, can recover a classical scaling behavior at critical points where β vanishes, but with scaling dimensions for the fields that differ non-trivially from their canonical dimensions.

8 Thermal field theory

8.1 Quantum statistical mechanics

Before attempting to set up a formalism for quantum field theory at finite temperature, it is useful to briefly recall some basic results in quantum statistical mechanics. We will focus here on the description of systems in thermal equilibrium with the canonical ensemble. The basic object to consider is then the density matrix

$$\rho(\beta) = e^{-\beta H}, \quad (8.1)$$

depending on the inverse temperature:

$$\beta = \frac{1}{T}. \quad (8.2)$$

The thermal average of any observable A is then given by:

$$\langle A \rangle_\beta = \frac{\text{Tr}[\rho(\beta)A]}{\text{Tr}[\rho(\beta)]}. \quad (8.3)$$

Similarly, the thermal average of the correlation function of any string of observables A, B, \dots is given by:

$$\langle AB \dots \rangle_\beta = \frac{\text{Tr}[\rho(\beta)AB \dots]}{\text{Tr}[\rho(\beta)]}. \quad (8.4)$$

The partition function is defined as the trace of the density matrix:

$$Z(\beta) = \text{Tr}[\rho(\beta)] = \text{Tr}[e^{-\beta H}], \quad (8.5)$$

and the corresponding free energy is:

$$F(\beta) = -\frac{1}{\beta} \log Z(\beta). \quad (8.6)$$

The first basic case is that of a single-particle quantum state of energy ω . Considering then a system of many identical particles of this type, the level can be occupied by a number n of them. This corresponds to a state $|n\rangle$ with energy $n\omega$. For bosonic particles, $n = 0, 1, 2, \dots$ is arbitrary and the partition function is given by the Bose-Einstein distribution:

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-n\beta\omega} = \frac{1}{1 - e^{-\beta\omega}} = N_B(\beta, \omega). \quad (8.7)$$

For fermionic particles, on the other hand, $n = 0, 1$ is limited by the exclusion principle and the partition function is given by the Fermi-Dirac distribution:

$$Z(\beta) = \sum_{n=0}^1 e^{-n\beta\omega} = 1 + e^{-\beta\omega} = N_F(\beta, \omega). \quad (8.8)$$

The simplest concrete system behaving in this simple way is the harmonic oscillator of characteristic frequency ω . The excitation quanta are created and destroyed by operators a and a^\dagger satisfying $[a, a^\dagger] = 1$. The Hamiltonian is $H = \omega(N \pm 1/2)$, in terms of the number operator $N = a^\dagger a$. The generic excited state is then $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$, and has energy $(n \pm 1/2)\omega$ and excitation number n . From the correct choice of commutators and anticommutators, one automatically obtains the correct range of n in the bosonic and fermionic cases. But due to the zero-point energy, the partition function slightly differs from the above basic distributions. One finds $Z(\beta) = e^{\mp\beta\omega/2}N_{B,F}(\beta, \omega)$, that is:

$$Z_{\text{osc}}(\beta) = e^{\mp\beta\omega/2}(1 \mp e^{-\beta\omega})^{\mp 1}. \quad (8.9)$$

Another simple system is a gas of non-interacting particles, with some given dispersion relation $\omega = \omega(\vec{k})$. Putting the system in a box of size L , the momenta get quantized in units of $2\pi/L$: $\vec{k} = 2\pi\vec{r}/L$. The infinitely many independent quantum states can then be labelled by a triple of integers \vec{r} , with $\omega_{\vec{r}} = \omega(k_{\vec{r}})$. Each such state taken on its own would result in a partition function given by the appropriate basic statistical distribution $N_{B,F}(\beta, \omega_{\vec{r}})$ for the corresponding energy $\omega_{\vec{r}}$. Now, different such levels represent distinct systems which are not directly interacting. As a result, the total partition function is simply the product of all the partition functions for these modes: $Z(\beta) = \prod_{\vec{r}} N(\beta, \omega_{\vec{r}})$. This means that in the logarithm of the partition function, which gives the free energy, all these contributions simply add up: $\log Z(\beta) = \sum_{\vec{r}} \log N(\beta, \omega_{\vec{r}})$. Finally, one can now take the infinite volume limit and convert the sum $\sum_{\vec{r}}$ into an integral and one finds then $\log Z(\beta) = V \int d^3\vec{k}/(2\pi)^3 \log N(\beta, \omega(\vec{k}))$, that is:

$$\log Z_{\text{gas}}(\beta) = \mp V \int \frac{d^3\vec{k}}{(2\pi)^3} \log(1 \mp e^{-\beta\omega(\vec{k})}). \quad (8.10)$$

8.2 Finite temperature and Euclidean space

From the definition of the density matrix and the way thermal averages are computed with it, it is clear that there is a strong and direct analogy with the time evolution operator in quantum mechanics and the correlation functions that are computed with it. More precisely, we see that the problem of computing ordinary quantum correlation functions can be mapped to the problem of computing thermal averages for the same system at thermal equilibrium by making an analytic continuation to imaginary times, $t \rightarrow -i\tau$, and replacing the asymptotically large range $t \in [-T/2, T/2]$ with the finite range $\tau \in [0, \beta]$. This correspondence holds true independently of the number of degrees of freedom, and should thus work equally well in quantum mechanics, where a finite number of degrees of freedom are involved, and in quantum field theory, where infinitely many degrees of freedom arise.

Recall that in ordinary dynamical situations, the basic object defining correlation functions is the vacuum-to-vacuum evolution amplitude over a time T . In order to auto-

matically select the vacuum state, one takes the limit of large time T . One has then:

$$\begin{aligned} Z &= \langle \Omega | U(T) | \Omega \rangle \\ &= \int \mathcal{D}\phi \exp \left\{ i \int_{-T/2}^{T/2} dt L(t) \right\}. \end{aligned} \quad (8.11)$$

The corresponding vacuum energy is given by:

$$F = \frac{i}{T} \log Z. \quad (8.12)$$

Correlation functions are then computed as:

$$\begin{aligned} \langle \phi(x_1) \cdots \phi(x_n) \rangle &= \frac{\langle \Omega | T \phi(x_1) \cdots \phi(x_n) U(T) | \Omega \rangle}{\langle \Omega | U(T) | \Omega \rangle} \\ &= \frac{\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) \exp \left\{ i \int_{-T/2}^{T/2} dt L(t) \right\}}{\int \mathcal{D}\phi \exp \left\{ i \int_{-T/2}^{T/2} dt L(t) \right\}}. \end{aligned} \quad (8.13)$$

In thermal equilibrium situations, on the other hand, the basic object defining correlation functions is the trace of the evolution operator over Euclidean time β over all the states $|N\rangle$ of the exact theory:

$$\begin{aligned} Z(\beta) &= \sum_N \langle N | U(\beta) | N \rangle \\ &= \int_{\text{p.b.c.}} \mathcal{D}\phi \exp \left\{ - \int_0^\beta d\tau L_E(\tau) \right\}. \end{aligned} \quad (8.14)$$

