On triangle generation of finite groups of Lie type

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Abstract This paper is concerned with the (p_1, p_2, p_3)-generation of finite groups of Lie type, where we say a group is (p_1, p_2, p_3)-generated if it is generated by two elements of orders p_1, p_2 having product of order p_3. Given a triple (p_1, p_2, p_3) of primes, we say that (p_1, p_2, p_3) is rigid for a simple algebraic group G, if the sum of the dimensions of the subvarieties of elements of orders dividing p_1, p_2, p_3 in G is equal to $2 \dim G$. We conjecture that if (p_1, p_2, p_3) is a rigid triple for G then given a prime p, there are only finitely many positive integers r such that the finite group $G(p^r)$ is a (p_1, p_2, p_3)-group. We prove that the conjecture holds in many cases. Finally, we classify the rigid triples for simple algebraic groups. The conjecture together with this classification puts into context many results on Hurwitz $(2, 3, 7)$-generation in the literature, and motivates a new study of the (p_1, p_2, p_3)-generation problem for certain finite groups of Lie type of low rank.

1 Introduction

Let (p_1, p_2, p_3) be a triple of primes. We say that a group is a (p_1, p_2, p_3)-group if it is generated by two elements of respective orders p_1, p_2 whose product has order p_3. In other words a noncyclic group is a (p_1, p_2, p_3)-group if it is a homomorphic image of the triangle group T where

$$T = T_{p_1,p_2,p_3} = \langle x, y, z : x^{p_1} = y^{p_2} = z^{p_3} = xyz = 1\rangle.$$

Recall that a finite $(2, 3, 7)$-group is also called a Hurwitz group (see [4]).

In this paper we are concerned with the (p_1, p_2, p_3)-generation problem for finite groups of Lie type. Let G be a simple algebraic group of classical or exceptional type defined over an algebraically closed field of prime characteristic p and let $G_0 = G(p^r)$ be a finite group of Lie type arising from G. If $1/p_1 + 1/p_2 + 1/p_3 \geq 1$ then the only possible finite nonabelian simple image of T is the alternating group Alt_5 (see [4, p. 361]). We therefore assume hereafter that $1/p_1 + 1/p_2 + 1/p_3 < 1$ so that T is a hyperbolic triangle group. We then call (p_1, p_2, p_3) a *hyperbolic triple* of primes. Without loss of generality, we also suppose $p_1 \leq p_2 \leq p_3$.

There are many results in the literature about Hurwitz generation in finite groups of Lie type. The first is due to Macbeath who proved in [11] that

$\mathrm{PSL}_2(p^n)$ is a Hurwitz group if and only if $n = 1$ and $p \equiv 0, \pm 1 \mod 7$, or $n = 3$ and $p \equiv \pm 2, \pm 3 \mod 7$. Cohen showed in [3] that $\mathrm{PSU}_3(q)$ is never a Hurwitz group and $\mathrm{PSL}_3(q)$ is a Hurwitz group if and only if $q = 2$. It is known, see [21], that there are no Hurwitz groups amongst the groups $\mathrm{PSU}_4(q)$ and $\mathrm{PSL}_4(q)$, and by [20], given a prime p there is at most one positive integer r such that $\mathrm{PSU}_5(p^r)$ (respectively, $\mathrm{PSL}_5(p^r)$) is a Hurwitz group. The next known dimension n for which $\mathrm{PSL}_n(q)$ is Hurwitz is $n = 49$ and in fact $\mathrm{PSL}_{49}(q)$ is $(2, 3, 7)$-generated for any q (see [22]). Hurwitz generation for some classical groups of large rank is proved in [9, 10]. In [16], a large sample of quasisimple classical groups of low rank are shown to fail to be $(2, 3, 7)$-generated. For example, none of the quasisimple groups $\mathrm{SU}_n(q)$, $\mathrm{SL}_n(q)$ with $n = 4, 5, 6, 10$ are Hurwitz.
The Hurwitz groups amongst the exceptional groups of Lie types ${}^2G_2$, G_2, ${}^3D_4$ and ${}^2F_4$ have been determined (see [12] and [13]). Except in characteristic 3 for the Steinberg triality groups or possibly when the size of the underlying field is small, these groups are Hurwitz provided that they contain elements of order 7.

Turning to general hyperbolic triples (p_1, p_2, p_3) of primes, there are fewer results in the literature. It is proved in [14, Corollary 1] that given a prime p there is a unique positive integer r such that $\mathrm{PSL}_2(p^r)$ is a (p_1, p_2, p_3)-group. In fact, r is the smallest integer such that $\mathrm{lcm}(p_1, p_2, p_3)$ divides $|\mathrm{PSL}_2(p^r)|$.
For $\mathrm{PSU}_3(q)$ and $\mathrm{PSL}_3(q)$, it is shown in [15] that there is a dichotomy between the triples (p_1, p_2, p_3) for which $p_1 = 2$ and those for which $p_1 \neq 2$, in the following sense. For a given prime p and a fixed triple $(2, p_2, p_3)$, there are at most two (respectively, four) positive integers r such that $\mathrm{PSU}_3(p^r)$ (respectively, $\mathrm{PSL}_3(p^r)$) is a $(2, p_2, p_3)$-group. This contrasts with the situation when $p_1 \neq 2$, where for a given prime p and a fixed triple (p_1, p_2, p_3) of odd primes, there are infinitely many positive integers r such that $\mathrm{PSU}_3(p^r)$ (respectively, $\mathrm{PSL}_3(p^r)$) is a (p_1, p_2, p_3)-group. (Indeed if $G_0 = \mathrm{PSU}_3(q)$ or $\mathrm{PSL}_3(q)$, $p_1 > 2$ and $\mathrm{lcm}(p_1, p_2, p_3)$ divides $|G_0|$, then the probability that a randomly chosen homomorphism in $\mathrm{Hom}(T_{p_1,p_2,p_3}, G_0)$ is an epimorphism tends to 1 as $|G_0| \to \infty$.)

We formulate a conjecture below which places these results into a more conceptual framework, and also motivates a further specific study of the (p_1, p_2, p_3)-generation problem for finite groups of Lie type.

To set up the conjecture we need the following result. In the statement, given a hyperbolic triple (p_1, p_2, p_3) of primes, we let $\delta^G_{p_i}$ be the maximal dimension in G of a conjugacy class of G of elements of order p_i. Furthermore, throughout the paper, given a prime p, we let $\mathbb{F}$ denote an algebraically closed field of characteristic p. Recall that a quasisimple group is a perfect group whose quotient by its centre is simple.

Proposition 1. *Let $G_0 = G(p^r)$ be a finite quasisimple group of Lie type, and let G be the corresponding algebraic group over $\mathbb{F}$. If*

$$\delta^G_{p_1} + \delta^G_{p_2} + \delta^G_{p_3} < 2 \dim G$$

then G_0 is not a (p_1, p_2, p_3)-group.

Most of the above proposition follows from [19]. We give a full proof for completeness in §2. Proposition 1 motivates the following definition.

Definition We say that the hyperbolic triple (p_1, p_2, p_3) is *rigid* for a simple algebraic group G if

$$\delta^G_{p_1} + \delta^G_{p_2} + \delta^G_{p_3} = 2 \dim G.$$

We are now ready to formulate the conjecture.

Conjecture *Let p be a prime and let $G_0 = G(p^r)$ denote a finite quasisimple group of Lie type with corresponding algebraic group G defined over $\mathbb{F}$. Suppose that (p_1, p_2, p_3) is a rigid hyperbolic triple of primes for G. Then there are only finitely many positive integers r such that $G(p^r)$ is a (p_1, p_2, p_3)-group.*

We will classify all rigid triples of primes in Theorem 3 below. For example in $\mathrm{PSL}_2(\mathbb{F})$ all triples are rigid; and in $\mathrm{PSL}_3(\mathbb{F})$ the triple (p_1, p_2, p_3) is rigid if and only if $p_1 = 2$. Hence the conjecture holds for $\mathrm{PSL}_2(q), \mathrm{PSU}_3(q), \mathrm{PSL}_3(q)$ by results in [14, 15] discussed above. As another illustration, by Theorem 3 below the only rigid triple for $\mathrm{SL}_{10}(\mathbb{F})$ is (2,3,7), and so the conjecture also holds for $\mathrm{SU}_{10}(q), \mathrm{SL}_{10}(q)$. Also, since by Theorem 3 a rigid triple for $\mathrm{SL}_n(\mathbb{F})$ with $n \in \{2, 3\}$ remains rigid for $\mathrm{PSL}_n(\mathbb{F})$ it follows that the conjecture also holds for $\mathrm{SL}_2(q), \mathrm{SU}_3(q)$ and $\mathrm{SL}_3(q)$. More generally, using Theorem 3 and the concept of linear rigidity defined in [19] (originally introduced in [5]) we prove that the conjecture holds in almost all cases.

Theorem 1. *The conjecture holds for all finite groups G_0 of Lie type except possibly when G_0 is one of the following:*

$$\mathrm{Sp}_{2m}(q) \text{ with } m \leq 13, \quad \mathrm{PSp}_4(q), \quad G_2(q).$$

We also derive the following result.

Theorem 2. *The conjecture holds for $G_0 = \mathrm{PSp}_4(q)$ and $(p_1, p_2) = (2, 3)$.*

It is natural to refine the triples which are not rigid for a simple algebraic group G into reducible and nonrigid triples, as follows. We say that the triple (p_1, p_2, p_3) is *reducible* if $\delta^G_{p_1} + \delta^G_{p_2} + \delta^G_{p_3} < 2 \dim G$, and *nonrigid* if $\delta^G_{p_1} + \delta^G_{p_2} + \delta^G_{p_3} > 2 \dim G$.

Below is the classification of the reducible, rigid and nonrigid triples of primes for simple algebraic groups G defined over an algebraically closed field $\mathbb{F}$ of prime characteristic p. Recall that if G is not of simply connected or adjoint type then either G is abstractly isomorphic to $\mathrm{SL}_n(\mathbb{F})/C$ where $C \leq Z(\mathrm{SL}_n(\mathbb{F}))$, or G is of type D_m, $p \neq 2$ and G is abstractly isomorphic to $\mathrm{SO}_{2m}(\mathbb{F})$ or a half-spin group $\mathrm{HSpin}_{2m}(\mathbb{F})$ where m is even in the latter case.

Theorem 3. *The following hold:*

(i) A complete list of reducible hyperbolic triples (p_1, p_2, p_3) of primes, with $p_1 \leq p_2 \leq p_3$, for simple algebraic groups of simply connected or adjoint type is given in Table 1.

(ii) *A complete list of rigid hyperbolic triples* (p_1, p_2, p_3) *of primes, with* $p_1 \leq p_2 \leq p_3$*, for simple algebraic groups of simply connected or adjoint type is given in Table 2.*

(iii) *The classification of triples of primes for* $\mathrm{SO}_n(\mathbb{F})$ *is the same as for* $\mathrm{PSO}_n(\mathbb{F})$*.*

(iv) *The classification of triples of primes for* $\mathrm{HSpin}_{2m}(\mathbb{F})$ *is the same as for* $\mathrm{Spin}_{2m}(\mathbb{F})$*.*

(v) *If* $C \leq Z(\mathrm{SL}_n(\mathbb{F}))$ *contains an involution then the classification of triples of primes for* $\mathrm{SL}_n(\mathbb{F})/C$ *is the same as for* $\mathrm{PSL}_n(\mathbb{F})$*. Otherwise, the classification of triples of primes for* $\mathrm{SL}_n(\mathbb{F})/C$ *is the same as for* $\mathrm{SL}_n(\mathbb{F})$*.*

Table 1: Reducible triples

G	p	(p_1, p_2, p_3)
$\mathrm{SL}_2(\mathbb{F})$	$p \neq 2$	$p_1 = 2$
$\mathrm{Sp}_4(\mathbb{F})$	$p \neq 2$	$p_1 = 2$, $p_2 = 3$
$\mathrm{Sp}_6(\mathbb{F})$	$p \neq 2$	$p_1 = 2$, $p_2 = 3$
		or $p_1 = 2$, $p_2 = p_3 = 5$
$\mathrm{Sp}_{2m}(\mathbb{F})$, $m \in \{4, 5, 6, 7, 8, 9, 11\}$	$p \neq 2$	$p_1 = 2$, $p_2 = 3$, $p_3 = 7$

