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The p-local splitting of $\Sigma\mathbb{C}P^\infty$ and co-H-structures

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$$\Sigma\mathbb{C}P^\infty$$

Common Name

Suspension of the Infinite Complexe Projective Space

Category

Pointed Topological Spaces
CW-Complexes

CW-Decomposition

connected with one zero-cell (the base point) and a single cell in each odd dimension ≥ 3:

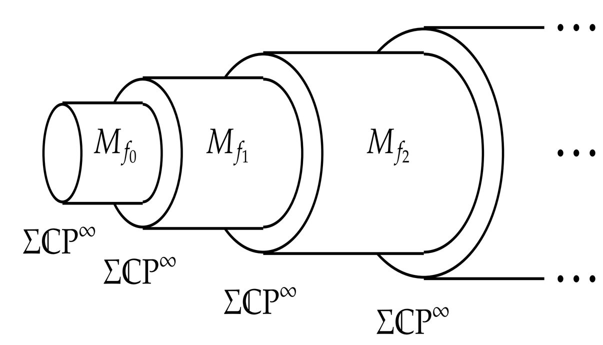
$$e^0 \cup e^3 \cup e^5 \cup e^7 \cup \cdots$$

General Setting

Let p be an odd prime. There is a p-local splitting of $\Sigma\mathbb{C}P^\infty$ as a wedge of $p-1$ topological spaces, described by C.A. McGibbon [1]. Namely, there is a homotopy equivalence

$$\Sigma\mathbb{C}P^\infty_{(p)} \simeq \bigvee_{j=1}^{p-1} K_j$$

where each space K_j is built as a mapping telescope of a sequence $\{f_i\}_{i\geq 0}$ of well-chosen self-maps of $\Sigma\mathbb{C}P^\infty$. It's only a model, but one could imagine that the K_j's look as drawn on the right. Furthermore their integral homology is given by the formula



$$\widetilde{H}_q(K_j,\mathbb{Z}) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } q = 2n+1 \text{ and } n(\geq 1) \equiv j \pmod{p-1} \\ 0 & \text{otherwise.} \end{cases}$$

Aim Of The Project

Studying possible co-H-structures on the topological spaces K_j. In particular, the question we investigate is to know whether or not the spaces K_1 to K_{p-2} can bear a coassociative coproduct.

Co-H-Structures

Every space K_j, $j = 1, \ldots p-1$ has a co-H-space structure inherited from the suspension co-H-stucture on $\Sigma\mathbb{C}P^\infty$, say θ. Concretely, using the canonical inclusions ι_j and retractions q_j, we obtain the following coproduct on the spaces K_j:

$$K_j \xhookrightarrow{\iota_j} \Sigma\mathbb{C}P^\infty_{(p)} \xrightarrow{\theta_{(p)}} \Sigma\mathbb{C}P^\infty_{(p)} \vee \Sigma\mathbb{C}P^\infty_{(p)} \xrightarrow{q_j \vee q_j} K_j \vee K_j$$

The space K_{p-1} has one of the nicest co-H-structure one can imagine, in the sense that it has the homotopy type of a suspension. Unfortunately, one (african) swallow does not make a summer and the other spaces $K_1, \ldots, K_{p-2}$ do not have the homotopy type of a suspension. In fact, the main result below says that these spaces can't even be endowed with a coproduct having as nice co-H-structures properties as coassociativity or co-H-group structrure.

To sum up:

$$\Sigma\mathbb{C}P^\infty_{(p)} \simeq \underbrace{K_1 \vee \cdots \vee K_{p-2}} \vee \underbrace{K_{p-1}}$$

Do not possess any coassociative coproduct! — Has the homotopy type of a suspension, thus is a co-H-group.

The Main Result

Theorem. *Let $j \in \mathbb{N}_{p-2}$, then the space K_j does not possess any coassociative coproduct.*

Main Steps Of The Proof

- → Assume that K_j possesses a coassociative coproduct and hope to find a contradiction!
- → Use the Bott-Samelson Theorem to see that $H_*(\Omega K_j;\mathbb{F}_p)$ is a primitively generated Hopf algebra.
- → Deduce that its dual Hopf algebra $H^*(\Omega K_j;\mathbb{F}_p)$ has only trivial p^{th} powers.
- → Use the fact that $H^*(\mathbb{C}P^\infty;\mathbb{F}_p) \cong \mathbb{F}_p[x]$ with $|x| = 2$ and use the commutativity of the Steenrod reduced powers $\mathcal{P}^i$ with the suspension isomorphism Σ and the cohomology suspension monomorphism σ^*.
- → Then letting a generator $k_j \in H^{2j+1}(K_j;\mathbb{F}_p)$ going round the following diagram provides a **contradiction** to the previous observation concerning the p^{th} powers in $H^*(\Omega K_j;\mathbb{F}_p)$.