The corresponding free energy is then computed as:

$$F(\beta) = -\frac{1}{\beta} \log Z(\beta). \quad (8.15)$$

Thermal correlation functions of observables can then be computed as ordinary correlation functions, but with a periodic Euclidean time. Physically relevant observables for a system in thermal equilibrium are of course time-independent, but one can nevertheless define more generally the correlation function of a string of fields at arbitrary Euclidean time:

$$\begin{aligned} \langle \phi(x_1) \cdots \phi(x_n) \rangle_\beta &= \frac{\sum_N \langle N | T \phi(x_1) \cdots \phi(x_n) U(\beta) | N \rangle}{\sum_N \langle N | U(\beta) | N \rangle} \\ &= \frac{\int_{\text{p.b.c.}} \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) \exp \left\{ - \int_0^\beta d\tau L_E(\tau) \right\}}{\int_{\text{p.b.c.}} \mathcal{D}\phi \exp \left\{ - \int_0^\beta d\tau L_E(\tau) \right\}}. \end{aligned} \quad (8.16)$$

8.3 Matsubara modes

The fact that finite temperature correlation functions are defined by a trace over the whole set of states of the theory restricts the fields to have periodic boundary conditions along the finite Euclidean time interval $\tau \in [0, \beta]$. More precisely, $\phi(\beta)$ should be equivalent to $\phi(0)$, in the sense that these two operators should create the same states. We will now see that this implies that:

$$\phi(\beta) = \pm\phi(0), \quad (8.17)$$

where the two signs apply respectively to bosons and fermions. The negative sign for fermions has the same origin as the sign flip one gets on fermions after a rotation of 2π around any axis. It has no observable effect on the states created by the fields, since only fermion bilinears are observables. However, we will see that it has important consequences on the behavior of correlation functions.

To derive in a systematic way the above boundary conditions, consider the propagator defined by the 2-point function at finite temperature over a time τ :

$$\Delta_\beta(\vec{x}, \vec{y}; \tau, 0) = \langle \phi(\vec{x}, \tau) \phi(\vec{y}, 0) \rangle_\beta = Z(\beta)^{-1} \text{Tr} [T \phi(\vec{x}, \tau) \phi(\vec{y}, 0) U(\beta)], \quad (8.18)$$

$$\Delta_\beta(\vec{x}, \vec{y}; \tau, \beta) = \langle \phi(\vec{x}, \tau) \phi(\vec{y}, \beta) \rangle_\beta = Z(\beta)^{-1} \text{Tr} [T \phi(\vec{x}, \tau) \phi(\vec{y}, \beta) U(\beta)]. \quad (8.19)$$

By using the properties of the Euclidean time evolution operator $U(\beta)$ and the cyclicity of the trace one finds the following Kubo-Martin-Schwinger relation:

$$\begin{aligned} \text{Tr} [\phi(\vec{x}, \tau) \phi(\vec{y}, 0) U(\beta)] &= \text{Tr} [\phi(\vec{x}, \tau) U(\beta) U(-\beta) \phi(\vec{y}, 0) U(\beta)] \\ &= \text{Tr} [\phi(\vec{x}, \tau) U(\beta) \phi(\vec{y}, \beta)] \\ &= \text{Tr} [\phi(\vec{y}, \beta) \phi(\vec{x}, \tau) U(\beta)]. \end{aligned} \quad (8.20)$$

Moreover, recalling the usual definition of time-ordered product for bosons and fermions, one has

$$T \phi(\vec{x}, \tau) \phi(\vec{y}, 0) = \phi(\vec{x}, \tau) \phi(\vec{y}, 0), \quad T \phi(\vec{x}, \tau) \phi(\vec{y}, \beta) = \pm \phi(\vec{y}, \beta) \phi(\vec{x}, \tau). \quad (8.21)$$

It follows then that:

$$\Delta_\beta(\vec{x}, \vec{y}; \tau, \beta) = \pm \Delta_\beta(\vec{x}, \vec{y}; \tau, 0). \quad (8.22)$$

This derivation proves the advocated periodicity of bosonic fields and antiperiodicity of fermionic fields.

It is now clear that the only difference between ordinary dynamical situations and finite temperature thermal equilibrium is that one has to go to compact Euclidean time, and impose periodic and antiperiodic boundary conditions for bosons and fermions on the time interval β . This will result in a discretization of the allowed frequencies ω for generic off-shell propagation, in units of the basic Matsubara frequency π/β . This means that the Fourier transform of the propagators becomes a series for the Euclidean time coordinate, whereas it remains an integral for the spatial coordinates:

$$\Delta_\beta(\vec{x}, \tau) = \frac{1}{\beta} \sum_n \int \frac{d^3 \vec{k}}{(2\pi)^3} e^{i(\vec{k} \cdot \vec{x} + \omega_n \tau)} \Delta_\beta(\vec{k}, \omega_n). \quad (8.23)$$

The form of the propagator in Fourier space is unchanged compared to the ordinary case, because the Green-function differential equation satisfied by the propagator is local. The periodic and antiperiodic boundary conditions imply on the other hand even frequencies for bosons and odd frequencies for fermions. More precisely, one finds:

$$\text{bosons : } \Delta_{\beta}(\vec{k}, \omega_n) = \frac{1}{\vec{k}^2 + \omega_n^2 + m^2}, \quad \omega_n = \frac{2n\pi}{\beta}, \quad (8.24)$$

$$\text{fermions : } \Delta_{\beta}(\vec{k}, \omega_n) = \frac{\vec{\gamma}_E \cdot \vec{k} + \gamma_E^0 \omega_n - m}{\vec{k}^2 + \omega_n^2 + m^2}, \quad \omega_n = \frac{(2n+1)\pi}{\beta}. \quad (8.25)$$

We see from the above results that in the limit of vanishing temperature, namely $\beta \rightarrow \infty$, the thermodynamical formulation reduces simply to the analytic continuation to Euclidean space of the ordinary dynamical formulation. The basic Matsubara frequency π/β becomes very small and the discrete spectrum of frequencies degenerates to a continuous one.

Another important observation is that the presence of a finite temperature influences the behavior of the theory only in the IR, at scales below the temperature. In the far UV, indeed, the discretization of the frequencies becomes negligible and one has exactly the same behavior as at zero temperature. This implies in particular that the structure of divergences is independent of the temperature. One can then set-up renormalized perturbation theory at finite temperature exactly in the same way as at zero temperature.

An important novelty arising at finite temperature is that the structure of imaginary parts of the amplitude gets modified. More precisely, new cuts arise in the amplitudes, due to the fact that the vacuum state is now actually not the empty vacuum without real particles, but really a thermal bath containing real particle. These can then join to the asymptotic state of a scattering process to give new contributions.

8.4 Free energy of free fields

As a simple application of the formalism, let us consider the computation of the free energy associated to free bosonic and fermionic quantum fields. We know that a free relativistic quantum field corresponds to an infinite collection of harmonic oscillators labelled by the momentum \vec{k} with characteristic frequency determined by the relativistic dispersion relation

$$\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}. \quad (8.26)$$

We expect then to reproduce the free energy of a gas of free particles, but with an extra term corresponding to the zero-point energies of the oscillators. Using the path-integral definition of the partition function, it is straightforward to show that this is indeed the result that one obtains by applying the imaginary time formalism.