Table 2: Rigid triples

G	p	(p_1, p_2, p_3)
$\mathrm{SL}_2(\mathbb{F})$	$p = 2$	any
	$p \neq 2$	$p_1 > 2$
$\mathrm{SL}_3(\mathbb{F})$	any	$p_1 = 2$
$\mathrm{SL}_4(\mathbb{F})$	any	$p_1 = 2$, $p_2 = 3$
$\mathrm{SL}_5(\mathbb{F})$	any	$p_1 = 2$, $p_2 = 3$
$\mathrm{SL}_6(\mathbb{F})$	$p \neq 2$	$p_1 = 2$, $p_2 = 3$
$\mathrm{SL}_{10}(\mathbb{F})$	$p \neq 2$	$p_1 = 2$, $p_2 = 3$, $p_3 = 7$
$\mathrm{PSL}_2(\mathbb{F})$	any	any
$\mathrm{PSL}_3(\mathbb{F})$	any	$p_1 = 2$
$\mathrm{PSL}_4(\mathbb{F})$	any	$p_1 = 2$, $p_2 = 3$
$\mathrm{PSL}_5(\mathbb{F})$	any	$p_1 = 2$, $p_2 = 3$
$\mathrm{Sp}_4(\mathbb{F})$	$p = 2$	$(p_1, p_2) \in \{(2, 3), (3, 3)\}$
	$p \neq 2$	$p_1 = p_2 = 3$
		or $p_1 = 2$, $p_2 > 3$
$\mathrm{Sp}_6(\mathbb{F})$	$p \neq 2$	$p_1 = 2$, $p_2 = 5$, $p_3 \geq 7$
$\mathrm{Sp}_8(\mathbb{F})$	$p \neq 2$	$p_1 = 2$, $p_2 = 3$, $p_3 > 7$
		or $p_1 = 2$, $p_2 = p_3 = 5$
$\mathrm{Sp}_{10}(\mathbb{F})$	$p \neq 2$	$p_1 = 2$, $p_2 = 3$, $p_3 > 7$
$\mathrm{Sp}_{2m}(\mathbb{F})$, $m \in \{10, 12, 13\}$	$p \neq 2$	$p_1 = 2$, $p_2 = 3$, $p_3 = 7$
$\mathrm{PSp}_4(\mathbb{F})$	any	$(p_1, p_2) \in \{(2, 3), (3, 3)\}$
$\mathrm{Spin}_{11}(\mathbb{F})$	$p \neq 2$	$p_1 = 2$, $p_2 = 3$, $p_3 = 7$
$\mathrm{Spin}_{12}(\mathbb{F})$	$p \neq 2$	$p_1 = 2$, $p_2 = 3$, $p_3 = 7$
$G_2(\mathbb{F})$	any	$p_1 = 2$, $p_2 = p_3 = 5$

Using Proposition 1, Table 1 gives us a list of examples of finite quasisimple groups of Lie type that are not (p_1, p_2, p_3)-generated. If q is odd then it is not a surprise that $\mathrm{SL}_2(q)$ is not a $(2, p_2, p_3)$-group, as the only involution in $\mathrm{SL}_2(q)$ is central. As another example if q is odd and $p_3 \geq 7$ is a prime then $\mathrm{Sp}_4(q)$ and $\mathrm{Sp}_6(q)$ are never $(2, 3, p_3)$-groups. Also if q is odd then $\mathrm{Sp}_6(q)$ is never a $(2, 5, 5)$-group. With a single exception, the non-Hurwitz finite symplectic groups that are derived from Table 1 are already listed in [16]. As a new contribution, if q is odd then by Table 1, $\mathrm{Sp}_{22}(q)$ is not a Hurwitz group.

We now turn to the rigid triples given in Table 2. In view of Theorems 1-3, the conjecture remains open for simple groups $G_0 = G(p^r)$ only in the following cases:

$$G = \mathrm{PSp}_4(\mathbb{F}) \text{ with } (p_1, p_2) = (3, 3)$$

$$G = G_2(\mathbb{F}) \text{ with } (p_1, p_2, p_3) = (2, 5, 5).$$

Concerning quasisimple groups with nontrivial centre the conjecture remains open only for the symplectic groups given in Table 2, and in particular it motivates, in its own right, a further study of Hurwitz generation of various symplectic groups, namely $\mathrm{Sp}_{20}(q), \mathrm{Sp}_{24}(q), \mathrm{Sp}_{26}(q)$ for q odd.

It is important to notice that the converse to the conjecture does not hold in general, in the sense that we can have a simple algebraic group G defined over $\mathbb{F}$, and a nonrigid hyperbolic triple (p_1, p_2, p_3) of primes for G for which there are only finitely many positive integers r such that $G(p^r)$ is a (p_1, p_2, p_3)-group. For example by [16] $\mathrm{SL}_7(p^r)$ is never a Hurwitz group, although the triple $(2, 3, 7)$ is nonrigid for $\mathrm{SL}_7(\mathbb{F})$. Other similar classical examples can be given using [16]. As a final illustration, ${}^3D_4(3^r)$ is never a Hurwitz group (see [13]), but the triple $(2, 3, 7)$ is nonrigid for the simple adjoint group of type D_4 over an algebraically closed field of characteristic 3. It would be interesting to investigate a possible converse to the conjecture. Ideally such a result would give necessary and sufficient conditions for $G(p^r)$ to be a (p_1, p_2, p_3)-group only for finitely many r. We leave this task for a future paper.

The layout of the paper is as follows. In §2 we prove Proposition 1. In §3 we prove Theorems 1 and 2. Finally we derive Theorem 3 for exceptional groups in §4 and for classical groups in §5.

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2 Proof of Proposition 1

Let G be a simple algebraic group over an algebraically closed field $\mathbb{F}$ of prime characteristic p. The main ingredient in the proof of Proposition 1 is the following result, combined with the classification of triples of primes given in Theorem 3 if p is a bad prime or G is of exceptional type. Recall that p is a bad prime if G is of type B_l, C_l, D_l and $p = 2$, or of type G_2, F_4, E_6, E_7 and $p \in \{2, 3\}$, or of type E_8 and $p \in \{2, 3, 5\}$. A prime p is said to be good if it is not bad. Also in the statement below, by an irreducible subgroup of a classical group G, we mean a subgroup acting irreducibly on the natural module for G.

Proposition 2.1. *Suppose that G is of classical type and that p is a good prime for G. If g_1, g_2, g_3 are elements of G such that $g_1g_2g_3 = 1$ and $\langle g_1, g_2\rangle$ is an irreducible subgroup of G then*

$$\dim g_1^G + \dim g_2^G + \dim g_3^G \geq 2\dim G.$$

The proof of Proposition 2.1 is based on the following three lemmas. The first lemma is Scott's formula.

Lemma 2.1. *(Scott [17]). Let H be a group acting linearly on a finite-dimensional vector space V. For X a subgroup or element of H, let $v(X) = v(X, V)$ denote the codimension of the fixed-point space of X in V. Also, write $v(X^*)$ for $v(X, V^*)$, where V^* is the dual of V. Suppose H is generated by elements $x_1, \ldots, x_s$ with $x_1 \cdots x_s = 1$. Then*

$$\sum_{i=1}^{s} v(x_i) \geq v(H) + v(H^*).$$

In the next lemma, we apply Scott's formula to the adjoint module $L(G)$.

Lemma 2.2. *Let $L(G)$ denote the Lie algebra of G. Let g_1, g_2, g_3 be elements of G such that $g_1g_2g_3 = 1$. Put $H = \langle g_1, g_2\rangle$. Suppose that for $i = 1, 2, 3$ we have*

$$\dim C_G(g_i) = \dim C_{L(G)}(g_i).$$

Assume also that $C_{L(G)}(H) = 0$. Then

$$\dim g_1^G + \dim g_2^G + \dim g_3^G \geq 2\dim G.$$

Proof. We apply Scott's formula to the module $L(G)$ for H. Since $C_{L(G)}(H) = 0$, we get

$$\sum_{i=1}^{3} \text{codim } C_{L(G)}(g_i) \geq 2\dim L(G).$$

The result follows from the equality $\dim L(G) = \dim G$ and the assumption that for $i = 1, 2, 3$ we have $\dim C_{L(G)}(g_i) = \dim C_G(g_i)$. □

The next lemma ensures that the hypotheses of Lemma 2.2 hold when G is of symplectic or orthogonal type, and $p \neq 2$.

Lemma 2.3. *Let G be of symplectic or orthogonal type over an algebraically closed field of characteristic $p > 2$. Then*

(i) For all g in G we have $\dim C_G(g) = \dim C_{L(G)}(g)$.

(ii) If H is an irreducible subgroup of G then $C_{L(G)}(H) = 0$.

Proof. Part (i) is shown in [18, p. 38].

Let us now consider part (ii). We write V for the natural module for G. Suppose first that G is of symplectic type. Since $p \neq 2$, H fixes a non-degenerate alternating bilinear form f on V. As H is irreducible, using Schur's lemma, f is in fact (up to scalars) the unique nonzero bilinear form on V fixed by H. Because $p \neq 2$, we have $L(G) \cong S^2(V)$ as a G-module. Also $S^2(V)$ is isomorphic to the

space of symmetric bilinear forms. It follows that $C_{L(G)}(H) = 0$.
Finally suppose that G is of orthogonal type. As $p \neq 2$, we have $L(G) \cong \bigwedge^2(V)$, the space of alternating forms on V. But H fixes up to scalars a unique (non-degenerate) bilinear form on V which is symmetric. It follows that $C_{L(G)}(H) = 0$. □

Proof of Proposition 2.1. Let $H = \langle g_1, g_2 \rangle$. Suppose first that G is of linear type in dimension n. Then H acts by conjugation on the algebra $V = M_n(\mathbb{F})$ consisting of $n \times n$ matrices over $\mathbb{F}$. Since H is an irreducible subgroup of G, we have $v(H, V) = v(H, V^*) = n^2 - 1$ by Schur's lemma. The result now follows from Scott's formula.
If G is not of linear type, then the result follows from Lemmas 2.2 and 2.3. This completes the proof of Proposition 2.1.

Proof of Proposition 1. If G is of classical type and p is a good prime for G then the result follows from Proposition 2.1.
The remaining cases follow from Theorem 3 which shows that there are no reducible triples of primes if G is of symplectic or orthogonal type and $p = 2$, or if G is of exceptional type. This completes the proof of Proposition 1.

3 Proof of Theorems 1 and 2

For the proof of Theorem 1, we prove the following two propositions.

Proposition 3.1. *The conjecture holds for $G_0 = \mathrm{SU}_n(q)/C$ or $\mathrm{SL}_n(q)/C$, where C is any central subgroup.*

Proposition 3.2. *Let $n \in \{11, 12\}$. If $p \neq 2$ then $\mathrm{Spin}_n(p^r)$ and $\mathrm{HSpin}_n(p^r)$ are never Hurwitz groups.*

Let G be a simple algebraic group of classical or exceptional type defined over an algebraically closed field $\mathbb{F}$ of prime characteristic p. By Theorem 3, if G is not of linear or unitary type then there are rigid triples (p_1, p_2, p_3) of primes for G only if G is of type C_m and $m \leq 13$, or G is a spin or half-spin group in dimension 11 or 12, $p \neq 2$ and $(p_1, p_2, p_3) = (2, 3, 7)$, or G is of type G_2 and $(p_1, p_2, p_3) = (2, 5, 5)$. Also if G is of type B_2 and $p = 2$ then (p_1, p_2, p_3) is rigid for G only if $p_2 = 3$. Hence Theorem 1 follows from Propositions 3.1 and 3.2 together with the fact that the Suzuki groups ${}^2B_2(2^{2r+1})$ are never $(2, 3)$-generated or $(3, 3)$-generated and the Ree groups of type ${}^2G_2$ are never $(2, 5)$-generated (as they respectively do not contain elements of orders 3 and 5).

The main ingredient in the proof of Proposition 3.1 is the use of linear rigidity, a notion introduced in [19]. In the statement below, K denotes any field.

Definition Let $g_1, \ldots, g_s \in \mathrm{GL}_n(K)$ with $g_1 \cdots g_s = 1$. We say $(g_1, \ldots, g_s)$ is a *linearly rigid tuple* if the following holds: for any $h_1, \ldots, h_s \in \mathrm{GL}_n(K)$ with $h_1 \cdots h_s = 1$ such that h_i is conjugate to g_i for each i, there exists $g \in \mathrm{GL}_n(K)$ with $g_i = g h_i g^{-1}$ for all i.

We need the following lemma which follows from [19, Theorem 2.3].

Lemma 3.1. *Let $G = \mathrm{SL}_n(\mathbb{F})$ and let g_1, g_2, g_3 be elements of G such that $g_1g_2g_3 = 1$ and $\langle g_1, g_2\rangle$ acts irreducibly on the natural module for G. If*

$$\dim g_1^G + \dim g_2^G + \dim g_3^G = 2\dim G$$

then the triple (g_1, g_2, g_3) is linearly rigid.

