Homotopical Algebra
Solution Sketches for Exercise Set 13

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Exercise 1

We'll follow the hint.

Since $V(Y')$ is cofibrant, i. e. $V(\varnothing_{\mathbf{K}}) \cong \varnothing_{\mathbf{M}} \to V(Y')$ is a cofibration, the lifting problem

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
Y' \arrow[r, "\varepsilon_y"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

can be solved to obtain an $s\colon Y' \to X$ with $fs = \varepsilon_Y$.

Now we have $fsf' = \varepsilon_Y f' = f\varepsilon_X$, so we have a lifting problem

$$\begin{tikzcd}
X' \amalg X' \arrow[rrr, "\varepsilon_X + sf'"] \arrow[d, "j"'] & & & X \arrow[d, "f"] \\
CX' \arrow[r, "p"'] \arrow[urrr, dashed] & X' \arrow[r, "f'"'] & Y' \arrow[r, "\varepsilon_Y"'] & Y
\end{tikzcd}$$

which can be solved because $V(j)$ is a cofibration. Thus we obtain a map $H\colon CX' \to X$ which is the desired "left homotopy" between ε_X and sf'. Applying V to it (and using that V commutes with coproducts as a left adjoint), we obtain a left homotopy between $V(\varepsilon_X)$ and $V(sf')$ (in $\mathbf{M}$). Thus, $V(sf')$ is a weak equivalence since $V(\varepsilon_X)$ is and we obtain a $\tilde{\varepsilon}_X \coloneqq sf' \coloneqq X' \to X$ with $f\tilde{\varepsilon}_X = fsf' = \varepsilon_Y f'$ as in the hint.

Now we apply 2-oo-6 to $V(X') \xrightarrow{V(f')} V(Y') \xrightarrow{V(s)} V(X) \xrightarrow{V(f)} V(Y)$: $V(s)V(f') = V(sf')$ and $V(f)V(s) = V(fs) = V(\varepsilon_Y)$ are weak equivalences by the above discussion, so by 2-oo-6, $V(f)$ is a weak equivalence.

Finally, as described in the hint, we will prove the 2-oo-6 property for weak equivalences in a model category by showing that it holds for isomorphisms in any category: Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ be a sequence of composable arrows s. t. gf and

hg are isomorphisms. Then we have $\mathrm{Id}_X = (gf)^{-1}(gf) = ((gf)^{-1}g)f$. Moreover, $(hg)f(gf)^{-1}g = h(gf)(gf)^{-1}g = h\,\mathrm{Id}_Z\, g = hg\,\mathrm{Id}_Y$, so $f(gf)^{-1}g = \mathrm{Id}_Y$ since hg is an isomorphism. Thus, f is an isomorphism with inverse $(gf)^{-1}g$. Now g is an isomorphism by 2-oo-3 for isomorphisms since gf is, and then h is an isomorphism by 2-oo-3 since hg is. Lastly, hgf is an isomorphism as a composition of isomorphisms.

Exercise 2

This should be the straightforward formal dual of Exercise 1...

Exercise 3

In this exercise, we will identify $G\mathrm{sSet}$ with sSet^{BG}, $G\mathrm{Set}^{\Delta^{\mathrm{op}}}$ and $\mathrm{Set}^{\Delta^{\mathrm{op}}\times BG}$.