Consider first the simplest case of a free massive complex boson, with a Lagrangian given by:

$$\mathcal{L} = \partial_{\mu} \phi^* \partial^{\mu} \phi - m^2 \phi^* \phi. \quad (8.27)$$

The corresponding Euclidean action can be rewritten, after integration by parts, in the form:

$$S_E = \int_0^\beta d\tau \int d^3\vec{x} \left(|\partial_\tau \phi|^2 + |\vec{\nabla} \phi|^2 + m^2 |\phi|^2 \right). \quad (8.28)$$

The partition function is then given by a Gaussian path-integral:

$$\begin{aligned} Z(\beta) &= \int_{\mathbb{P}} \mathcal{D}\phi \int_{\mathbb{P}} \mathcal{D}\phi^* e^{-S_E[\beta, \phi, \phi^*]} \\ &= \det_{\mathbb{P}}^{-1} \left(-\partial_\tau^2 - \vec{\nabla}^2 + m^2 \right). \end{aligned} \quad (8.29)$$

Using the relation $\log \det M = \text{Tr} \log M$, the free energy $F(\beta) = -1/\beta \log Z(\beta)$ is finally found to be:

$$F(\beta) = \frac{1}{\beta} \text{Tr}_{\mathbb{P}} \log \left(-\partial_\tau^2 - \vec{\nabla}^2 + m^2 \right). \quad (8.30)$$

To properly compute these formal expressions, it is convenient to put the system in a box of finite size L . The fields can then be decomposed in Fourier modes with discrete energies $\omega_n = 2n\pi/\beta$ and discrete momenta $\vec{k}_{\vec{r}} = 2\pi\vec{r}/L$, as:

$$\phi(\vec{x}, \tau) = \sqrt{\frac{\beta}{V}} \sum_{n, \vec{r}} e^{i(\vec{k}_{\vec{r}} \cdot \vec{x} + \omega_n \tau)} \phi_{n, \vec{r}}. \quad (8.31)$$

Plugging this back into the action and performing the integrals over time and space, one finds that:

$$S_E = \sum_{n, \vec{r}} \phi_{n, \vec{r}} \left[\beta^2 (\omega_n^2 + \omega_{\vec{r}}^2) \right] \phi_{n, \vec{r}}. \quad (8.32)$$

The measure of the functional integral is now:

$$\mathcal{D}\phi \mathcal{D}\phi^* = \prod_{n, \vec{r}} d\phi_{n, \vec{r}} d\phi_{n, \vec{r}}^*. \quad (8.33)$$

Performing the Gaussian integral for each mode, one finds

$$Z = \mathcal{N} \prod_{n, \vec{r}} \left[\beta^2 (\omega_n^2 + \omega_{\vec{r}}^2) \right]^{-1}. \quad (8.34)$$

The β -independent multiplicative constant \mathcal{N} is irrelevant, since it drops from all the thermal averages. One can then fix it to the value $\mathcal{N} = \prod_{n', \vec{r}} \beta^2 \omega_{n'}^2$, where $n' \neq 0$, in such a way to be left with converging products. This gives:

$$\begin{aligned} Z &= \prod_{\vec{r}} \left[\beta^2 \omega_{\vec{r}}^2 \prod_{n'} \left(1 + \frac{\omega_{\vec{r}}^2}{\omega_{n'}^2} \right) \right]^{-1} = \prod_{\vec{r}} \left[(\beta \omega_{\vec{r}})^2 \prod_{n=1}^{\infty} \left(1 + \left(\frac{\beta \omega_{\vec{r}}}{2n\pi} \right)^2 \right)^2 \right]^{-1} \\ &= \prod_{\vec{r}} \left[2 \sinh \frac{\beta \omega_{\vec{r}}}{2} \right]^{-2}. \end{aligned} \quad (8.35)$$

This has as expected the form of an infinite product of partition functions for bosonic oscillators. For the corresponding free energy, one arrives at:

$$F(\beta) = \frac{2}{\beta} \sum_{\vec{r}} \log \left[2 \sinh \frac{\beta \omega_{\vec{r}}}{2} \right]. \quad (8.36)$$

Taking finally the limit of large volume, and decomposing the hyperbolic sine, this result becomes:

$$F(\beta) = 2V \int \frac{d^3 \vec{k}}{(2\pi)^3} \left[\frac{\omega(\vec{k})}{2} + \frac{1}{\beta} \log \left(1 - e^{-\beta \omega(\vec{k})} \right) \right]. \quad (8.37)$$

This has the expected form. The overall factor of 2 is due to the fact that there are two independent kinds of degrees of freedom: the particles and the anti-particles.

Consider next the slightly more complicated case of a free massive Dirac fermion, with a Lagrangian given by:

$$\mathcal{L} = i\bar{\psi} \not{\partial} \psi - m\bar{\psi} \psi. \quad (8.38)$$

The corresponding Euclidean action is:

$$S_E = \int_0^\beta d\tau \int d^3 \vec{x} \bar{\psi} \left(-i\gamma_E^0 \partial_\tau - i\vec{\gamma}_E \cdot \vec{\nabla} + m \right) \psi. \quad (8.39)$$

The partition function is then given by a Gaussian path-integral:

$$\begin{aligned} Z(\beta) &= \int_{\mathbb{A}} \mathcal{D}\psi \int_{\mathbb{A}} \mathcal{D}\bar{\psi} e^{-S_E[\beta, \psi, \bar{\psi}]} \\ &= \det_{\mathbb{A}} \left(-i\gamma_E^0 \partial_\tau - i\vec{\gamma}_E \cdot \vec{\nabla} + m \right). \end{aligned} \quad (8.40)$$

Using again the relation $\log \det M = \text{Tr} \log M$, one can write the corresponding free energy $F(\beta) = -1/\beta \log Z(\beta)$ as:

$$F(\beta) = -\frac{1}{\beta} \text{Tr}_{\mathbb{A}} \log \left(-i\gamma_E^0 \partial_\tau - i\vec{\gamma}_E \cdot \vec{\nabla} + m \right). \quad (8.41)$$

To compute these expressions, we put as before the system in a box of finite size L . The fields can then be decomposed in Fourier modes with discrete energies $\omega_n = (2n+1)\pi/\beta$ and discrete momenta $\vec{k}_{\vec{r}} = 2\pi\vec{r}/L$, as:

$$\psi(\vec{x}, \tau) = \frac{1}{\sqrt{V}} \sum_{n, \vec{r}} e^{i(\vec{k}_{\vec{r}} \cdot \vec{x} + \omega_n \tau)} \psi_{n, \vec{r}}. \quad (8.42)$$

Plugging this back into the action and performing the integrals over time and space, one finds that:

$$S_E = \sum_{n, \vec{r}} \bar{\psi}_{n, \vec{r}} \left[\beta (\gamma_E^0 \omega_n + \vec{\gamma}_E \cdot \vec{k}_{\vec{r}} + m) \right] \psi_{n, \vec{r}}. \quad (8.43)$$

The measure of the functional integral is now:

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} = \prod_{n, \vec{r}} d\psi_{n, \vec{r}} d\bar{\psi}_{n, \vec{r}}. \quad (8.44)$$

Performing the Gaussian integral for each mode, one finds

$$\begin{aligned} Z &= \mathcal{N} \prod_{n, \vec{r}} \det \left[\beta (\gamma_E^0 \omega_n + \vec{\gamma}_E \cdot \vec{k}_{\vec{r}} + m) \right] \\ &= \mathcal{N} \prod_{n, \vec{r}} \left[\beta^2 (\omega_n^2 + \omega_{\vec{r}}^2) \right]^2. \end{aligned} \quad (8.45)$$