Proposition 3.1 is a direct consequence of the following lemma.

Lemma 3.2. *(i) If (p_1, p_2, p_3) is a rigid triple of primes for $\mathrm{SL}_n(\mathbb{F})$ then up to conjugacy in $\mathrm{GL}_n(\mathbb{F})$, there are only finitely many irreducible subgroups of $\mathrm{SL}_n(\mathbb{F})$ that are (p_1, p_2, p_3)-generated.*

(ii) Let $C \leq Z(\mathrm{SL}_n(\mathbb{F}))$. If (p_1, p_2, p_3) is a rigid triple of primes for $\mathrm{SL}_n(\mathbb{F})/C$ then up to isomorphism, there are only finitely many irreducible subgroups of $\mathrm{SL}_n(\mathbb{F})/C$ that are (p_1, p_2, p_3)-generated.

Proof. (i) We denote by $|g|$ the order of an element $g \in \mathrm{SL}_n(\mathbb{F})$. Let

$$\mathcal{T} = \big\{(g_1, g_2, g_3) \in \mathrm{SL}_n(\mathbb{F})^3 : g_1g_2g_3 = 1, |g_i| = p_i, \langle g_1, g_2\rangle \text{ irreducible}\big\}$$

and for a fixed triple (C_1, C_2, C_3) of conjugacy classes of $\mathrm{SL}_n(\mathbb{F})$ consisting respectively of elements of orders p_1, p_2, p_3, let

$$\mathcal{T}_{C_1,C_2,C_3} = \{(g_1, g_2, g_3) \in \mathcal{T} : g_i \in C_i\}.$$

Since (p_1, p_2, p_3) is a rigid triple of primes, it follows from Proposition 2.1 and Lemma 3.1 that every element of $\mathcal{T}$ is linearly rigid. Hence if $\mathcal{T}_{C_1,C_2,C_3} \neq \emptyset$ then $\mathrm{GL}_n(\mathbb{F})$ is transitive on $\mathcal{T}_{C_1,C_2,C_3}$. Since the number of conjugacy classes C_i of elements of order p_i in $\mathrm{SL}_n(\mathbb{F})$ is finite, it follows that $\mathrm{GL}_n(\mathbb{F})$ has finitely many orbits on $\mathcal{T}$. Therefore up to conjugacy in $\mathrm{GL}_n(\mathbb{F})$, there are only finitely many irreducible subgroups of $\mathrm{SL}_n(\mathbb{F})$ that are (p_1, p_2, p_3)-generated.

(ii) Suppose now that (p_1, p_2, p_3) is a rigid triple of primes for $\mathrm{SL}_n(\mathbb{F})/C$. Suppose that g_1, g_2, g_3 are elements of $\mathrm{SL}_n(\mathbb{F})$ such that $g_1g_2g_3 = c$ for some $c \in C$, g_iC has order p_i, and $\langle g_1, g_2\rangle$ is irreducible. Replacing g_3 by g_3c^{-1} we then have $g_1g_2g_3 = 1$, the order $|g_i|$ divides np_i, $\langle g_1, g_2\rangle$ is irreducible, and

$$\sum \dim g_i^{\mathrm{SL}_n(\mathbb{F})} = 2\dim \mathrm{SL}_n(\mathbb{F}).$$

(The latter assertion follows from the rigidity of (p_1, p_2, p_3) for $\mathrm{SL}_n(\mathbb{F})/C$ together with Proposition 2.1 applied to $\mathrm{SL}_n(\mathbb{F})/C$.)
Set

$$\begin{aligned}\hat{\mathcal{T}} \;=\; & \{(g_1, g_2, g_3) \in \mathrm{SL}_n(\mathbb{F})^3 : g_1g_2g_3 = 1,\ |g_i| \text{ divides } np_i,\ \langle g_1, g_2\rangle \text{ irreducible},\\ & \textstyle\sum \dim g_i^{\mathrm{SL}_n(\mathbb{F})} = 2\dim \mathrm{SL}_n(\mathbb{F})\}.\end{aligned}$$

By Lemma 3.1 every element of $\hat{T}$ is linearly rigid, and a similar argument to that given in (i) above shows that $\mathrm{GL}_n(\mathbb{F})$ has finitely many orbits on $\hat{\mathcal{T}}$. Therefore up to isomorphism there are only finitely many irreducible subgroups of $\mathrm{SL}_n(\mathbb{F})/C$ that are (p_1, p_2, p_3)-generated. □

Proof of Proposition 3.2. Let $G = \mathrm{Spin}_n(\mathbb{F})$ or $\mathrm{HSpin}_n(\mathbb{F})$. We describe in the two tables below the elements g in G of order $u \in \{2,3,7\}$ for which the conjugacy class in G has maximal dimension. In the tables below, J_i denotes a unipotent Jordan block of size i, and ω is an element of $\mathbb{F}$ of order u. Also if $G = \mathrm{Spin}_n(\mathbb{F})$ we let $\overline{g}$ be the image of g under the canonical map $G \to \mathrm{SO}_n(\mathbb{F})$; and if $G = \mathrm{HSpin}_n(\mathbb{F})$ we let $\hat{g}$ be the image of g in $\mathrm{PSO}_n(\mathbb{F})$, and let $\overline{g}$ be a preimage of $\hat{g}$ in $\mathrm{SO}_n(\mathbb{F})$.

u	$\overline{g} \in \mathrm{SO}_{11}(\mathbb{F})$	$\dim g^{\mathrm{Spin}_{11}(\mathbb{F})}$	$\dim \overline{g}^{\mathrm{SL}_{11}(\mathbb{F})}$
2	$(-I_4, I_7)$	28	56
$3 = p$	$J_3^3 \oplus J_1^2$	36	78
$3 \neq p$	$(\omega I_3, \omega^{-1} I_3, I_5)$	36	78
	$(\omega I_4, \omega^{-1} I_4, I_3)$	36	80
$7 = p$	$J_7 \oplus J_3 \oplus J_1$	46	100
$7 \neq p$	$(\omega I_2, \omega^{-1} I_2, \omega^2, \omega^{-2}, \omega^3, \omega^{-3}, I_3)$	46	100
	$(\omega I_2, \omega^{-1} I_2, \omega^2 I_2, \omega^{-2} I_2, \omega^3, \omega^{-3}, I_1)$	46	102

u	$\overline{g} \in \mathrm{SO}_{12}(\mathbb{F})$	$\dim g^G$	$\dim \overline{g}^{\mathrm{SL}_{12}(\mathbb{F})}$
2	$(\pm I_4, \mp I_8)$	32	64
$3 = p$	J_3^4	44	96
$3 \neq p$	$(\omega I_4, \omega^{-1} I_4, I_4)$	44	96
$7 = p$	$J_7 \oplus J_5$	56	122
$7 \neq p$	$(\omega I_2, \omega^{-1} I_2, \omega^2 I_2, \omega^{-2} I_2, \omega^3, \omega^{-3}, I_2)$	56	122

Rules for calculating $\dim C_G(g)$ are given in the proof of Lemma 5.1.4. Note also that an involution $(-I_{2i}, I_{n-2i})$ in $\mathrm{SO}_n(\mathbb{F})$ corresponds to an involution in G if and only if i is even.

Let $G_0 = G(p^r)$ and suppose for a contradiction that G_0 is a $(2,3,7)$-group. Then there is a triple (g_1, g_2, g_3) of elements of G_0 of respective orders $2,3,7$ such that $g_1g_2g_3 = 1$ and $G_0 = \langle g_1, g_2 \rangle$. Since $(2,3,7)$ is a rigid triple of primes for G, by Proposition 2.1 g_i^G has maximal dimension. Now by the above tables

$$\sum \dim \overline{g_i}^{\mathrm{SL}_n(\mathbb{F})} < 2 \dim \mathrm{SL}_n(\mathbb{F}).$$

Hence by Proposition 2.1, the group $\langle g_1, g_2 \rangle$ maps onto a reducible subgroup of $\mathrm{SL}_n(\mathbb{F})$ or $\mathrm{PSL}_n(\mathbb{F})$, a contradiction.

Proof of Theorem 2. Let $G = \mathrm{PSp}_4(\mathbb{F})$. In [7, Proposition 6.2] it is shown that $\mathrm{PSp}_4(2^r)$ and $\mathrm{PSp}_4(3^r)$ are never $(2,3)$-generated. We can therefore assume that $p > 3$. An easy check reveals that if $p_3 \geq 7$ then the rigid triple $(2,3,p_3)$ for G remains rigid for $\mathrm{PSL}_4(\mathbb{F})$. The result now follows from Lemma 3.2.

4 Proof of Theorem 3 for exceptional groups

In the rest of the paper, we prove Theorem 3. Let us fix some notation. We let G be a simple algebraic group defined over an algebraically closed field $\mathbb{F}$ of prime characteristic p.
Given an element g in G we let

$$d_g^G = \dim C_G(g) \quad \text{and} \quad \delta_g^G = \dim g^G = \mathrm{codim}\ C_G(g).$$

Finally, given a prime number u, we denote the maximal dimension of a conjugacy class of G of elements of order u by δ_u or δ_u^G. We denote by d_u or d_u^G the dimension of the corresponding centralizer.

Lemma 4.1. *Theorem 3 holds if G is of exceptional type.*

Proof. We first suppose that G is of adjoint type. Given a prime number u, Lawther gives in [6] the maximal dimension δ_u of a conjugacy class of G of elements of order u. In particular, we have the following table.

G	$\dim G$	prime u	δ_u	G	$\dim G$	prime u	δ_u
G_2	14	$u = 2$	8	E_7	133	$u = 2$	70
		$u \in \{3, 5\}$	10			$u = 3$	90
		$u \geq 7$	12			$u = 5$	106
F_4	52	$u = 2$	28			$u \geq 7$	≥ 114
		$u = 3$	36	E_8	248	$u = 2$	128
		$u = 5$	40			$u = 3$	168
		$u \geq 7$	≥ 44			$u = 5$	200
E_6	78	$u = 2$	40			$u \geq 7$	≥ 212
		$u = 3$	54				
		$u = 5$	62				
		$u \geq 7$	≥ 66				

The result follows from the table.

Suppose now that G is of simply connected type. Without loss of generality, we can assume that G is of type E_6 or E_7 and $p \neq 3$ or 2 respectively, as these are the only cases where the simply connected and the adjoint groups are not abstractly isomorphic. We denote by H the corresponding simple algebraic group of adjoint type.
Suppose first that G is of type E_6. If $u \neq 3$ then clearly $\delta_u^G = \delta_u^H$. If $u = 3$ then by the above table, we have $\delta_3^H = 54$. Also by [2, Table 2], we have $\delta_3^G = 54$. Hence the result follows as above.
Suppose finally that G is of type E_7. If $u \neq 2$ then clearly $\delta_u^G = \delta_u^H$. If $u = 2$ then by [1, Table 6], we have $\delta_2^G = 64$. Hence we get the following table.

$\dim G$	δ_2^G	δ_3^G	δ_5^G	δ_u^G $(u \geq 7)$
133	64	90	106	≥ 114

It follows that every hyperbolic triple of primes is nonrigid for G. $\square$

5 Proof of Theorem 3 for classical groups

We now classify hyperbolic triples of primes for classical simple algebraic groups G defined over an algebraically closed field $\mathbb{F}$ of prime characteristic p. For clarity we give below a table describing the classical simple algebraic groups of simply connected type and adjoint type arising from the types A_l, B_l, C_l, D_l. We denote by n the dimension of the natural module for G.

Lie Type	Simply connected group	Adjoint group	Dimension of G
A_{n-1}	$\mathrm{SL}_n(\mathbb{F})$	$\mathrm{PSL}_n(\mathbb{F})$	n^2-1
C_m $m \geq 2$	$\mathrm{Sp}_{2m}(\mathbb{F})$	$\mathrm{PSp}_{2m}(\mathbb{F})$	$n^2/2+n/2$ $(n=2m)$
D_m $m \geq 4$	$\mathrm{SO}_{2m}(\mathbb{F})$ if $p=2$ $\mathrm{Spin}_{2m}(\mathbb{F})$ if $p \neq 2$	$\mathrm{PSO}_{2m}(\mathbb{F})$	$n^2/2-n/2$ $(n=2m)$
B_m, $p \neq 2$ $m \geq 3$	$\mathrm{Spin}_{2m+1}(\mathbb{F})$	$\mathrm{SO}_{2m+1}(\mathbb{F})$	$n^2/2-n/2$ $(n=2m+1)$

To prove Theorem 3 for classical groups, given a prime u, we find δ_u^G - or when this is awkward, a class of large (possibly not maximal) dimension of elements of G of order u - where G is of simply connected type. When the corresponding natural module is of small dimension, this data is recorded in Tables 3-9 in §5.5.