a) Consider the "inclusion" $i\colon * \to BG$ of the terminal category to BG which sends the unique object of $*$ to the unique object in BG and the unique morphism in $*$ to the identity in BG. This induces the "restriction functor" $i^*\colon G\mathrm{sSet} \cong \mathrm{sSet}^{BG} \to \mathrm{sSet}^* \cong \mathrm{sSet}$ which is the "underlying simplicial set" functor (under the above mentioned identification). Now since sSet is bicomplete, i^* has a left adjoint $i_!$ and a right adjoint i_* given by left resp. right Kan extensions.
Now we will describe the values of $i_!$ more explicitly to see that it agrees with the adjoint given in the exercise. Recall that given $X \in \mathrm{sSet}$, $(i_!X)(t)$ is given by a colimit over the category i/t for all $t \in BG$ which we will use to compute (values of) $i_!$. i_* can be described analogously using the description of $(i_*X)(t)$ as a limit over the category t/i.
Since BG has only one object (call it t_0), we will only have to see what i/t_0 is. This category has pairs of the form $(c \in *, f\colon i(c) \to t_0)$ as objects. Now $*$ has a unique object (call it $*$) which is mapped to t_0 under i and $\mathrm{Hom}_{BG}(t_0, t_0)$ can be identified with G, so $\mathrm{ob}(i/t_0)$ can be identified with pairs of the form $(*, i(*) = t_0 \xrightarrow{g\in G} t_0)$, i. e. with G.
Now morphisms in i/t_0 from $(*, i(*) = t_0 \xrightarrow{g} t_0)$ to $(*, i(*) \xrightarrow{h} t_0 \to t_0)$ is given by morphisms $\phi\colon * \to *$ in $*$ s. t. $g = h \circ i(\phi)$. Since only possible such ϕ is Id_*, we see that there are no non-identity morphisms in i/t_0, which means that i/t_0 is isomorphic to the discrete category given by (the underlying set of) G.
Thus we have $(i_!X)(t_0) = X^{\amalg G} \cong G \times X$. Moreover, recall that for a morphism g in BG, $(i_!X)(g)$ is induced by the functor $i/t_0 \to i/t_0$ which maps $(*, i(*) = t_0 \xrightarrow{h} t_0)$ to $(*, i(*) = t_0 \xrightarrow{gh} t_0)$. Hence the G-action on $i_!X$ agrees with the "obvious" action on $G \times X$ induced by the action of G on itself.
As mentioned above, a similar argument can be used to identify i_* with $\mathrm{Map}(G, -)$.

b) Exercise 1 (resp. its dual) imply that the acyclicity condition for left (resp. right) induced model structures holds if (a), (b) and (c) (resp. their duals) hold. Thus, (since the categories in consideration satisfy the technical conditions of the theorem) the indicated induced model structures exist by [HKRS17, Corollary 3.3.4] if

we can show (a), (b) and (c) (resp. their duals).
To this end, we observe that for all $X \in G\mathrm{sSet}$, $U(X)$ is cofibrant since every simplical set in the Quillen-Kan model structure. Thus we can take $X' = X$, $\varepsilon_X = \mathrm{Id}_X$ for (a) and $f' = f$ for a morphism $f\colon X \to Y$ in $G\mathrm{sSet}$ for (b). Moreover, considering $\Delta[1]$ as a simplicial set with a trivial G-action, we see that all the maps in the factorization $X \amalg X \xrightarrow{(\mathrm{Id},0)+(\mathrm{Id},1)} X \times \Delta[1] \xrightarrow{\pi_0} X$ of ∇_X are G-equivariant, so it yields a "cylinder" as required in (c).
For the dual situation, we will actually need to replace X with a G-simplicial set whose underlying simplicial set is a Kan complex. For this, we first observe that a G-action on a simplicial set X can equivalently be described as a simplicial map $G \times X \to X$ satisfiying certain conditions encoding "associativity" and "unitality" of the action where we consider G as a simplicial set given by $\coprod_G \Delta[0]$. We also observe that $|G \times X| \cong |\coprod_G X| \cong \coprod_G |X| \cong G \times |X| \cong |G| \times |X|$.
Now, using the unit of the adjunction $|-| \dashv \mathrm{Sing}$, preservation properties of (co)limits and the action map of X, we get a natural map

$$G \times \mathrm{Sing}\,|X| \to \mathrm{Sing}\,|G| \times \mathrm{Sing}\,|X| \cong \mathrm{Sing}\,|G \times X| \to \mathrm{Sing}\,|X|.$$

One can check that this actually defines an action of G on $\mathrm{Sing}\,|X|$ and that this lifts the (endo)functor $\mathrm{Sing}\,|-|$ to $G\mathrm{sSet}$. Thus we obtain the dual of (a) and (b) by applying $\mathrm{Sing}\,|-|$ and using the unit of $|-| \dashv \mathrm{Sing}$ since $U(\mathrm{Sing}\,|X|)$ is a Kan complex. For "path objects", we observe again that equipping $\Delta[1]$ with the trivial G-action, $\mathrm{Sing}\,|X| \xrightarrow{\mathrm{const}} \mathrm{Map}(\Delta[1], \mathrm{Sing}\,|X|) \xrightarrow{(\mathrm{ev}_0,\mathrm{ev}_1)} \mathrm{Sing}\,|X| \times \mathrm{Sing}\,|X|$ yields the required object.