The β -independent multiplicative constant \mathcal{N} is as before irrelevant, and it is convenient to fix it to the value $\mathcal{N} = 16 (\prod_{n, \vec{r}} \beta^4 \omega_n^4)^{-1}$. This gives:

$$\begin{aligned} Z &= \prod_{\vec{r}} \left[4 \prod_n \left(1 + \frac{\omega_{\vec{r}}^2}{\omega_n^2} \right) \right]^2 = \prod_{\vec{r}} \left[4 \prod_{n=-\infty}^{\infty} \left(1 + \left(\frac{\beta \omega_{\vec{r}}}{(2n+1)\pi} \right)^2 \right) \right]^2 \\ &= \prod_{\vec{r}} \left[2 \cosh \frac{\beta \omega_{\vec{r}}}{2} \right]^4. \end{aligned} \quad (8.46)$$

This has as expected the form of an infinite product of partition functions for fermionic oscillators. For the corresponding free energy, one arrives at:

$$F(\beta) = -\frac{4}{\beta} \sum_{\vec{r}} \log \left[2 \cosh \frac{\beta \omega_{\vec{r}}}{2} \right]. \quad (8.47)$$

Taking finally the limit of large volume, and decomposing the hyperbolic cosine, this result becomes:

$$F(\beta) = -4V \int \frac{d^3 \vec{k}}{(2\pi)^3} \left[\frac{\omega(\vec{k})}{2} + \frac{1}{\beta} \log \left(1 + e^{-\beta \omega(\vec{k})} \right) \right]. \quad (8.48)$$

This has again the expected form. The overall factor of 4 is due to the fact that there are four independent kinds of degrees of freedom: the particles and the anti-particles, each with their two independent spin polarizations.

8.5 Finite temperature effective potential

For a generic interacting theory, the generating functional of correlation functions $Z[J]$ in Euclidean space with periodic time $\tau \in [0, \beta]$ corresponds to the partition function $Z[\beta, J]$ in the presence of an external field J coupling to the field variables. The generating functional $W[J]$ of connected correlation functions corresponds then to the Helmholtz free energy functional $F[\beta, J] = -1/\beta \log Z[\beta, J]$. Finally, the quantum effective action $\Gamma[\phi_{\text{cl}}]$ obtained by Legendre transform corresponds to the Gibbs free energy functional defined as $G[\beta, \phi_{\text{cl}}] = F[\beta, J] + \int J \phi_{\text{cl}}$. This means that we can extend the definition of effective action and effective potential to finite temperature, and use these quantities to determine by the least action principle the mean field associated to the equilibrium state of the theory at finite temperature. This allows in particular to generalize the computation of the free energy of free field theories to interacting field theories, by simply extending in the obvious way all the computational machinery concerning the quantum effective potential.

The prototype example is the self-interacting scalar field theory, with Lagrangian given by:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4. \quad (8.49)$$

Going to Euclidean space this gives:

$$S_E = \int_0^\beta d\tau \int d^3 \vec{x} \left[(\partial_\tau \phi)^2 + (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 \right]. \quad (8.50)$$

To compute the quantum effective potential, we split as usual the field into a classical background part and a quantum fluctuating part:

$$\phi = \phi_{\text{cl}} + \eta. \quad (8.51)$$

The quadratic part of the action for η is then controlled by a kinetic operator of the form:

$$\frac{\delta^2 S_E}{\delta \phi^2}[\phi_{\text{cl}}] = -\partial_\tau^2 - \vec{\nabla}^2 + M^2(\phi_{\text{cl}}), \quad (8.52)$$

where

$$M^2(\phi_{\text{cl}}) = m^2 + \frac{\lambda}{2} \phi_{\text{cl}}^2. \quad (8.53)$$

The finite temperature 1-loop correction to the effective potential as a function of the background field ϕ_{cl} is then given by the following expression:

$$\begin{aligned} V_{\text{eff}}^1(\phi_{\text{cl}}) &= \frac{1}{2} \text{Tr} \log \left(-\partial_\tau^2 - \vec{\nabla}^2 + M^2(\phi_{\text{cl}}) \right) \\ &= \frac{1}{2\beta} \sum_n \int \frac{d^3 \vec{k}}{(2\pi)^3} \log \left(\omega_n^2 + \Omega^2(\vec{k}, \phi_{\text{cl}}) \right), \end{aligned} \quad (8.54)$$

where:

$$\Omega(\vec{k}, \phi_{\text{cl}}) = \sqrt{\vec{k}^2 + M^2(\phi_{\text{cl}})}. \quad (8.55)$$

This can be rewritten in the form:

$$V_{\text{eff}}^1 = \int \frac{d^3 \vec{k}}{(2\pi)^3} I(\Omega(\vec{k}, \phi_{\text{cl}})), \quad (8.56)$$

in terms of the function:

$$I(\Omega) = \frac{1}{2\beta} \sum_n \log \left(\omega_n^2 + \Omega^2 \right). \quad (8.57)$$

Since we are interested only in the Ω -dependent part of this function, we can study its first derivative. This is easier to compute, and one finds:

$$\begin{aligned} I'(\Omega) &= \frac{1}{\beta} \sum_n \frac{\Omega}{\Omega^2 + \omega_n^2} = \sum_n \frac{\beta \Omega}{(\beta \Omega)^2 + (2\pi n)^2} \\ &= \frac{1}{2} \coth \frac{\beta \Omega}{2}. \end{aligned} \quad (8.58)$$

Integrating once this result, and dropping an irrelevant integration constant, it follows that:

$$\begin{aligned} I(\Omega) &= \frac{1}{\beta} \log \left[2 \sinh \frac{\beta\Omega}{2} \right] \\ &= \frac{\Omega}{2} + \frac{1}{\beta} \log \left[1 - e^{-\beta\Omega} \right]. \end{aligned} \quad (8.59)$$

Plugging this back into the expression for V_{eff} , the first term gives the zero-temperature result, whereas the second describes the finite-temperature corrections. Indeed, the zero-temperature effective potential can be written in the following form, by performing the energy integral with the help of the residue theorem:

$$\begin{aligned} V_{\text{eff}}^{1(0)} &= \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \log \left(k^2 + M^2(\vec{k}, \phi_{\text{cl}}) \right) + \text{const.} \\ &= \frac{1}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \int \frac{dk_0}{(2\pi)} \log \left(k_0^2 + \Omega^2(\vec{k}, \phi_{\text{cl}}) \right) + \text{const.} \\ &= \frac{1}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \int_0^1 d\alpha \int \frac{dk_0}{(2\pi)} \frac{\Omega^2(\vec{k}, \phi_{\text{cl}})}{k_0^2 + \alpha \Omega^2(\vec{k}, \phi_{\text{cl}})} \\ &= \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\Omega(\vec{k}, \phi_{\text{cl}})}{2}. \end{aligned} \quad (8.60)$$

The temperature-dependent correction can instead be written in the following way, after switching to the dimensionless integration variable $x = \beta k$:

$$V_{\text{eff}}^{1(\beta)} = \frac{1}{2\pi^2 \beta^4} \int_0^{+\infty} dx x^2 \log \left[1 - e^{-\sqrt{x^2 + (\beta M)^2}} \right], \quad (8.61)$$