5.1 Elements of prime order

Let G be a classical algebraic group of simply connected type with natural module of dimension n.

Lemma 5.1.1. *The values of δ_2^G and δ_3^G are as in the table below.*

G	δ_2^G	δ_3^G
$\mathrm{SL}_n(\mathbb{F})$	$\lfloor n^2/2 \rfloor$ *if $p=2$ or $n \not\equiv 2 \mod 4$* $\frac{n^2}{2}-2$ *if $p \neq 2$ and $n \equiv 2 \mod 4$*	$\lfloor 2n^2/3 \rfloor$
$\mathrm{Sp}_n(\mathbb{F})$ $n=2m$	m^2+m *if $p=2$* $m^2-\epsilon$ *if $p \neq 2$*	$\lfloor 2(2m^2+m)/3 \rfloor$
$\mathrm{SO}_{2m}(\mathbb{F})$, $p=2$ $n=2m$	$m^2-\epsilon$	$\lfloor 2(2m^2-m)/3 \rfloor$
$\mathrm{Spin}_n(\mathbb{F})$, $p \neq 2$ $n=2m$	$m^2-\epsilon$ *if $m \not\equiv 2 \mod 4$* m^2-4 *if $m \equiv 2 \mod 4$*	$\lfloor 2(2m^2-m)/3 \rfloor$
$\mathrm{Spin}_n(\mathbb{F})$, $p \neq 2$ $n=2m+1$	m^2+m *if $m \not\equiv 1,2 \mod 4$* m^2+m-2 *if $m \equiv 1,2 \mod 4$*	$\lfloor 2(2m^2+m)/3 \rfloor$

In the table $\epsilon \in \{0,1\}$ is such that $m \equiv \epsilon \mod 2$.

Proof. The statement concerning elements of order 3, or elements of order 2 when $\mathrm{char}(\mathbb{F})=2$ follows from the proof of [7, Proposition 4.1].

We now assume $p \neq 2$. For $G = \mathrm{SL}_n(\mathbb{F})$, one checks that the minimal dimension of the centralizer of an involution is achieved by

$$g_1 = \begin{cases} (-I_{(n-\tau)/2}, I_{(n+\tau)/2}) & \text{if } n \equiv \tau \mod 4, \tau \in \{\pm 1\} \\ (-I_{n/2}, I_{n/2}) & \text{if } n \equiv 0 \mod 4 \\ (-I_{\frac{n}{2}+1}, I_{\frac{n}{2}-1}) & \text{if } n \equiv 2 \mod 4. \end{cases}$$

For $G = \mathrm{Sp}_n(\mathbb{F})$, one checks that the minimal dimension of the centralizer of an involution is achieved by

$$g_1 = (-I_{m-\epsilon}, I_{m+\epsilon}).$$

Finally, an involution $g = (-I_{2i}, I_{n-2i})$ in $\mathrm{SO}_n(\mathbb{F})$ lifts to an involution in $\mathrm{Spin}_n(\mathbb{F})$ if and only if i is even. One can therefore check that the minimal dimension of the centralizer of an involution in $\mathrm{Spin}_n(\mathbb{F})$ is achieved by

$$g_1 = \begin{cases} (-I_m, I_{m+\tau}) & \text{if } m \equiv 0 \mod 4 \\ (-I_{m-1}, I_{m+1+\tau}) & \text{if } m \equiv 1 \mod 4 \\ (-I_{m+2}, I_{m-2+\tau}) & \text{if } m \equiv 2 \mod 4 \\ (-I_{m+1}, I_{m-1+\tau}) & \text{if } m \equiv 3 \mod 4 \end{cases}$$

where $\tau \in \{0,1\}$ is such that $n = 2m+\tau$. (Note that here we describe the element g_1 by its image in $\mathrm{SO}_n(\mathbb{F})$.) $\square$

We now describe some general elements of G of prime order $u \geq 5$. In the statement below, J_i denotes a unipotent Jordan block of size i, and we let ω be an element of $\mathbb{F}$ of order u.

Definition Say $n \equiv \zeta \mod 5$ and $n \equiv \gamma \mod 7$ where $0 \leq \zeta < 5$ and $0 \leq \gamma < 7$. We define the elements $g_a, g_b, g_c, g_d, g_e, g_f$ of G of prime order $u \geq 5$ as follows.

G	u	$u = p$	$u \neq p$
$\mathrm{SL}_n(\mathbb{F})$	$u \geq n$	$g_a = J_n$	$g_b = b_\omega$
	$u < n, u \geq 5$	$g_c = J_5^{\frac{n-\zeta}{5}} \oplus J_\zeta$	$g_d = d_{\omega,1}$
	$u < n, u \geq 7$	$g_e = J_7^{\frac{n-\gamma}{7}} \oplus J_\gamma$	$g_f = f_\omega$
$\mathrm{Sp}_n(\mathbb{F})$ $n = 2m$	$u \geq n$	$g_a = J_n$	$g_b = b_\omega$
	$u < n, u \geq 5$	$g_c = \begin{cases} J_5^{\frac{n-\zeta}{5}} \oplus J_\zeta & \text{if } \zeta \text{ even} \\ J_5^{\frac{n-\zeta-5}{5}} \oplus J^2_{\frac{\zeta+5}{2}} & \text{if } \zeta \text{ odd} \end{cases}$	$g_d = d_{\omega,2}$
	$u < n, u \geq 7$	$g_e = \begin{cases} J_7^{\frac{n-\gamma}{7}} \oplus J_\gamma & \text{if } \gamma \text{ even} \\ J_7^{\frac{n-\gamma-7}{7}} \oplus J^2_{\frac{\gamma+7}{2}} & \text{if } \gamma \text{ odd} \end{cases}$	$g_f = f_\omega$
$\mathrm{SO}_n(\mathbb{F})$ $(p=2)$ $\mathrm{Spin}_n(\mathbb{F})$	$u \geq n$	$g_a = \begin{cases} J_n & \text{if } n \text{ odd} \\ J_{n-1} \oplus J_1 & \text{if } n \text{ even} \end{cases}$	$g_b = b_\omega$
	$u < n, u \geq 5$	$g_c = \begin{cases} J_5^{\frac{n-\zeta}{5}} \oplus J_\zeta & \text{if } \zeta \text{ odd} \\ J_5^{\frac{n-\zeta}{5}} \oplus J^2_{\frac{\zeta}{2}} & \text{if } \zeta \text{ even} \end{cases}$	$g_d = d_{\omega,2}$
	$u < n, u \geq 7$	$g_e = \begin{cases} J_7^{\frac{n-\gamma}{7}} \oplus J_\gamma & \text{if } \gamma \text{ odd} \\ J_7^{\frac{n-\gamma}{7}} \oplus J^2_{\frac{\gamma}{2}} & \text{if } \gamma \text{ even} \end{cases}$	$g_f = f_\omega$

Here the following hold:

$$b_\omega = \begin{cases} (\omega, \omega^{-1}, \ldots, \omega^{\frac{n}{2}}, \omega^{-\frac{n}{2}}) & \text{if } n \text{ even} \\ (\omega, \omega^{-1}, \ldots, \omega^{\frac{n-1}{2}}, \omega^{\frac{1-n}{2}}, 1) & \text{if } n \text{ odd} \end{cases}$$

$$d_{\omega,1} = (\omega I_{\frac{n-\zeta}{5}}, \omega^{-1} I_{\frac{n-\zeta}{5}}, \omega^2 I_{\frac{n-\zeta}{5}}, \omega^{-2} I_{\frac{n-\zeta}{5}}, I_{\frac{n+4\zeta}{5}})$$

$$d_{\omega,2} = \begin{cases} (\omega I_{\frac{n-4}{4}}, \omega^{-1} I_{\frac{n-4}{4}}, \omega^2 I_{\frac{n-4}{4}}, \omega^{-2} I_{\frac{n-4}{4}}, I_4) & \text{if } n \equiv 0 \mod 4 \\ (\omega I_{\frac{n-1}{4}}, \omega^{-1} I_{\frac{n-1}{4}}, \omega^2 I_{\frac{n-1}{4}}, \omega^{-2} I_{\frac{n-1}{4}}, 1) & \text{if } n \equiv 1 \mod 4 \\ (\omega I_{\frac{n-2}{4}}, \omega^{-1} I_{\frac{n-2}{4}}, \omega^2 I_{\frac{n-2}{4}}, \omega^{-2} I_{\frac{n-2}{4}}, I_2) & \text{if } n \equiv 2 \mod 4 \\ (\omega I_{\frac{n-3}{4}}, \omega^{-1} I_{\frac{n-3}{4}}, \omega^2 I_{\frac{n-3}{4}}, \omega^{-2} I_{\frac{n-3}{4}}, I_3) & \text{if } n \equiv 3 \mod 4 \end{cases}$$

$$f_\omega = (\omega I_{\frac{n-\gamma}{7}}, \omega^{-1} I_{\frac{n-\gamma}{7}}, \omega^2 I_{\frac{n-\gamma}{7}}, \omega^{-2} I_{\frac{n-\gamma}{7}}, \omega^3 I_{\frac{n-\gamma}{7}}, \omega^{-3} I_{\frac{n-\gamma}{7}}, I_{\frac{n+6\gamma}{7}}).$$

Also we slightly abuse notation by describing elements of $\mathrm{Spin}_n(\mathbb{F})$ by their images in $\mathrm{SO}_n(\mathbb{F})$.

Note that up to conjugation the elements $g_a, g_b, g_c, g_d, g_e, g_f$ are uniquely defined (see [8, Theorem 3]). In the following three lemmas, we give the dimensions of their conjugacy classes.

Lemma 5.1.2. *Suppose $G = \mathrm{SL}_n(\mathbb{F})$. Then*

(i) $\delta^G_{g_a} = \delta^G_{g_b} = n^2 - n$.

(ii) $\delta^G_{g_c} = (4n^2 + \zeta(\zeta-5))/5$ *and* $\delta^G_{g_d} = 4(n^2 - \zeta^2)/5$.

(iii) $\delta^G_{g_e} = (6n^2 + \gamma(\gamma-7))/7$ *and* $\delta^G_{g_f} = 6(n^2 - \gamma^2)/7$.

Lemma 5.1.3. *Suppose $G = \mathrm{Sp}_n(\mathbb{F})$. Then*

(i) $\delta^G_{g_a} = \delta^G_{g_b} = n^2/2$.

(ii) $$\delta^G_{g_c} = \begin{cases} (4n^2+4n+\zeta(\zeta-4))/10 & \text{if } \zeta \text{ is even} \\ (4n^2+4n-28)/10 & \text{if } \zeta = 1 \\ (4n^2+4n-8)/10 & \text{if } \zeta = 3. \end{cases}$$

(iii) $$\delta^G_{g_d} = \begin{cases} (3n^2+12n-96)/8 & \text{if } n \equiv 0 \mod 4 \\ (3n^2+8n-28)/8 & \text{if } n \equiv 2 \mod 4. \end{cases}$$

(iv) $$\delta^G_{g_e} = \begin{cases} (6n^2+6n+\gamma(\gamma-6))/14 & \text{if } \gamma \text{ is even} \\ (6n^2+6n-40)/14 & \text{if } \gamma = 1 \\ (6n^2+6n-44)/14 & \text{if } \gamma = 3 \\ (6n^2+6n-12)/14 & \text{if } \gamma = 5. \end{cases}$$

(v) $\delta^G_{g_f} = 3(n^2 + n - \gamma(\gamma+1))/7$.

Lemma 5.1.4. *Suppose $G = \mathrm{Spin}_n(\mathbb{F})$ or $\mathrm{SO}_n(\mathbb{F})$ $(p = 2)$. Then*

(i) $\delta^G_{g_a} = \delta^G_{g_b} = (n^2 - 2n + \varsigma)/2$ *where $\varsigma \in \{0,1\}$ is such that $n \equiv \varsigma \mod 2$.*

(ii) $$\delta^G_{g_c} = \begin{cases} (4n^2-4n+\zeta(\zeta-6)+5)/10 & \text{if } \zeta \text{ is odd} \\ (4n^2-4n)/10 & \text{if } \zeta = 0 \\ (4n^2-4n-8)/10 & \text{if } \zeta = 2 \\ (4n^2-4n-28)/10 & \text{if } \zeta = 4. \end{cases}$$

(iii) $$\delta^G_{g_d} = \begin{cases} (3n^2+4n-64)/8 & \text{if } n \equiv 0 \mod 4 \\ (3n^2-2n-1)/8 & \text{if } n \equiv 1 \mod 4 \\ (3n^2-12)/8 & \text{if } n \equiv 2 \mod 4 \\ (3n^2+2n-33)/8 & \text{if } n \equiv 3 \mod 4. \end{cases}$$

(iv) $$\delta^G_{g_e} = \begin{cases} (6n^2-6n+\gamma(\gamma-8)+7)/14 & \text{if } \gamma \text{ is odd} \\ (6n^2-6n)/14 & \text{if } \gamma = 0 \\ (6n^2-6n-12)/14 & \text{if } \gamma = 2 \\ (6n^2-6n-44)/14 & \text{if } \gamma = 4 \\ (6n^2-6n-40)/14 & \text{if } \gamma = 6. \end{cases}$$

(v) $\delta^G_{g_f} = 3(n^2 - n - \gamma(\gamma-1))/7$.