c) Note that $U \circ \mathrm{Triv}_G \cong \mathrm{Id}_{\mathrm{sSet}}$. Thus Triv_G maps (acyclic) cofibrations to (acyclic) cofibrations in the left induced model structure and (acyclic) fibrations to (acyclic) fibrations in the right induced model structure. Hence the adjunctions mentioned in the exercise (which are adjunctions by a straightforward calculation or the observation that $\mathrm{Triv}_G \cong \Delta_{BG}$ and $(-)_G \cong \mathrm{colim}_{BG}$ resp. $(-)^G \cong \lim_{BG}$) are indeed Quillen adjunctions.

Exercise 4

a) The following solution is based on [Bro74, I.4, Lemma 4]. See also [Hir03, Section 13] for a more detailed treatment of properness.
First we will factor into w factors which are easier to analyze. Let $A \amalg A \rightarrowtail CA \overset{\sim}{\twoheadrightarrow} A$

be a good cylinder. Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "w"] \arrow[d, tail, "i_1"'] & & B \arrow[d, tail, "i_1"] \arrow[drrr, equal, bend left] & \\
A \amalg A \arrow[rr, "\mathrm{Id}_A \amalg w"] \arrow[d, tail] & & A \amalg B \arrow[d, tail] \arrow[ddr, "w+\mathrm{Id}_B"] & \\
CA \arrow[rr] \arrow[dr, two heads] & & Z \arrow[dr, dashed, "p"'] & \\
& A \arrow[rr, "w"'] & & B
\end{tikzcd}$$

where Z is defined to be the pushout in the bottom square it sits in and the dashed arrow $p\colon Z \to B$ is induced by the universal property of the pushout. Note that the upper vertical arrows are cofibrations by the cofibrancy assumption on A and B, and $A \amalg B \rightarrowtail Z$ is a cofibration since cofibrations are preserved under pushouts. Moreover, the upper square is also a pushout square by a straightforward verification of the universal property, so by pasting the two pushout diagrams, Z can be identified with the pushout $CA \amalg_A B$. Thus the composite

$$k\colon B \overset{i_1}{\rightarrowtail} A \amalg B \rightarrowtail Z$$

is an acyclic cofibration since

$$A \overset{i_1}{\rightarrowtail} A \amalg A \rightarrowtail CA$$

is (and acyclic cofibrations are preserved under pushouts). Hence p is a weak equivalence by 2-oo-3 as $pk = \mathrm{Id}_B$.

Let j be the composite

$$A \overset{i_0'}{\rightarrowtail} A \amalg B \rightarrowtail Z$$

which is a cofibration. Then the diagram

$$\begin{tikzcd}
A \arrow[d, tail, "i_0"'] \arrow[dr, tail, "i_0'"] \arrow[ddrr, bend left, "w"] & & \\
A \amalg A \arrow[r, "\mathrm{Id}_A \amalg w"'] & A \amalg B \arrow[d, tail] \arrow[dr, "w+\mathrm{Id}_B"] & \\
& Z \arrow[r, "p"'] & B
\end{tikzcd}$$

commutes, so we have $w = jp$. In particular, j is an *acyclic* cofibration by 2-oo-3. Now the pushout Y of $jp = w\colon A \to B$ along $i\colon A \to X$ can be realized as an iterated pushout

$$\begin{tikzcd}
A \arrow[r, tail, "j", "\sim"'] \arrow[d, tail, "i"'] & Z \arrow[r, "p"] \arrow[d, tail, "i'"] & B \arrow[d, tail] \\
X \arrow[r, tail, "\sim"] & X' \arrow[r, "\tilde{w}"'] & Y
\end{tikzcd}$$