This integral cannot be computed in closed form in terms of simple functions. However, one can derive its high-temperature behavior by expanding in powers of βM . One finds:

$$V_{\text{eff}}^{1(\beta)} \simeq -\frac{\pi^2}{90} \frac{1}{\beta^4} + \frac{1}{24} \frac{M^2}{\beta^2} + \dots \quad (8.62)$$

The full effective potential at 1-loop order is finally given by:

$$V_{\text{eff}} = \frac{1}{2} m^2 \phi_{\text{cl}}^2 + \frac{1}{4!} \lambda \phi_{\text{cl}}^4 + V_{\text{eff}}^{1(0)}(\phi_{\text{cl}}) + V_{\text{eff}}^{1(\beta)}(\phi_{\text{cl}}). \quad (8.63)$$

In the high-temperature limit, this can be approximated with:

$$V_{\text{eff}} \simeq \left(-\frac{\pi^2}{90} T^4 + \frac{1}{24} m^2 T^2 \right) + \frac{1}{2} \left(m^2 + \frac{\lambda}{24} T^2 \right) \phi_{\text{cl}}^2 + \frac{1}{4!} \lambda \phi_{\text{cl}}^4. \quad (8.64)$$

From the form of the result, we see that temperature effects give a positive contribution to the effective mass. This means that symmetries tend to be restored at high-temperatures, as expected. For instance, in the case where $m^2 < 0$ and the Z_2 parity symmetry $\phi \rightarrow -\phi$ is spontaneously broken at zero temperature, one finds a second-order phase transition at some finite temperature, separating a low-temperature phase where the symmetry is spontaneously broken with $\phi_{\text{cl}} \neq 0$, and a high-temperature phase where the symmetry is

restored with $\phi_{\text{cl}} = 0$. The critical point can be estimated by using the high-temperature approximation, and corresponds to the temperature where the mass term changes sign. This happens for:

$$T_{\text{cr}} \simeq \sqrt{\frac{24 m^2}{-\lambda}}. \quad (8.65)$$

9 Tunneling and vacuum decay

9.1 Semi-classical tunneling in quantum mechanics

Phenomena involving the quantum tunneling through a potential barrier can be efficiently studied in quantum mechanics in the semi-classical approximation. Using the standard WKB approximation, one can for instance compute the probability per unit time for single degree of freedom of mass m described by a variable q to tunnel through a potential barrier $V(q)$. Denoting by q_i and q_f the fixed entrance and exit points, which are conventionally taken to satisfy $V(q_i) = V(q_f) = 0$, the result for this decay rate is simply:

$$\Gamma \sim \exp \left\{ -2 \int_{q_i}^{q_f} dq \sqrt{2mV(q)} \right\}. \quad (9.1)$$

This result can be generalized to the case of n degrees of freedom of mass m described by n variables \vec{q} , tunneling through a potential barrier $V(\vec{q})$. Denoting by \vec{q}_i the fixed entrance point and by \vec{q}_f the now arbitrary exit point, satisfying again $V(\vec{q}_i) = V(\vec{q}_f) = 0$, the result becomes in this case:

$$\Gamma \sim \exp \left\{ -2 \min \int_{\vec{q}_i}^{\vec{q}_f} |d\vec{q}| \sqrt{2mV(\vec{q})} \right\}. \quad (9.2)$$

The minimization is over all the possible paths connecting the initial point \vec{q}_i and any final point \vec{q}_f with the same vanishing potential energy. It corresponds to the fact that the tunneling rate depends only on the shape of the section of the potential barrier along some most favorable tunneling path in the n -dimensional space of variables.

In order to generalize this kind of expression to more complicated situations, and in particular to systems with an infinite number of degrees of freedom like quantum field theories, it is convenient to rely on an alternative understanding of tunneling processes within the path-integral formulation. In this formulation, however, one immediately faces the difficulty that a tunneling process corresponds to a classically forbidden path. It is therefore unclear how to formulate the semi-classical approximation for such a genuinely quantum phenomenon from the functional integral over all the possible paths. This is related to the fact that tunneling processes are non-perturbative effects in \hbar , with a non-analytic behavior of the type $e^{-S_E/\hbar}$, which cannot be captured by a Taylor expansion in \hbar . The form of the known result suggests however that it should nevertheless be possible to attribute the effect to some kind of dominant path with action S_E , although this cannot be a classical path.

To make progress in understanding which path gives the dominant contribution to tunneling processes, let us recall the Jacobi form of the least action principle. This states that the classical trajectories solving the equation of motion with a given energy,

$$m\ddot{\vec{q}} = -V'(\vec{q}) \quad \text{with} \quad \frac{1}{2}m\dot{\vec{q}}^2 + V(\vec{q}) = E, \quad (9.3)$$

minimize the functional

$$S = \int_{\vec{q}_a}^{\vec{q}_b} |d\vec{q}| \sqrt{2m(E - V(\vec{q}))}. \quad (9.4)$$

We observe now that the exponent in the formula for the tunneling rate is the minimum of a function that has the same form as S , but with vanishing energy and inverted potential. We can then conclude that the trajectory associated to the tunneling is the classical trajectory that the particle would follow in the inverted potential $-V$, with vanishing conserved energy:

$$m \ddot{\vec{q}} = V'(\vec{q}) \quad \text{with} \quad \frac{1}{2} m \dot{\vec{q}}^2 - V(\vec{q}) = 0. \quad (9.5)$$

Notice finally that these equations coincide with the Euclidean version of the ordinary equations of motion and vanishing energy conditions of the particle, with Euclidean time $\tau = -it$. We can then recall that the Euclidean equations of motion can be obtained as the Euler-Lagrange equations for the Euclidean action:

$$S_E = \int_{t_i}^{t_f} d\tau \left(\frac{1}{2} m \dot{\vec{q}}^2(\tau) + V(\vec{q}(\tau)) \right). \quad (9.6)$$

Moreover, using the equation fixing the energy, it follows that the exponent of the tunneling probability can be rewritten as:

$$2 \min \int_{\vec{q}_i}^{\vec{q}_f} |d\vec{q}| \sqrt{2mV(\vec{q})} = 2 \text{extr} S_E. \quad (9.7)$$

Using this result, and the fact that the action S_E for the reversed path is identical by time-reversal symmetry, one can finally rewrite the tunneling probability as

$$\Gamma \sim e^{-\bar{S}_E}. \quad (9.8)$$

The quantity \bar{S}_E denotes the Euclidean action for the trajectory $\vec{q}(\tau)$ starting from \vec{q}_i , going up to \vec{q}_f and finally bouncing back to \vec{q}_i . This is a solution of the Euclidean equations of motion, which is generically referred to as bounce. In the particular case where the points \vec{q}_i and \vec{q}_f correspond to degenerate minima of $V(\vec{q})$, its basic first half going from \vec{q}_i to \vec{q}_f is called instanton, and the other half going from \vec{q}_f to \vec{q}_i anti-instanton. Using the vanishing energy condition one finds:

$$\begin{aligned} \bar{S}_E &= \int d\tau \left(\frac{1}{2} m \dot{\vec{q}}(\tau)^2 + V(\vec{q}(\tau)) \right) \\ &= m \int d\tau \dot{\vec{q}}(\tau)^2. \end{aligned} \quad (9.9)$$

9.2 Path-integral description of tunneling processes

The above results suggest that tunneling processes should admit a simple semiclassical interpretation in Euclidean space, in the sense that it should be possible to interpret the associated exponentially suppressed probability amplitude as coming from a classical Euclidean path weighted by its classical action.