Proof of Lemmas 5.1.2-5.1.4. This follows from the calculation of $d_g^G = \dim C_G(g)$ for $g \in \{g_a, \dots, g_f\}$. Suppose first that g is a unipotent element. As the order u of g is greater than 5, we have $p \geq 5$. Hence by [8, Theorem 3] $g = \oplus_i J_i^{r_i}$ where r_i is even for each odd i if $G = \mathrm{Sp}_n(\mathbb{F})$, and r_i is even for each even i if $G = \mathrm{Spin}_n(\mathbb{F})$, and

$$d_g^G = \begin{cases} \sum_i ir_i^2 + 2\sum_{i<j} ir_ir_j - 1 & \text{if } G = \mathrm{SL}_n(\mathbb{F}) \\ \frac{1}{2}\sum_i ir_i^2 + \sum_{i<j} ir_ir_j + \frac{1}{2}\sum_{i \text{ odd}} r_i & \text{if } G = \mathrm{Sp}_n(\mathbb{F}) \\ \frac{1}{2}\sum_i ir_i^2 + \sum_{i<j} ir_ir_j - \frac{1}{2}\sum_{i \text{ odd}} r_i & \text{if } G = \mathrm{Spin}_n(\mathbb{F}). \end{cases}$$

Suppose now that g is a semisimple element. It is straightforward to derive the structure of $C_G(g)$ and to calculate d_g^G. For example suppose $G = \mathrm{Sp}_n(\mathbb{F})$ and $g = (\omega_1 I_{i_1}, \omega_1^{-1} I_{i_1}, \omega_2 I_{i_2}, \omega_2^{-1} I_{i_2}, \dots, \omega_k I_{i_k}, \omega_k^{-1} I_{i_k}, I_{n-2k})$ where ω_i, ω_i^{-1} are distinct u^{th} roots of 1. Then $C_G(g) = \mathrm{GL}_{i_1}(\mathbb{F}) \times \mathrm{GL}_{i_2}(\mathbb{F}) \times \cdots \times \mathrm{GL}_{i_k}(\mathbb{F}) \times \mathrm{Sp}_{n-2k}(\mathbb{F})$ and so

$$d_g^G = \frac{1}{2}((n-2k)^2 + (n-2k)) + \sum_{j=1}^{k} i_j^2.$$

5.2 Proof of Theorem 3 for simply connected groups

We prove Theorem 3 for the simply connected classical groups in a series of lemmas. Recall that $p = \mathrm{char}(\mathbb{F})$.

Lemma 5.2.1. *Suppose $(p, p_1) \neq (2, 2)$. Assume $G = \mathrm{SL}_n(\mathbb{F})$ with $n \leq 14$, or $G = \mathrm{Sp}_n(\mathbb{F})$ with $n \leq 12$. Then Theorem 3 holds.*

Proof. The result follows from Tables 3 and 5 in §5.5 which list for a given prime u the maximal dimension of a conjugacy class of G of elements of order u. □

Lemma 5.2.2. *Suppose $p > 2$ and $(p_1, p_2) = (2, 3)$. Assume $G = \mathrm{SL}_n(\mathbb{F})$ with $n \geq 15$, or $G = \mathrm{Sp}_n(\mathbb{F})$ with $n \geq 14$, or $G = \mathrm{Spin}_n(\mathbb{F})$. Then Theorem 3 holds.*

Proof. Let

$$\alpha = \begin{cases} 34 & \text{if } G = \mathrm{SL}_n(\mathbb{F}) \\ 48 & \text{if } G = \mathrm{Sp}_n(\mathbb{F}) \\ 27 & \text{if } G = \mathrm{Spin}_n(\mathbb{F}). \end{cases}$$

Suppose first that $n \leq \alpha$. We use Tables 4, 6, 7, 8 and 9 in §5.5 which list the maximal dimension of a conjugacy class of an element of order $u \in \{2, 3, 7\}$. They also give a representative g in G of prime order $u \geq 11$ whose conjugacy class has large dimension. The result follows from the tables.

Suppose now that $n > \alpha$. If $n \leq p_3$ then there is a regular element g in G of order p_3. Hence $\delta_{p_3}^G = \dim G - \mathrm{rank}\, G$ and by applying Lemma 5.1.1 we get $\delta_2^G + \delta_3^G + \delta_{p_3}^G > 2\dim G$. Hence the triple $(2, 3, p_3)$ is nonrigid. Assume finally that $n > p_3$. Consider an element g in G of order p_3 of the form g_e if $p_3 = p$, and of the form g_f if $p_3 \neq p$. Then Lemmas 5.1.1-5.1.4 yield $\delta_2^G + \delta_3^G + \delta_g^G > 2\dim G$, and so the triple $(2, 3, p_3)$ is nonrigid. □

Lemma 5.2.3. *Suppose $p > 2$, $p_1 = 2$ and $p_2 \geq 5$. Assume $G = \mathrm{SL}_n(\mathbb{F})$ with $n \geq 15$, or $G = \mathrm{Sp}_n(\mathbb{F})$ with $n \geq 14$, or $G = \mathrm{Spin}_n(\mathbb{F})$. Then Theorem 3 holds.*

Proof. We prove that the triple $(2, p_2, p_2)$ is nonrigid. It is then clear that in general the triple $(2, p_2, p_3)$ is nonrigid.

If $G = \mathrm{Spin}_n(\mathbb{F})$ with $n \leq 9$ then the result follows at once from Tables 8 and 9. We therefore suppose that $n \geq 10$ if $G = \mathrm{Spin}_n(\mathbb{F})$. If $n \leq p_2$ then there exists a regular element g in G of order p_2, and by applying Lemma 5.1.1 we get $\delta_2^G + 2\,\delta_{p_2}^G > 2\dim G$. Hence the triple $(2, p_2, p_2)$ is nonrigid. Assume finally that $n > p_2$. Consider an element g in G of order p_2 of the form g_c if $p_2 = p$, and of the form g_d if $p_2 \neq p$. Then Lemmas 5.1.1-5.1.4 yield $\delta_2^G + 2\delta_g^G > 2\dim G$, and so the triple $(2, p_2, p_2)$ is nonrigid. □

Lemma 5.2.4. *Suppose $p = p_1 = 2$. Then Theorem 3 holds for simply connected classical groups G.*

Proof. By Lemma 5.1.1 the maximal dimension of an involution class in G is at least equal to the maximal dimension of an involution class in a simply connected group H of the same type as G but defined over an algebraically closed field of odd characteristic. The result now follows from the proofs of Lemmas 5.2.1-5.2.3 where we classify the triple $(2, p_2, p_3)$ for H. Indeed, if $(2, p_2, p_3)$ is nonrigid for H then it is also nonrigid for G, and if it is rigid for H then it remains rigid for G if and only if $\delta_2^G = \delta_2^H$. Finally if $(2, p_2, p_3)$ is reducible for H then the difference $\delta_2^G - \delta_2^H$ is positive and depending on its value the triple $(2, p_2, p_3)$ becomes either rigid or nonrigid for G. □

Lemma 5.2.5. *Suppose $p_1 \geq 3$. Assume $G = \mathrm{SL}_n(\mathbb{F})$ with $n \geq 15$, or $G = \mathrm{Sp}_n(\mathbb{F})$ with $n \geq 14$, or $G = \mathrm{Spin}_n(\mathbb{F})$, or $G = \mathrm{SO}_n(\mathbb{F})$ with $n = 2m$ and $p = 2$. Then Theorem 3 holds.*

Proof. We prove that the triple (p_1, p_2, p_3) is nonrigid. Suppose first that $(p_1, p_2, p_3) \neq (3, 3, 5)$. Assume G is not of symplectic type or $n \notin \{14, 16, 18, 22\}$ or $(p_1, p_2, p_3) \neq (3, 3, 7)$. Since $\delta_{p_1}^G > \delta_2^G$, the result follows from the fact that the triple $(2, p_2, p_3)$ is not reducible for G by previous lemmas. If $G = \mathrm{Sp}_n(\mathbb{F})$ where $n \in \{14, 16, 18, 22\}$ and $(p_1, p_2, p_3) = (3, 3, 7)$ then the result follows at once from Table 6.
Suppose finally that $(p_1, p_2, p_3) = (3, 3, 5)$. If G is of type D_4 then the result follows from Table 8. For the other cases, one can choose an element g of G of order 5 of the form g_c or g_d and simply apply Lemmas 5.1.1-5.1.4 to show that $2\delta_3^G + \delta_g^G > 2\dim G$. □

This completes the proof of Theorem 3 for simply connected groups.

5.3 Proof of Theorem 3 for adjoint groups

Let H be the simply connected group corresponding to the adjoint group G. We have $\delta_2^G \geq \delta_2^H$ and for a prime $u \geq 3$, δ_u^G is equal to δ_u^H. More precisely, by [7, Proposition 4.1] we have $\delta_2^{\mathrm{PSL}_n(\mathbb{F})} = \lfloor n^2/2 \rfloor$, $\delta_2^{\mathrm{PSp}_{2m}(\mathbb{F})} = \delta_2^{\mathrm{SO}_{2m+1}(\mathbb{F})} = m^2 + m$ and finally $\delta_2^{\mathrm{PSO}_{2m}(\mathbb{F})} = m^2 - \epsilon$ where $\epsilon \in \{0, 1\}$ is such that $m \equiv \epsilon$ mod 2. The result now follows from the classification of triples of primes for the corresponding simply connected group H, established in §5.2.

5.4 Proof of Theorem 3 for other groups

Suppose first that $G = \mathrm{SL}_n(\mathbb{F})/C$ where $C \leq Z(\mathrm{SL}_n(\mathbb{F}))$. For a prime $u \geq 3$ we have $\delta_u^G = \delta_u^{\mathrm{SL}_n(\mathbb{F})}$, and if C does not contain an involution then $\delta_2^G = \delta_2^{\mathrm{SL}_n(\mathbb{F})}$, otherwise $\delta_2^G = \delta_2^{\mathrm{PSL}_n(\mathbb{F})}$. The result now follows from the classification of triples of primes for $\mathrm{SL}_n(\mathbb{F})$ and $\mathrm{PSL}_n(\mathbb{F})$ established in §§5.2 and 5.3.

Suppose finally that G is of type D_m and not of simply connected or adjoint type. If $G = \mathrm{SO}_{2m}(\mathbb{F})$ then the result follows from the fact that $\delta_2^G = \delta_2^{\mathrm{PSO}_{2m}(\mathbb{F})}$, and if G is a half-spin group the result then follows from the fact that $\delta_2^G = \delta_2^{\mathrm{Spin}_{2m}(\mathbb{F})}$.

5.5 Tables of conjugacy classes in groups of low rank

We list in Tables 3-9 some representatives g of prime order u in simply connected classical groups G of low rank defined over $\mathbb{F}$. We let n denote the dimension of the natural module for G. For each g we give the dimension δ_g^G of its conjugacy class. We assume that $p = \mathrm{char}(\mathbb{F}) > 2$ if $u = 2$. The elements described have conjugacy classes of maximal dimension provided that they are of order $u \in \{2, 3, 5, 7\}$, or G is of linear type and $n \leq 14$, or G is of symplectic type and $n \leq 12$. If $u > 7$ the classes given are of sufficiently large dimension for us to check in the above proofs that every hyperbolic triple (p_1, p_2, p_3) of primes with $p_3 > 7$ is nonrigid for G, unless G is of linear type and $n \leq 6$ or G is of symplectic type and $n \leq 10$.