$$\begin{array}{ccccc}
\mathbb{F}_p < x^j > \cong H^{2j}(\mathbb{C}P^\infty;\mathbb{F}_p) & \xrightarrow[\cong]{\Sigma} & H^{2j+1}(\Sigma\mathbb{C}P^\infty;\mathbb{F}_p) \cong H^{2j+1}(K_j;\mathbb{F}_p) & \xhookrightarrow{\sigma^*} & H^{2j}(\Omega K_j;\mathbb{F}_p) \\
\downarrow \mathcal{P}^j & \circlearrowleft & \downarrow \mathcal{P}^j & \circlearrowleft & \downarrow \mathcal{P}^j \\
\mathbb{F}_p < x^{jp} > \cong H^{2jp}(\mathbb{C}P^\infty;\mathbb{F}_p) & \xrightarrow[\Sigma]{\cong} & H^{2jp+1}(\Sigma\mathbb{C}P^\infty;\mathbb{F}_p) \cong H^{2jp+1}(K_j;\mathbb{F}_p) & \xhookrightarrow[\sigma^*]{} & H^{2j}(\Omega K_j;\mathbb{F}_p)\,.
\end{array}$$

$$\begin{array}{ccccc}
x^j & \xleftarrow{\cong} & k_j & \longmapsto & \sigma^*(k_j) \\
\downarrow \mathcal{P}^j & & \downarrow \mathcal{P}^j & & \downarrow \mathcal{P}^j \\
x^{pj} \neq 0 & \xmapsto{\cong} & \mathcal{P}^j(k_j) \neq 0 & \longmapsto & \sigma^*\mathcal{P}^j(k_j) = \mathcal{P}^j\sigma^*(k_j) = (\sigma^*(k_j))^p = 0 \;\lightning \\
 & & & & \not\Downarrow \; 0
\end{array}$$

■ **CQFD**

MATHEMATICAL TOOLBOX

The following "tools" from algebraic topology have been used:

- Co-H-Spaces
- Homology and Cohomology of Spaces
- Loop Spaces and Suspensions
- Topological Localizations
- Hopf Algebras
- The Steenrod Algebra
- LS-Category
- Serre Spectral Sequences

THE HOMOLOGY OF K_j

The homology of the spaces K_j is distributed according to the following pattern:

	K_1	K_2	K_3	$\cdots$	K_{p-2}	K_{p-1}
H_1						
H_3	$\mathbb{Z}_{(p)}$					
H_5		$\mathbb{Z}_{(p)}$				
H_7			$\mathbb{Z}_{(p)}$			
$\vdots$				$\ddots$		
$H_{2(p-2)+1}$					$\mathbb{Z}_{(p)}$	
$H_{2(p-1)+1}$						$\mathbb{Z}_{(p)}$
H_{2p+1}	$\mathbb{Z}_{(p)}$					
H_{2p+3}		$\mathbb{Z}_{(p)}$				
$\vdots$			$\ddots$			

WHY IS K_{p-1} DIFFERENT ?

It is a fact, K_{p-1} has a nicer co-H-structure than the other $p-2$ spaces. Nonetheless, this apparently farcical behaviour of its can entirely be explained by the degrees in witch* its homology (and thus cohomology) is concentrated.

Why is it a suspension?

This follows from work of D. Sullivan [2] and which provides, for N dividing $p-1$, the homotopy equivalence

$$\Sigma BS^{2N-1}_{(p)} \simeq \bigvee_{i=1}^{(p-1)/N} K_{Ni}\,.$$

Thus taking $N = p-1$ yields the result. But there are other bridges to cross to prove it.

COPRODUCT

A coproduct is a pointed continuous map $\theta : X \longrightarrow X \vee X$ which makes the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\theta} & X \vee X \\
 & \searrow^{\text{diagonal map}} & \downarrow \\
 & & X \times X
\end{array}$$

commute up to homotopy. It is coassociative if

$$(\mathrm{Id} \vee \theta)\theta \simeq (\theta \vee \mathrm{Id})\theta\,.$$

p-LOCALIZATION

p-localization is a process that associates to a topological space X another topological space $X_{(p)}$ such that its homology is:

$$\widetilde{H}_*(X_{(p)}) \cong \widetilde{H}_*(X) \otimes \mathbb{Z}_{(p)}$$

References

[1] C.A. McGibbon. Stable properties of rank 1 loop structures. Topology, 20(2):109-118, 1981.

[2] D. P. Sullivan. Genetics of homotopy theory and the Adams conjecture. Ann. of Math. (2), 100:1-79, 1974.