For concreteness, let us consider a system with n degrees of freedom described by some variables \vec{q} , with a potential $V(\vec{q})$ admitting a local minimum with conventionally

vanishing energy at some point \vec{q}_0 , so that $V(\vec{q}_0) = V'(\vec{q}_0) = 0$, which is separated by a finite energy barrier from some region of negative energy, delimited by a surface of points \vec{q}_1 where $V(\vec{q}_1) = 0$. In order to estimate the probability of decay by tunneling through the barrier starting at \vec{q}_0 , we use then the following strategy. We imagine to construct a family of potentials where a parameter can be smoothly varied to turn the minimum in \vec{q}_0 from a stable to a metastable minimum. We compute then the exact energy E_0 of the ground state in the region where the minimum is stable and analytically continue the result to the region where it becomes metastable. Finally, we identify the imaginary part that arises with the decay rate for tunneling through the barrier:

$$\Gamma_0 = -2 \operatorname{Im} E_0. \quad (9.10)$$

The starting point to compute the energy of the ground state is the evolution kernel over some large Euclidean time T on the state $|\vec{q}_0\rangle$:

$$Z_0(T) = \langle \vec{q}_0 | e^{-HT} | \vec{q}_0 \rangle. \quad (9.11)$$

This is given by a functional integral with centered boundary conditions, over configurations starting and ending at the point q_0 :

$$Z_0(T) = \int_{\text{c.b.c.}} \mathcal{D}\vec{q} \exp \left\{ - \int_{-T/2}^{T/2} d\tau L_E(\tau) \right\}. \quad (9.12)$$

In the limit of large T , one deduces then that:

$$E_0 = -\frac{1}{T} \log Z_0(T). \quad (9.13)$$

In the semi-classical approximation, the main contributions to the functional integral come from those trajectories that make the Euclidean action stationary. In order to properly normalize the result, notice first that if the minimum of the potential $V(\vec{q})$ at \vec{q}_0 were an absolute stable minimum, then there would be only one possible classical trajectory for the Euclidean problem, where the particle stays at the maximum \vec{q}_0 of the inverted potential without moving. The corresponding classical action is then zero, and the functional integral over small Gaussian fluctuations around this trajectory is that of a harmonic oscillator with frequency $\omega_0 = \sqrt{V''(\vec{q}_0)}$, which for large T is given by:

$$Z_0^{\text{osc}}(T) = \frac{m\omega_0}{\pi} e^{-\omega_0 T/2}. \quad (9.14)$$

The corresponding ground-state energy is then found to be just the usual zero-point energy. Indeed, in the large- T limit one finds:

$$E_0^{\text{osc}} = \frac{\omega_0}{2}. \quad (9.15)$$

In the case where the minimum of $V(\vec{q})$ at \vec{q}_0 is only a local metastable minimum, there exists an additional bounce solution $\vec{q}(\tau)$ to the classical Euclidean equations of motions, where the particle starts with zero velocity and potential energy at \vec{q}_0 , rolls down the inverted potential and reaches some turning point \vec{q}_1 where the potential energy again

vanishes and the velocities invert, and finally comes back to \vec{q}_0 with vanishing final potential energy and velocity. Notice that this trajectory takes indeed an asymptotically large Euclidean time, since \vec{q}_0 is a stationary point. As already seen, the corresponding Euclidean action is positive and given by the following expression:

$$\bar{S}_E = m \int_{-T/2}^{T/2} d\tau \dot{\vec{q}}(\tau)^2. \quad (9.16)$$

The contribution to $Z_0(T)$ from such a basic bouncing path with a given center in Euclidean time is weighted by the semi-classical factor involving its action, namely $e^{-\bar{S}_E}$, times a prefactor which involves the determinant of the kinetic operator for quadratic fluctuations around this path. This is given by the basic harmonic oscillator result mentioned above times a relative factor K given by:

$$K = \sqrt{\frac{\bar{S}_E}{2\pi m}} \frac{\det'^{-1/2}(-m\partial_\tau^2 + V''(\vec{q}))}{\det^{-1/2}(-m\partial_\tau^2 + V''(\vec{q}_0))}. \quad (9.17)$$

The prefactor arises from the contribution of the zero-mode of the operator $-m\partial_\tau^2 + V''(\vec{q})$, whose existence is implied by time-translation invariance. More precisely, taking the time derivative of the Euclidean equation of motion satisfied by the bounce one deduces that $(-m\partial_\tau^2 + V''(\vec{q}))\dot{\vec{q}} = 0$. This shows that the cited zero-mode is proportional to the derivative of the bounce solution. More precisely, using the expression for the bounce action we see that the properly normalized solution is $\vec{q}_{\lambda_1}(\tau) = \sqrt{m/\bar{S}_E}\dot{\vec{q}}(\tau)$, which has $\lambda_1 = 0$. The integral over that mode must then be done separately in the functional integral, since it is divergent. In fact, the result is proportional to the integral over the total time, which will be done in the end, and a careful treatment of the normalization of the path-integral measure shows that the proportionality factor is $\sqrt{\bar{S}_E/(2\pi m)}$.

Notice now that one can actually compose n such basic bounce solutions centered at well separated successive times to form an multiple bounce solution. The corresponding action will be n times the basic bounce action \bar{S}_E , and the integration over Gaussian fluctuations will give the harmonic oscillator result times the product of n copies of the factor K . Finally, one has to integrate over all the possible center times for these n bounces, keeping their order fixed. This gives a factor $T^n/n!$. Summing over all the possible values of n , one finds then the following result for $Z_0(T)$:

$$\begin{aligned} Z_0(T) &= Z_0^{\text{osc}}(T) \sum_{n=0}^{\infty} \frac{T^n}{n!} K^n e^{-n\bar{S}_E} \\ &= Z_0^{\text{osc}} \exp \left\{ T K e^{-\bar{S}_E} \right\} \\ &= \frac{m\omega_0}{\pi} \exp \left\{ -T \left(\frac{\omega_0}{2} - K e^{-\bar{S}_E} \right) \right\}. \end{aligned} \quad (9.18)$$