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n	$u=2$	$u=3$	$u=5$	$u=7$	$u=11$	$u=13$	$u\geq 17$
2	$\delta_g^G=0$	$\delta_g^G=2$					
3	$\delta_g^G=4$	$\delta_g^G=6$					
4	$\delta_g^G=8$	$\delta_g^G=10$	$\delta_g^G=12$				
5	$\delta_g^G=12$	$\delta_g^G=16$	$\delta_g^G=20$				
6	$\delta_g^G=16$	$\delta_g^G=24$	$g=\begin{cases}(\omega,\omega^{-1},\omega^2,\omega^{-2},I_2) & \text{if } p\neq 5\\ J_5\oplus J_1 & \text{if } p=5\end{cases}$ $\delta_g^G=28$	$\delta_g^G=30$			
7	$\delta_g^G=24$	$\delta_g^G=32$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2,\omega^{-2},1) & \text{if } p\neq 5\\ J_5\oplus J_2 & \text{if } p=5\end{cases}$ $\delta_g^G=38$	$\delta_g^G=42$			
8	$\delta_g^G=32$	$\delta_g^G=42$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2,\omega^{-2},I_2) & \text{if } p\neq 5\\ J_5\oplus J_3 & \text{if } p=5\end{cases}$ $\delta_g^G=50$	$g=\begin{cases}(\omega,\omega^{-1},\dots,\omega^3,\omega^{-3},I_2) & \text{if } p\neq 7\\ J_7\oplus J_1 & \text{if } p=7\end{cases}$ $\delta_g^G=54$	$\delta_g^G=56$		
9	$\delta_g^G=40$	$\delta_g^G=54$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2I_2,\omega^{-2}I_2,1) & \text{if } p\neq 5\\ J_5\oplus J_4 & \text{if } p=5\end{cases}$ $\delta_g^G=64$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2,\omega^{-2}, & \text{if } p\neq 7\\ \omega^3,\omega^{-3},1) & \\ J_7\oplus J_2 & \text{if } p=7\end{cases}$ $\delta_g^G=68$	$\delta_g^G=72$		
10	$\delta_g^G=48$	$\delta_g^G=66$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2I_2,\omega^{-2}I_2,I_2) & \text{if } p\neq 5\\ J_5^2 & \text{if } p=5\end{cases}$ $\delta_g^G=80$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2, & \text{if } p\neq 7\\ \omega^2,\omega^{-2},\omega^3,\omega^{-3},I_2) & \\ J_7\oplus J_3 & \text{if } p=7\end{cases}$ $\delta_g^G=84$	$\delta_g^G=90$		
11	$\delta_g^G=60$	$\delta_g^G=80$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2I_2,\omega^{-2}I_2,I_3) & \text{if } p\neq 5\\ J_5^2\oplus J_1 & \text{if } p=5\end{cases}$ $\delta_g^G=96$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2I_2,\omega^{-2}I_2, & \text{if } p\neq 7\\ \omega^3,\omega^{-3},1) & \\ J_7\oplus J_4 & \text{if } p=7\end{cases}$ $\delta_g^G=102$	$\delta_g^G=110$		
12	$\delta_g^G=72$	$\delta_g^G=96$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_2,\omega^{-2}I_2,I_2) & \text{if } p\neq 5\\ J_5^2\oplus J_2 & \text{if } p=5\end{cases}$ $\delta_g^G=114$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2I_2,\omega^{-2}I_2, & \text{if } p\neq 7\\ \omega^3,\omega^{-3},I_2) & \\ J_7\oplus J_5 & \text{if } p=7\end{cases}$ $\delta_g^G=122$	$g=\begin{cases}(\omega,\omega^{-1},\dots,\omega^5,\omega^{-5},I_2) & \text{if } p\neq 11\\ J_{11}\oplus J_1 & \text{if } p=11\end{cases}$ $\delta_g^G=130$	$\delta_g^G=132$	
13	$\delta_g^G=84$	$\delta_g^G=112$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_2,\omega^{-2}I_2,I_3) & \text{if } p\neq 5\\ J_5^2\oplus J_3 & \text{if } p=5\end{cases}$ $\delta_g^G=134$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2I_2,\omega^{-2}I_2, & \text{if } p\neq 7\\ \omega^3I_2,\omega^{-3}I_2,1) & \\ J_7\oplus J_6 & \text{if } p=7\end{cases}$ $\delta_g^G=144$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2, & \text{if } p\neq 11\\ \omega^2,\omega^{-2},\dots,\omega^5,\omega^{-5},1) & \\ J_{11}\oplus J_2 & \text{if } p=11\end{cases}$ $\delta_g^G=152$	$\delta_g^G=156$	
14	$\delta_g^G=96$	$\delta_g^G=130$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_3,\omega^{-2}I_3,I_2) & \text{if } p\neq 5\\ J_5^2\oplus J_4 & \text{if } p=5\end{cases}$ $\delta_g^G=156$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2I_2,\omega^{-2}I_2, & \text{if } p\neq 7\\ \omega^3I_2,\omega^{-3}I_2,I_2) & \\ J_7^2 & \text{if } p=7\end{cases}$ $\delta_g^G=168$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2, & \text{if } p\neq 11\\ \omega^2,\omega^{-2},\dots,\omega^5,\omega^{-5},I_2) & \\ J_{11}\oplus J_3 & \text{if } p=11\end{cases}$ $\delta_g^G=176$	$\delta_g^G=180$	$\delta_g^G=182$

Table 3: Maximal dimension of a conjugacy class of an element of prime order u in $G=\mathrm{SL}_n(\mathbb{F})$, $\mathrm{char}(\mathbb{F})>0$, $n\leq 14$

$p=\mathrm{char}(\mathbb{F})>2$ if $u=2$.
$\omega\in\mathbb{F}$ denotes an element of order u.

Table 4: Possible dimension for a conjugacy class of an element of prime order $u \neq 5$ in $G = \mathrm{SL}_n(\mathbb{F})$, $\mathrm{char}(\mathbb{F}) > 0$, $15 \leq n \leq 34$

n	$\dim G$	$u = 2$	$u = 3$	$u \geq 7$
15	224	$\delta_g^G = 112$	$\delta_g^G = 150$	$g = \begin{cases} (\omega I_2, \omega^{-1} I_2, \omega^2 I_2, \omega^{-2} I_2, \omega^3 I_2, \omega^{-3} I_2, I_3) & \text{if } p \neq u \\ J_7^2 \oplus J_1 & \text{if } p = u \end{cases}$ $\delta_g^G = 192$
16	255	$\delta_g^G = 128$	$\delta_g^G = 170$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_2, \omega^{-2} I_2, \omega^3 I_2, \omega^{-3} I_2, I_2) & \text{if } p \neq u \\ J_7^2 \oplus J_2 & \text{if } p = u \end{cases}$ $\delta_g^G = 218$
17	288	$\delta_g^G = 144$	$\delta_g^G = 192$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_2, \omega^{-2} I_2, \omega^3 I_2, \omega^{-3} I_2, I_3) & \text{if } p \neq u \\ J_7^2 \oplus J_3 & \text{if } p = u \end{cases}$ $\delta_g^G = 246$
18	323	$\delta_g^G = 160$	$\delta_g^G = 216$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_3, \omega^{-2} I_3, \omega^3 I_2, \omega^{-3} I_2, I_2) & \text{if } p \neq u \\ J_7^2 \oplus J_4 & \text{if } p = u \end{cases}$ $\delta_g^G = 276$
19	360	$\delta_g^G = 180$	$\delta_g^G = 240$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_3, \omega^{-2} I_3, \omega^3 I_2, \omega^{-3} I_2, I_3) & \text{if } p \neq u \\ J_7^2 \oplus J_5 & \text{if } p = u \end{cases}$ $\delta_g^G = 308$
20	399	$\delta_g^G = 200$	$\delta_g^G = 266$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_3, \omega^{-2} I_3, \omega^3 I_3, \omega^{-3} I_3, I_2) & \text{if } p \neq u \\ J_7^2 \oplus J_6 & \text{if } p = u \end{cases}$ $\delta_g^G = 342$
21	440	$\delta_g^G = 220$	$\delta_g^G = 294$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_3, \omega^{-2} I_3, \omega^3 I_3, \omega^{-3} I_3, I_3) & \text{if } p \neq u \\ J_7^3 & \text{if } p = u \end{cases}$ $\delta_g^G = 378$
22	483	$\delta_g^G = 240$	$\delta_g^G = 322$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_3, \omega^{-2} I_3, \omega^3 I_3, \omega^{-3} I_3, I_4) & \text{if } p \neq u \\ J_7^3 \oplus J_1 & \text{if } p = u \end{cases}$ $\delta_g^G = 414$
23	528	$\delta_g^G = 264$	$\delta_g^G = 352$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_3, \omega^{-2} I_3, \omega^3 I_3, \omega^{-3} I_3, I_3) & \text{if } p \neq u \\ J_7^3 \oplus J_2 & \text{if } p = u \end{cases}$ $\delta_g^G = 452$
24	575	$\delta_g^G = 288$	$\delta_g^G = 384$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_3, \omega^{-2} I_3, \omega^3 I_3, \omega^{-3} I_3, I_4) & \text{if } p \neq u \\ J_7^3 \oplus J_3 & \text{if } p = u \end{cases}$ $\delta_g^G = 492$
25	624	$\delta_g^G = 312$	$\delta_g^G = 416$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_4, \omega^{-2} I_4, \omega^3 I_3, \omega^{-3} I_3, I_3) & \text{if } p \neq u \\ J_7^3 \oplus J_4 & \text{if } p = u \end{cases}$ $\delta_g^G = 534$
26	675	$\delta_g^G = 336$	$\delta_g^G = 450$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_4, \omega^{-2} I_4, \omega^3 I_3, \omega^{-3} I_3, I_4) & \text{if } p \neq u \\ J_7^3 \oplus J_5 & \text{if } p = u \end{cases}$ $\delta_g^G = 578$
27	728	$\delta_g^G = 364$	$\delta_g^G = 486$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_4, \omega^{-2} I_4, \omega^3 I_4, \omega^{-3} I_4, I_3) & \text{if } p \neq u \\ J_7^3 \oplus J_6 & \text{if } p = u \end{cases}$ $\delta_g^G = 624$
28	783	$\delta_g^G = 392$	$\delta_g^G = 522$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_4, \omega^{-2} I_4, \omega^3 I_4, \omega^{-3} I_4, I_4) & \text{if } p \neq u \\ J_7^4 & \text{if } p = u \end{cases}$ $\delta_g^G = 672$
29	840	$\delta_g^G = 420$	$\delta_g^G = 560$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_4, \omega^{-2} I_4, \omega^3 I_4, \omega^{-3} I_4, I_5) & \text{if } p \neq u \\ J_7^4 \oplus J_1 & \text{if } p = u \end{cases}$ $\delta_g^G = 720$
30	899	$\delta_g^G = 448$	$\delta_g^G = 600$	$g = \begin{cases} (\omega I_5, \omega^{-1} I_5, \omega^2 I_4, \omega^{-2} I_4, \omega^3 I_4, \omega^{-3} I_4, I_4) & \text{if } p \neq u \\ J_7^4 \oplus J_2 & \text{if } p = u \end{cases}$ $\delta_g^G = 770$
31	960	$\delta_g^G = 480$	$\delta_g^G = 640$	$g = \begin{cases} (\omega I_5, \omega^{-1} I_5, \omega^2 I_4, \omega^{-2} I_4, \omega^3 I_4, \omega^{-3} I_4, I_5) & \text{if } p \neq u \\ J_7^4 \oplus J_3 & \text{if } p = u \end{cases}$ $\delta_g^G = 822$
32	1023	$\delta_g^G = 512$	$\delta_g^G = 682$	$g = \begin{cases} (\omega I_5, \omega^{-1} I_5, \omega^2 I_5, \omega^{-2} I_5, \omega^3 I_4, \omega^{-3} I_4, I_4) & \text{if } p \neq u \\ J_7^4 \oplus J_4 & \text{if } p = u \end{cases}$ $\delta_g^G = 876$
33	1088	$\delta_g^G = 544$	$\delta_g^G = 726$	$g = \begin{cases} (\omega I_5, \omega^{-1} I_5, \omega^2 I_5, \omega^{-2} I_5, \omega^3 I_4, \omega^{-3} I_4, I_5) & \text{if } p \neq u \\ J_7^4 \oplus J_5 & \text{if } p = u \end{cases}$ $\delta_g^G = 932$
34	1155	$\delta_g^G = 576$	$\delta_g^G = 770$	$g = \begin{cases} (\omega I_5, \omega^{-1} I_5, \omega^2 I_5, \omega^{-2} I_5, \omega^3 I_5, \omega^{-3} I_5, I_4) & \text{if } p \neq u \\ J_7^4 \oplus J_6 & \text{if } p = u \end{cases}$ $\delta_g^G = 990$

$p = \mathrm{char}(\mathbb{F}) > 2$ if $u = 2$.

$\omega \in \mathbb{F}$ denotes an element of order u.

The dimension given is maximal if $u = 2, 3, 7$.

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n	$u=2$	$u=3$	$u=5$	$u=7$	$u=11$	$u \geq 13$
4	$\delta_g^G = 4$	$\delta_g^G = 6$	$\delta_g^G = 8$			
6	$\delta_g^G = 8$	$\delta_g^G = 14$	$g = \begin{cases} (\omega, \omega^{-1}, \omega^2, \omega^{-2}, I_2) & \text{if } p \neq 5 \\ J_4 \oplus J_2 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 16$	$\delta_g^G = 18$		
8	$\delta_g^G = 16$	$\delta_g^G = 24$	$g = \begin{cases} (\omega I_2, \omega^{-1} I_2, \omega^2, \omega^{-2}, I_2) & \text{if } p \neq 5 \\ J_4^2 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 28$	$g = \begin{cases} (\omega, \omega^{-1}, \ldots, \omega^3, \omega^{-3}, I_2) & \text{if } p \neq 7 \\ J_6 \oplus J_2 & \text{if } p = 7 \end{cases}$ $\delta_g^G = 30$	$\delta_g^G = 32$	
10	$\delta_g^G = 24$	$\delta_g^G = 36$	$g = \begin{cases} (\omega I_2, \omega^{-1} I_2, \omega^2 I_2, \omega^{-2} I_2, I_2) & \text{if } p \neq 5 \\ J_5^2 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 44$	$g = \begin{cases} (\omega I_2, \omega^{-1} I_2, \omega^2, \omega^{-2}, \omega^3, \omega^{-3}, I_2) & \text{if } p \neq 7 \\ J_6 \oplus J_4 & \text{if } p = 7 \end{cases}$ $\delta_g^G = 46$	$\delta_g^G = 50$	
12	$\delta_g^G = 36$	$\delta_g^G = 52$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_2, \omega^{-2} I_2, I_2) & \text{if } p \neq 5 \\ J_5^2 \oplus J_2 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 62$	$g = \begin{cases} (\omega I_2, \omega^{-1} I_2, \omega^2 I_2, \omega^{-2} I_2, \omega^3, \omega^{-3}, I_2) & \text{if } p \neq 7 \\ J_6^2 & \text{if } p = 7 \end{cases}$ $\delta_g^G = 66$	$g = \begin{cases} (\omega, \omega^{-1}, \ldots, \omega^5, \omega^{-5}, I_2) & \text{if } p \neq 11 \\ J_{10} \oplus J_2 & \text{if } p = 11 \end{cases}$ $\delta_g^G = 70$	$\delta_g^G = 72$

Table 5: Maximal dimension of a conjugacy class of an element of prime order u in $G = \mathrm{Sp}_n(\mathbb{F})$, $\mathrm{char}(\mathbb{F}) > 0$, $n \in \{4, 6, 8, 10, 12\}$

$p = \mathrm{char}(\mathbb{F}) > 2$ if $u = 2$.
$\omega \in \mathbb{F}$ denotes an element of order u.

n	$\dim G$	$u=2$	$u=3$	$u=7$	$u\geq 11$
14	105	$\delta_g^G=48$	$\delta_g^G=70$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2I_2,\omega^{-2}I_2,\omega^3I_2,\omega^{-3}I_2,I_2) & \text{if } p\neq 7\\ J_7^2 & \text{if } p=7\end{cases}$ $\delta_g^G=90$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2,\omega^{-2},\dots,\omega^5,\omega^{-5},I_2) & \text{if } p\neq u\\ J_{10}\oplus J_4 & \text{if } p=u\end{cases}$ $\delta_g^G=94$
16	136	$\delta_g^G=64$	$\delta_g^G=90$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_2,\omega^{-2}I_2,\omega^3I_2,\omega^{-3}I_2,I_2) & \text{if } p\neq 7\\ J_7^2\oplus J_2 & \text{if } p=7\end{cases}$ $\delta_g^G=116$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\dots,\omega^3I_2,\omega^{-3}I_2,\omega^4,\omega^{-4},\omega^5,\omega^{-5}) & \text{if } p\neq u\\ J_{10}\oplus J_6 & \text{if } p=u\end{cases}$ $\delta_g^G=122$
18	171	$\delta_g^G=80$	$\delta_g^G=114$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_3,\omega^{-2}I_3,\omega^3I_2,\omega^{-3}I_2,I_2) & \text{if } p\neq 7\\ J_7^2\oplus J_4 & \text{if } p=7\end{cases}$ $\delta_g^G=146$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\dots,\omega^3I_2,\omega^{-3}I_2,\omega^4,\omega^{-4},\omega^5,\omega^{-5},I_2) & \text{if } p\neq u\\ J_{10}\oplus J_8 & \text{if } p=u\end{cases}$ $\delta_g^G=154$
20	210	$\delta_g^G=100$	$\delta_g^G=140$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_3,\omega^{-2}I_3,\omega^3I_3,\omega^{-3}I_3,I_2) & \text{if } p\neq 7\\ J_7^2\oplus J_6 & \text{if } p=7\end{cases}$ $\delta_g^G=180$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\dots,\omega^5I_2,\omega^{-5}I_2) & \text{if } p\neq u\\ J_{10}^2 & \text{if } p=u\end{cases}$ $\delta_g^G=190$
22	253	$\delta_g^G=120$	$\delta_g^G=168$	$g=\begin{cases}(\omega I_4,\omega^{-1}I_4,\omega^2I_3,\omega^{-2}I_3,\omega^3I_3,\omega^{-3}I_3,I_2) & \text{if } p\neq 7\\ J_7^2\oplus J_6\oplus J_2 & \text{if } p=7\end{cases}$ $\delta_g^G=216$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\dots,\omega^5I_2,\omega^{-5}I_2,I_2) & \text{if } p\neq u\\ J_{11}^2 & \text{if } p=u\end{cases}$ $\delta_g^G=230$
24	300	$\delta_g^G=144$	$\delta_g^G=200$	$g=\begin{cases}(\omega I_4,\omega^{-1}I_4,\omega^2I_4,\omega^{-2}I_4,\omega^3I_3,\omega^{-3}I_3,I_2) & \text{if } p\neq 7\\ J_7^2\oplus J_6\oplus J_4 & \text{if } p=7\end{cases}$ $\delta_g^G=256$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_2,\omega^{-2}I_2,\dots,\omega^5I_2,\omega^{-5}I_2,I_2) & \text{if } p\neq u\\ J_{11}^2\oplus J_2 & \text{if } p=u\end{cases}$ $\delta_g^G=272$
26	351	$\delta_g^G=168$	$\delta_g^G=234$	$g=\begin{cases}(\omega I_4,\omega^{-1}I_4,\dots,\omega^3I_4,\omega^{-3}I_4,I_2) & \text{if } p\neq 7\\ J_7^2\oplus J_6^2 & \text{if } p=7\end{cases}$ $\delta_g^G=300$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_3,\omega^{-2}I_3, & \text{if } p\neq u\\ \omega^3I_2,\omega^{-3}I_2,\dots,\omega^5I_2,\omega^{-5}I_2,I_2) & \\ J_{11}^2\oplus J_4 & \text{if } p=u\end{cases}$ $\delta_g^G=318$

Table 6: Possible dimension for a conjugacy class of an element of prime order $u\neq 5$ in $G=\mathrm{Sp}_n(\mathbb{F})$, $\mathrm{char}(\mathbb{F})>0$, $14\leq n\leq 26$

$p=\mathrm{char}(\mathbb{F})>2$ if $u=2$.
$\omega\in\mathbb{F}$ denotes an element of order u.
The dimension given is maximal if $u=2,3,7$.

Table 7: Possible dimension for a conjugacy class of an element of prime order $u \neq 5$ in $G = \mathrm{Sp}_n(\mathbb{F})$, $\mathrm{char}(\mathbb{F}) > 0$, $28 \leq n \leq 48$

n	$\dim G$	$u=2$	$u=3$	$u \geq 7$
28	406	$\delta_g^G = 196$	$\delta_g^G = 270$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_4, \omega^{-2} I_4, \omega^3 I_4, \omega^{-3} I_4, I_4) & \text{if } p \neq u \\ J_7^4 & \text{if } p = u \end{cases}$ $\delta_g^G = 348$
30	465	$\delta_g^G = 224$	$\delta_g^G = 310$	$g = \begin{cases} (\omega I_5, \omega^{-1} I_5, \omega^2 I_4, \omega^{-2} I_4, \omega^3 I_4, \omega^{-3} I_4, I_4) & \text{if } p \neq u \\ J_7^4 \oplus J_2 & \text{if } p = u \end{cases}$ $\delta_g^G = 398$
32	528	$\delta_g^G = 256$	$\delta_g^G = 352$	$g = \begin{cases} (\omega I_5, \omega^{-1} I_5, \omega^2 I_5, \omega^{-2} I_5, \omega^3 I_4, \omega^{-3} I_4, I_4) & \text{if } p \neq u \\ J_7^4 \oplus J_4 & \text{if } p = u \end{cases}$ $\delta_g^G = 452$
34	595	$\delta_g^G = 288$	$\delta_g^G = 396$	$g = \begin{cases} (\omega I_5, \omega^{-1} I_5, \omega^2 I_5, \omega^{-2} I_5, \omega^3 I_5, \omega^{-3} I_5, I_4) & \text{if } p \neq u \\ J_7^4 \oplus J_6 & \text{if } p = u \end{cases}$ $\delta_g^G = 510$
36	666	$\delta_g^G = 324$	$\delta_g^G = 444$	$g = \begin{cases} (\omega I_5, \omega^{-1} I_5, \omega^2 I_5, \omega^{-2} I_5, \omega^3 I_5, \omega^{-3} I_5, I_6) & \text{if } p \neq u \\ J_7^4 \oplus J_6 \oplus J_2 & \text{if } p = u \end{cases}$ $\delta_g^G = 570$
38	741	$\delta_g^G = 360$	$\delta_g^G = 494$	$g = \begin{cases} (\omega I_6, \omega^{-1} I_6, \omega^2 I_5, \omega^{-2} I_5, \omega^3 I_5, \omega^{-3} I_5, I_6) & \text{if } p \neq u \\ J_7^4 \oplus J_6 \oplus J_4 & \text{if } p = u \end{cases}$ $\delta_g^G = 634$
40	820	$\delta_g^G = 400$	$\delta_g^G = 546$	$g = \begin{cases} (\omega I_6, \omega^{-1} I_6, \omega^2 I_6, \omega^{-2} I_6, \omega^3 I_5, \omega^{-3} I_5, I_6) & \text{if } p \neq u \\ J_7^4 \oplus J_6^2 & \text{if } p = u \end{cases}$ $\delta_g^G = 702$
42	903	$\delta_g^G = 440$	$\delta_g^G = 602$	$g = \begin{cases} (\omega I_6, \omega^{-1} I_6, \omega^2 I_6, \omega^{-2} I_6, \omega^3 I_6, \omega^{-3} I_6, I_6) & \text{if } p \neq u \\ J_7^6 & \text{if } p = u \end{cases}$ $\delta_g^G = 774$
44	990	$\delta_g^G = 484$	$\delta_g^G = 660$	$g = \begin{cases} (\omega I_7, \omega^{-1} I_7, \omega^2 I_6, \omega^{-2} I_6, \omega^3 I_6, \omega^{-3} I_6, I_6) & \text{if } p \neq u \\ J_7^6 \oplus J_2 & \text{if } p = u \end{cases}$ $\delta_g^G = 848$
46	1081	$\delta_g^G = 528$	$\delta_g^G = 720$	$g = \begin{cases} (\omega I_7, \omega^{-1} I_7, \omega^2 I_7, \omega^{-2} I_7, \omega^3 I_6, \omega^{-3} I_6, I_6) & \text{if } p \neq u \\ J_7^6 \oplus J_4 & \text{if } p = u \end{cases}$ $\delta_g^G = 926$
48	1176	$\delta_g^G = 576$	$\delta_g^G = 784$	$g = \begin{cases} (\omega I_7, \omega^{-1} I_7, \omega^2 I_7, \omega^{-2} I_7, \omega^3 I_7, \omega^{-3} I_7, I_6) & \text{if } p \neq u \\ J_7^6 \oplus J_6 & \text{if } p = u \end{cases}$ $\delta_g^G = 1008$

$p = \mathrm{char}(\mathbb{F}) > 2$ if $u = 2$.

$\omega \in \mathbb{F}$ denotes an element of order u.