In the large- T limit it follows then that the energy of the ground state is given by:

$$E_0 = \frac{\omega_0}{2} - K e^{-\bar{S}_E}. \quad (9.19)$$

The computed tunneling corrections is exponentially suppressed, and therefore much smaller than the subleading corrections that have been neglected in deriving the first

term. However, the crucial point is that these tiny effects are the dominant source for the imaginary part of the energy, and represent the leading terms in the tunneling rate:

$$\Gamma_0 = 2 \operatorname{Im} K e^{-\bar{S}_E} . \quad (9.20)$$

The fact that K has an imaginary part, and that it is actually purely imaginary, is related to the fact that there exist precisely 1 negative eigenvalue, whose square root must be analytically continued to a purely imaginary result. The generic presence of this unstable mode can be understood as follows. We already saw that the operator $(-m \partial_\tau^2 + V''(\vec{q}))$ has a mode $\vec{q}_{\lambda_1}(\tau) = \sqrt{m/\bar{S}_E} \dot{\vec{q}}(\tau)$ with eigenvalue $\lambda_1 = 0$. However, this cannot be the lowest-lying eigenmode, because it turns out to have a node in correspondence of the turning point of the bounce solution. There must then exist a node-less mode $\vec{q}_{\lambda_0}(\tau)$ with a smaller negative eigenvalue $\lambda_0 < 0$. On the other hand, all the other eigenmodes $\vec{q}_{\lambda_i}(\tau)$ with more than one node will have higher positive eigenvalues $\lambda_i > 0$, with $i > 1$. A careful analytic continuation shows then that the correct prescription is:

$$\operatorname{Im} K \rightarrow \frac{1}{2} |K| . \quad (9.21)$$

It follows finally that the tunneling rate is given by the following expression:

$$\Gamma_0 = \sqrt{\frac{\bar{S}_E}{2\pi m}} \sqrt{\left| \frac{\det(-m \partial_\tau^2 + V''(\vec{q}_0))}{\det'(-m \partial_\tau^2 + V''(\vec{q}))} \right|} e^{-\bar{S}_E} . \quad (9.22)$$

In most of the applications, the prefactor in the above expression is irrelevant, and only the non-perturbative exponential matters. As we have seen, the exponent is simply the Euclidean action of the basic bounce solution of the Euclidean equations of motions, and reproduces the standard semi-classical suppression factor, which is entirely determined by the shape of the potential barrier.

9.3 Tunneling in quantum field theory

The path-integral approach to tunneling processes described in the previous section for quantum mechanical systems can now be generalized in a straightforward way to quantum field theories. Let us consider for concreteness the theory of a single self-interacting real scalar field ϕ , with action:

$$S = \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] (x) . \quad (9.23)$$

Suppose now that $V(\phi)$ has a local metastable minimum with vanishing energy at some point ϕ_0 , separated by some finite potential barrier from some region of negative energy starting at a point ϕ_1 with vanishing energy. The tunneling probability per unit time and unit volume can be computed exactly as before, by considering the Euclidean path-integral where paths are weighted by the Euclidean action:

$$S_E = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \right] . \quad (9.24)$$

The corresponding Euclidean equations of motion are:

$$-\square_E \phi + V'(\phi) = 0. \quad (9.25)$$

The relevant bounce solution $\bar{\phi}(x)$ starts from the initial point ϕ_0 with vanishing potential energy and velocity, goes then to some configuration $\phi'_1(\vec{x})$ for which the positive-definite gradient energy is compensated by a negative potential energy to yield again vanishing total energy and the velocity inverts, and finally goes back to the initial point ϕ_0 with vanishing final velocity and energy. The associated action can be simplified as follows. Consider the 1-parameter family of scaled trajectories

$$\phi_\lambda(x) = \bar{\phi}(x/\lambda). \quad (9.26)$$

The corresponding Euclidean action is found to be:

$$S_E[\phi_\lambda] = \int d^4x \left[\frac{1}{2} \lambda^2 (\partial_\mu \bar{\phi})^2 + \lambda^4 V(\bar{\phi}) \right]. \quad (9.27)$$

The stationary bounce trajectory corresponds to $\lambda = 1$. This means that the above function of λ must be stationary at $\lambda = 1$, $\delta S_E / \delta \lambda[\bar{\phi}] = 0$, and we deduce that

$$\int d^4x \left[(\partial_\mu \bar{\phi})^2 + 4V(\bar{\phi}) \right] = 0. \quad (9.28)$$

This relation implies that the bounce action is positive and can be rewritten in the following way:

$$\bar{S}_E = \frac{1}{4} \int d^4x (\partial_\mu \bar{\phi})^2. \quad (9.29)$$

The path-integral over Gaussian fluctuations can be computed in a way similar to before. The resulting normalized factor K is in this case found to be:

$$K = \frac{\bar{S}_E^2}{4\pi^2} \frac{\det'^{-1/2}(-\square_E + V''(\bar{\phi}))}{\det^{-1/2}(-\square_E + V''(\phi_0))}. \quad (9.30)$$

The first factor arises exactly in the same way as before from the zero-modes implied by translational invariance. In this case, there are four of them, associated to both time and space translations, of the form $\phi_{\lambda_1 \mu} \propto \partial_\mu \bar{\phi}$. In this case, one has to eventually integrate over the location of the bounces not only in time but also in space, and this will produce a factor of T times a factor of the total volume V .

The crucial question is now whether there is a negative eigenvalue mode, and whether there is just one, so that K is as before purely imaginary and the bounce contributes to the decay rate. The fact that there must exist at least one unstable mode with negative eigenvalue can be easily proven by looking at the operator governing the quadratic fluctuation in the parameter λ describing the family of trajectory considered above. Indeed, using the formulae derived in the previous paragraph, one finds that $\delta^2 S_E / \delta \lambda^2[\bar{\phi}] < 0$, implying that fluctuations in the scale of the bounce lead to an instability. This obviously implies that the full kinetic operator $\delta^2 S_E / \delta \phi^2[\bar{\phi}]$ for Gaussian fluctuations around the

bounce has at least one negative eigenvalue. Concerning the uniqueness of such a negative eigenvalue, one can prove it in certain limits, but not in full generality. Finally, we have then as before that $\text{Im}K \rightarrow 1/2|K|$.

In the end, the final formula for the tunneling rate in quantum field theory is given by the following expression:

$$\frac{\Gamma_0}{V} = \frac{\bar{S}_E^2}{4\pi^2} \sqrt{\left| \frac{\det(-\square_E + V''(\phi_0))}{\det'(-\square_E + V''(\bar{\phi}))} \right|} e^{-\bar{S}_E}. \quad (9.31)$$

This formula is written in terms of bare quantities. However, all the UV divergences can be renormalized through local counter-terms in the standard way. Indeed, the above computation of the tunneling rate can be directly mapped to the computation of the Euclidean version of the quantum effective action Γ_E in the particular background of the bounce solution $\bar{\phi}$ and the constant configuration ϕ_0 . More precisely, using the usual relation $\det M = \exp \text{tr} \log M$, one can rewrite the above result as:

$$\frac{\Gamma_0}{V} = \frac{\bar{S}_E^2}{4\pi^2} e^{-(\tilde{\Gamma}_E[\bar{\phi}] - \tilde{\Gamma}_E[\phi_0])}, \quad (9.32)$$

where:

$$\tilde{\Gamma}_E[\phi] = S_E[\phi] - \frac{1}{2} \text{Re tr}' \log \left(-\square_E + V''(\phi) \right). \quad (9.33)$$

9.4 Vacuum decay and bubbles

In a quantum field theory, the process of tunneling through a potential barrier from a metastable vacuum associated to a value ϕ_+ of the field to a stable vacuum associated to a value ϕ_- of the field, with $V(\phi_+) > V(\phi_-)$, is interpreted as a vacuum decay process, in which the false vacuum is converted into the true vacuum through a phase transition. More precisely, quantum fluctuations allow tunneling and the consequent formation and expansion of bubbles of true vacuum, with a certain probability per unit time and volume. One can then define the life-time of a metastable vacuum state exactly as for quantum mechanical systems with a finite number of degrees of freedom, as the inverse of the decay rate:

$$\tau = \frac{1}{\Gamma}. \quad (9.34)$$

The situation is perfectly analogous to a system in thermal equilibrium that is overheated beyond the critical temperature at which a liquid-vapor phase transition occurs. At such temperatures the stable state is the vapor state, but the liquid state is still metastable, because there is a potential barrier associated to the formation of bubbles due to their surface tension. Thermal fluctuations allow however the creation and expansion of bubbles of vapor state within the liquid state, with a certain probability per unit time and volume.