The dimension given is maximal if $u = 2, 3, 7$.

n	$\dim G$	$u=2$	$u=3$	$u=5$	$u\geq 7$
8	28	$\delta_g^G=16$	$\delta_g^G=18$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2,\omega^{-2},I_2) & \text{if } p\neq 5\\ J_5\oplus J_3 & \text{if } p=5\end{cases}$ $\delta_g^G=22$	$g=\begin{cases}(\omega,\omega^{-1},\omega^2,\omega^{-2},\omega^3,\omega^{-3},I_2) & \text{if } p\neq u\\ J_7\oplus J_1 & \text{if } p=u\end{cases}$ $\delta_g^G=24$
10	45	$\delta_g^G=24$	$\delta_g^G=30$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2I_2,\omega^{-2}I_2,I_2) & \text{if } p\neq 5\\ J_5^2 & \text{if } p=5\end{cases}$ $\delta_g^G=36$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2,\omega^{-2},\omega^3,\omega^{-3},I_2) & \text{if } p\neq u\\ J_7\oplus J_3 & \text{if } p=u\end{cases}$ $\delta_g^G=38$
12	66	$\delta_g^G=32$	$\delta_g^G=44$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_2,\omega^{-2}I_2,I_2) & \text{if } p\neq 5\\ J_5^2\oplus J_1^2 & \text{if } p=5\end{cases}$ $\delta_g^G=52$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2I_2,\omega^{-2}I_2,\omega^3,\omega^{-3},I_2) & \text{if } p\neq u\\ J_7\oplus J_5 & \text{if } p=u\end{cases}$ $\delta_g^G=56$ *
14	91	$\delta_g^G=48$	$\delta_g^G=60$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_3,\omega^{-2}I_3,I_2) & \text{if } p\neq 5\\ J_5^2\oplus J_3\oplus J_1 & \text{if } p=5\end{cases}$ $\delta_g^G=72$	$g=\begin{cases}(\omega I_2,\omega^{-1}I_2,\omega^2I_2,\omega^{-2}I_2,\omega^3I_2,\omega^{-3}I_2,I_2) & \text{if } p\neq u\\ J_7^2 & \text{if } p=u\end{cases}$ $\delta_g^G=78$
16	120	$\delta_g^G=64$	$\delta_g^G=80$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_3,\omega^{-2}I_3,I_4) & \text{if } p\neq 5\\ J_5^3\oplus J_1 & \text{if } p=5\end{cases}$ $\delta_g^G=96$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_2,\omega^{-2}I_2,\omega^3I_2,\omega^{-3}I_2,I_2) & \text{if } p\neq u\\ J_7^2\oplus J_1^2 & \text{if } p=u\end{cases}$ $\delta_g^G=102$
18	153	$\delta_g^G=80$	$\delta_g^G=102$	$g=\begin{cases}(\omega I_4,\omega^{-1}I_4,\omega^2I_3,\omega^{-2}I_3,I_4) & \text{if } p\neq 5\\ J_5^3\oplus J_3 & \text{if } p=5\end{cases}$ $\delta_g^G=122$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_3,\omega^{-2}I_3,\omega^3I_2,\omega^{-3}I_2,I_2) & \text{if } p\neq u\\ J_7^2\oplus J_3\oplus J_1 & \text{if } p=u\end{cases}$ $\delta_g^G=130$
20	190	$\delta_g^G=96$	$\delta_g^G=126$	$g=\begin{cases}(\omega I_4,\omega^{-1}I_4,\omega^2I_4,\omega^{-2}I_4,I_4) & \text{if } p\neq 5\\ J_5^4 & \text{if } p=5\end{cases}$ $\delta_g^G=152$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_3,\omega^{-2}I_3,\omega^3I_3,\omega^{-3}I_3,I_2) & \text{if } p\neq u\\ J_7^2\oplus J_5\oplus J_1 & \text{if } p=u\end{cases}$ $\delta_g^G=162$
22	231	$\delta_g^G=120$	$\delta_g^G=154$	$g=\begin{cases}(\omega I_5,\omega^{-1}I_5,\omega^2I_4,\omega^{-2}I_4,I_4) & \text{if } p\neq 5\\ J_5^4\oplus J_1^2 & \text{if } p=5\end{cases}$ $\delta_g^G=184$	$g=\begin{cases}(\omega I_3,\omega^{-1}I_3,\omega^2I_3,\omega^{-2}I_3,\omega^3I_3,\omega^{-3}I_3,I_4) & \text{if } p\neq u\\ J_7^3\oplus J_1 & \text{if } p=u\end{cases}$ $\delta_g^G=198$
24	276	$\delta_g^G=144$	$\delta_g^G=184$	$g=\begin{cases}(\omega I_5,\omega^{-1}I_5,\omega^2I_5,\omega^{-2}I_5,I_4) & \text{if } p\neq 5\\ J_5^4\oplus J_3\oplus J_1 & \text{if } p=5\end{cases}$ $\delta_g^G=220$	$g=\begin{cases}(\omega I_4,\omega^{-1}I_4,\omega^2I_3,\omega^{-2}I_3,\omega^3I_3,\omega^{-3}I_3,I_4) & \text{if } p\neq u\\ J_7^3\oplus J_3 & \text{if } p=u\end{cases}$ $\delta_g^G=236$
26	325	$\delta_g^G=168$	$\delta_g^G=216$	$g=\begin{cases}(\omega I_5,\omega^{-1}I_5,\omega^2I_5,\omega^{-2}I_5,I_6) & \text{if } p\neq 5\\ J_5^5\oplus J_1 & \text{if } p=5\end{cases}$ $\delta_g^G=260$	$g=\begin{cases}(\omega I_4,\omega^{-1}I_4,\omega^2I_4,\omega^{-2}I_4,\omega^3I_3,\omega^{-3}I_3,I_4) & \text{if } p\neq u\\ J_7^3\oplus J_5 & \text{if } p=u\end{cases}$ $\delta_g^G=278$

Table 8: Possible dimension for a conjugacy class of an element of prime order u in $G=\mathrm{Spin}_n(\mathbb{F})$, n even, $\mathrm{char}(\mathbb{F})>0$, $8\leq n\leq 26$

$p=\mathrm{char}(\mathbb{F})>2$ if $u=2$.
$\omega\in\mathbb{F}$ denotes an element of order u.
The dimension given is maximal if $u=2,3,5,7$.
* If $n=12$ and $u\geq 11$ then $\delta_u^G=60$.

n	$\dim G$	$u=2$	$u=3$	$u=5$	$u \geq 7$
7	21	$\delta_g^G = 12$	$\delta_g^G = 14$	$g = \begin{cases} (\omega, \omega^{-1}, \omega^2, \omega^{-2}, I_3) & \text{if } p \neq 5 \\ J_5 \oplus J_1^2 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 16$	$g = \begin{cases} (\omega, \omega^{-1}, \omega^2, \omega^{-2}, \omega^3, \omega^{-3}, 1) & \text{if } p \neq u \\ J_7 & \text{if } p = u \end{cases}$ $\delta_g^G = 18$
9	36	$\delta_g^G = 20$	$\delta_g^G = 24$	$g = \begin{cases} (\omega I_2, \omega^{-1} I_2, \omega^2, \omega^{-2}, I_3) & \text{if } p \neq 5 \\ J_5 \oplus J_3 \oplus J_1 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 28$	$g = \begin{cases} (\omega, \omega^{-1}, \omega^2, \omega^{-2}, \omega^3, \omega^{-3}, I_3) & \text{if } p \neq u \\ J_7 \oplus J_1^2 & \text{if } p = u \end{cases}$ $\delta_g^G = 30$
11	55	$\delta_g^G = 28$	$\delta_g^G = 36$	$g = \begin{cases} (\omega I_2, \omega^{-1} I_2, \omega^2 I_2, \omega^{-2} I_2, I_3) & \text{if } p \neq 5 \\ J_5^2 \oplus J_1 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 44$	$g = \begin{cases} (\omega I_2, \omega^{-1} I_2, \omega^2, \omega^{-2}, \omega^3, \omega^{-3}, I_3) & \text{if } p \neq u \\ J_7 \oplus J_3 \oplus J_1 & \text{if } p = u \end{cases}$ $\delta_g^G = 46$ *
13	78	$\delta_g^G = 40$	$\delta_g^G = 52$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_2, \omega^{-2} I_2, I_3) & \text{if } p \neq 5 \\ J_5^2 \oplus J_3 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 62$	$g = \begin{cases} (\omega I_2, \omega^{-1} I_2, \omega^2 I_2, \omega^{-2} I_2, \omega^3, \omega^{-3}, I_3) & \text{if } p \neq u \\ J_7 \oplus J_5 \oplus J_1 & \text{if } p = u \end{cases}$ $\delta_g^G = 66$
15	105	$\delta_g^G = 56$	$\delta_g^G = 70$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_3, \omega^{-2} I_3, I_3) & \text{if } p \neq 5 \\ J_5^3 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 84$	$g = \begin{cases} (\omega I_2, \omega^{-1} I_2, \omega^2 I_2, \omega^{-2} I_2, \omega^3 I_2, \omega^{-3} I_2, I_3) & \text{if } p \neq u \\ J_7^2 \oplus J_1 & \text{if } p = u \end{cases}$ $\delta_g^G = 90$
17	136	$\delta_g^G = 72$	$\delta_g^G = 90$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_3, \omega^{-2} I_3, I_3) & \text{if } p \neq 5 \\ J_5^3 \oplus J_1^2 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 108$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_2, \omega^{-2} I_2, \omega^3 I_2, \omega^{-3} I_2, I_3) & \text{if } p \neq u \\ J_7^2 \oplus J_3 & \text{if } p = u \end{cases}$ $\delta_g^G = 116$
19	171	$\delta_g^G = 88$	$\delta_g^G = 114$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_4, \omega^{-2} I_4, I_3) & \text{if } p \neq 5 \\ J_5^3 \oplus J_3 \oplus J_1 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 136$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_3, \omega^{-2} I_3, \omega^3 I_2, \omega^{-3} I_2, I_3) & \text{if } p \neq u \\ J_7^2 \oplus J_5 & \text{if } p = u \end{cases}$ $\delta_g^G = 146$
21	210	$\delta_g^G = 108$	$\delta_g^G = 140$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_4, \omega^{-2} I_4, I_5) & \text{if } p \neq 5 \\ J_5^4 \oplus J_1 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 168$	$g = \begin{cases} (\omega I_3, \omega^{-1} I_3, \omega^2 I_3, \omega^{-2} I_3, \omega^3 I_3, \omega^{-3} I_3, I_3) & \text{if } p \neq u \\ J_7^3 & \text{if } p = u \end{cases}$ $\delta_g^G = 180$
23	253	$\delta_g^G = 132$	$\delta_g^G = 168$	$g = \begin{cases} (\omega I_5, \omega^{-1} I_5, \omega^2 I_4, \omega^{-2} I_4, I_5) & \text{if } p \neq 5 \\ J_5^4 \oplus J_3 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 202$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_3, \omega^{-2} I_3, \omega^3 I_3, \omega^{-3} I_3, I_3) & \text{if } p \neq u \\ J_7^3 \oplus J_1^2 & \text{if } p = u \end{cases}$ $\delta_g^G = 216$
25	300	$\delta_g^G = 156$	$\delta_g^G = 200$	$g = \begin{cases} (\omega I_5, \omega^{-1} I_5, \omega^2 I_5, \omega^{-2} I_5, I_5) & \text{if } p \neq 5 \\ J_5^5 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 240$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_4, \omega^{-2} I_4, \omega^3 I_3, \omega^{-3} I_3, I_3) & \text{if } p \neq u \\ J_7^3 \oplus J_3 \oplus J_1 & \text{if } p = u \end{cases}$ $\delta_g^G = 256$
27	351	$\delta_g^G = 180$	$\delta_g^G = 234$	$g = \begin{cases} (\omega I_6, \omega^{-1} I_6, \omega^2 I_5, \omega^{-2} I_5, I_5) & \text{if } p \neq 5 \\ J_5^5 \oplus J_1^2 & \text{if } p = 5 \end{cases}$ $\delta_g^G = 280$	$g = \begin{cases} (\omega I_4, \omega^{-1} I_4, \omega^2 I_4, \omega^{-2} I_4, \omega^3 I_4, \omega^{-3} I_4, I_3) & \text{if } p \neq u \\ J_7^3 \oplus J_5 \oplus J_1 & \text{if } p = u \end{cases}$ $\delta_g^G = 300$

Table 9: Possible dimension for a conjugacy class of an element of prime order u in $G = \mathrm{Spin}_n(\mathbb{F})$, n odd, $\mathrm{char}(\mathbb{F}) > 2$, $7 \leq n \leq 27$

$p = \mathrm{char}(\mathbb{F})$.
$\omega \in \mathbb{F}$ denotes an element of order u.
The dimension given is maximal if $u = 2, 3, 5, 7$.
* If $n = 11$ and $u \geq 11$ then $\delta_u^G = 50$.