In order to make the interpretation in terms of bubbles somewhat more precise, notice that it is natural to assume that the bounce solution that is relevant for vacuum decay

should be invariant under the group of Euclidean rotations $SO(4)$. The problem can then be phrased in polar coordinates, and the bounce $\bar{\phi}$ depends non-trivially only on the Euclidean radial coordinate:

$$\rho = \sqrt{\tau^2 + |\vec{x}|^2}. \quad (9.35)$$

The Euclidean equation of motion satisfied by the bounce simplifies then to the following ordinary differential equation:

$$\left(\frac{d^2}{d\rho^2} + \frac{3}{\rho} \frac{d}{d\rho} \right) \bar{\phi} = V'(\bar{\phi}). \quad (9.36)$$

We must then look for a solution to this equation that corresponds to the second half of the bounce, starting at $\tau = 0$ from some non-trivial spherically symmetric configuration with vanishing velocity,

$$\partial_\tau \phi(\vec{x}, 0) = 0, \quad (9.37)$$

and reaching after an asymptotically large time the constant configuration corresponding to the the metastable vacuum:

$$\lim_{\tau \rightarrow \infty} \bar{\phi}(\vec{x}, \tau) = \phi_+. \quad (9.38)$$

Moreover, in order to get a finite action one also needs to require that:

$$\lim_{|\vec{x}| \rightarrow \infty} \bar{\phi}(\vec{x}, \tau) = \phi_+. \quad (9.39)$$

In terms of the variable ρ , the above conditions imply then the following boundary conditions:

$$\lim_{\rho \rightarrow 0} \frac{d}{d\rho} \phi(\rho) = 0, \quad \lim_{\rho \rightarrow \infty} \phi(\rho) = \phi_+. \quad (9.40)$$

Observe now that by interpreting ρ as a time and ϕ as a position, the above differential equation can be interpreted as the equation of motion for a particle of unit mass moving in a potential $-V$ and subject to some time-dependent damping force proportional to its velocity and inversely proportional to time, which is accounting for the effects due to the gradient energy of the original field. The boundary conditions imply on the other hand that the particle should start anywhere but with zero velocity and end after infinite time at ϕ_+ . It is easy to prove that such a motion exists and that it is unique. It corresponds to start at a point ϕ'_1 somewhere between ϕ_- and the point ϕ_1 defining the exit point of the barrier with vanishing potential energy, with a positive energy that is integrally dissipated during the rolling towards ϕ_+ , which is then reached after infinite time with zero energy and velocity. Schematically, one can think of the trajectory as follows, at least if the difference in energy of the two minima is not too big. It starts in the proximity of ϕ_- and remains in its vicinity until some characteristic time R , at which is rolls towards the point ϕ_+ in a time ΔR , and stays then in the vicinity of that point for infinite time.

In the original interpretation of the variable ρ as a radial coordinate in Euclidean space and ϕ as a field describing the type of phase, the bounce solution describes as heuristically expected a bubble separating an interior phase of true vacuum, corresponding to the value ϕ_- of the field, from an exterior phase of false vacuum corresponding to the value ϕ_+ of the field. The radius of the wall corresponds to the time R at which the rolling between the two extrema occurs, and the thickness of the wall corresponds to the typical time ΔR that this rolling takes.

The dynamics of the bubbles forming during the phase transition associated to the decay of a false vacuum into a true vacuum can be described by recalling what happens for the tunneling in quantum mechanical systems. In this simpler situation, a particle with zero energy can tunnel through the barrier and emerge at the other side at some given time, say $t = 0$, with vanishing energy, and start rolling down the potential. This stage of rolling down can then be approximately described by the classical real time equations of motions, with the boundary condition that the particle starts with vanishing potential energy and vanishing velocity. Notice now that these boundary conditions coincide with those arising at the turning point of the bounce. This means that the dynamical evolution of the particle after tunneling is governed by the analytic continuation of the bounce solution to real time after the turning point. In the context of quantum field theory, the same reasoning can be applied. The only differences are that at the turning point the field configuration has a non-trivial dependence on the spatial coordinates, and that its potential energy is negative and compensates the positive gradient energy. Again, one finds that the initial conditions at $t = 0$ for the real dynamical evolution of the bubble after tunneling coincide with the conditions obtained at the turning point of the bounce. One can then again identify the dynamical evolution of the field after tunneling with the analytic continuation of the bounce solution to real time after the turning point. Notice furthermore that the $SO(4)$ Euclidean rotation invariance of the bounce solution maps then to the $SO(1,3)$ Lorentz invariance of the bubble dynamics.

In the schematization of the bubble as a shell of radius R with small thickness ΔR , the trajectory is determined by the following simple equation:

$$-t^2 + |\vec{x}|^2 = R^2. \quad (9.41)$$

The radius of the bubble therefore expands from R to infinity as:

$$r(t) = \sqrt{R^2 + t^2}. \quad (9.42)$$

The radial velocity increases with time and asymptotically reaches the velocity of light:

$$v(t) = \frac{t}{\sqrt{R^2 + t^2}}. \quad (9.43)$$

As this expansion occurs, the energy gained by converting the false to the true vacuum in the gained volume is converted into gradient energy of the bubble.

Unfortunately, the bounce solution and its action cannot be computed in closed form for a general potential barrier, even using the assumption of $SO(4)$ symmetry. One can however estimate by simple scaling arguments the dependence of \bar{S}_E on the typical height

and width of the barrier. To do so, let us schematize the general situation of a potential involving arbitrarily many dimensionful parameters with the situation of a potential involving only two such dimensionful parameters, H and W with dimensions of energy, controlling respectively the typical height and the width of the barrier:

$$V(\phi) = H^4 v\left(\frac{\phi}{W}\right). \quad (9.44)$$

Defining then the dimensionless field $\pi = \phi/W$ and switching to a dimensionless integration variable $y^\mu = \Lambda x^\mu$, the Euclidean action can be written as:

$$S_E = \int d^4 y \left[\frac{1}{2} \frac{W^2}{\Lambda^2} (\partial_\mu \pi)^2 + \frac{H^4}{\Lambda^4} v(\pi) \right]. \quad (9.45)$$

Choosing now $\Lambda = H^2/W$, one obtains:

$$S_E = \frac{W^4}{H^4} \int d^4 y \left[\frac{1}{2} (\partial_\mu \pi)^2 + v(\pi) \right] \sim \frac{W^4}{H^4}. \quad (9.46)$$

This suggest that in general one has:

$$\bar{S}_E \sim \frac{\text{width of the potential barrier}}{\text{height of the potential barrier}}. \quad (9.47)$$

The bounce solution and its action can be found more explicitly in the limit where the difference in energy between the true and the false vacua is very small. This is called the thin-wall approximation, because in that case one indeed gets $\Delta R/R \ll 1$.

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