Note that the morphism $i' \colon Z \rightarrowtail X'$ is a cofibration as pushouts preserve cofibrations. Moreover, induced morphism $X \overset{\sim}{\rightarrowtail} X'$ is an acyclic cofibration because pushouts also preserve acyclic cofibrations. Hence it is enough to show that the morphism $\tilde{w} \colon X' \to Y$ is a weak equivalence as $\overline{w} \colon X \to Y$ is the composite

$$X \overset{\sim}{\rightarrowtail} X' \xrightarrow{\tilde{w}} Y.$$

Now we will try to "replace" X' by an object for which one can describe the pushout along $\tilde{w}$ more easily. For this we consider the commutative diagram

$$\begin{tikzcd}
B \arrow[r, tail, "k", "\sim"'] \arrow[d, tail, "i'k"'] & Z \arrow[d, tail, "h"] \arrow[ddr, tail, bend left, "i'"] & \\
X' \arrow[r, tail, "\sim"] \arrow[drr, equal, bend right] & T \arrow[dr, dashed, "f"] & \\
& & X'
\end{tikzcd}$$

where T is the pushout in the upper left square and the dashed arrow $f \colon T \to X'$ is induced by the universal property of the pushout. Note that $X' \overset{\sim}{\rightarrowtail} T$ is an acyclic cofibration since pushouts preserve acyclic cofibrations. Hence, since the lower triangle commutes, f is a weak equivalence by 2-oo-3.
Now consider the commutative diagram

$$\begin{tikzcd}
& & & B \arrow[dl, tail] \arrow[dr, tail] & \\
& Z \arrow[urr, "p"] \arrow[dl, tail, "h"'] \arrow[dr, tail, "i'"] & & & \\
& & & Y' \arrow[r, "f'"] & Y \\
T \arrow[urrr, "p'"] \arrow[rr, "f", "\sim"'] & & X' \arrow[urr, "\tilde{w}"'] & &
\end{tikzcd}$$

where Y' is defined to be the pushout in the rectangle at the back and $f' \colon Y' \to Y$ is the map induced on pushouts. We will now show that p' and f' are also weak equivalences to conclude that $\tilde{w}$ is also one by 2-oo-3.
To analyze p', we consider the iterated pushout diagram

$$\begin{tikzcd}
B \arrow[r, tail, "k", "\sim"'] \arrow[d, tail, "i'k"'] \arrow[rr, equal, bend left] & Z \arrow[r, "p"] \arrow[d, tail, "h"] & B \arrow[d, tail] \\
X' \arrow[r, tail, "\sim"] & T \arrow[r, "p'"'] & Y'
\end{tikzcd}.$$

Now, by pasting properties of pushout squares, the outer square is also a pushout which is in fact a pushout along the identity morphism of B. Thus Y' can be identified with X' and $B \rightarrowtail Y'$ with $i'k$. Moreover, $p'\colon T \to Y'$ is a priori given by $\mathrm{Id}_{X'} +_B i'kp$, but it can be identified with $\mathrm{Id}_{X'} +_B i' = f$ since $(i'kp)k = i'k\,\mathrm{Id}_B = (i')k$. Thus p' is indeed a weak equivalence.
As for f', it is a weak equivalence by an application of Ken Brown's lemma to the "cobase change" functor $(-) \amalg_p B\colon Z/\mathbf{M} \to B/\mathbf{M}$ given by taking the pushout along p:
Note that an object $u\colon Z \to Z'$ of $Z/\mathbf{M}$ is cofibrant iff u is a cofibration. Moreover, a morphism

$$\begin{tikzcd}
 & Z \arrow[dl, "u"'] \arrow[dr, "u'"] & \\
Z' \arrow[rr, "v"'] & & Z''
\end{tikzcd}$$

in $Z/\mathbf{M}$ is an acyclic cofibration iff v is an acyclic cofibration in $\mathbf{M}$.
Given such an acyclic cofibration between cofibrant objects in $Z/\mathbf{M}$, taking the pushout along p yields a (familiar) commutative diagram

$$\begin{tikzcd}
 & & & B \arrow[dl, tail] \arrow[dr, tail] & \\
 & Z \arrow[urr] \arrow[dl, tail] \arrow[dr, tail] & Z' \amalg_p B \arrow[rr, "v \amalg_p B"] & & Z'' \amalg_p B \\
Z' \arrow[rr, "v"'] \arrow[urr] & & Z'' \arrow[urr] & &
\end{tikzcd}.$$

Now the front and back rectangles are pushout squares by definition, so by pasting properties of pushout squares, the bottom one is also one.
Hence $v \amalg_p B$ is an acyclic cofibration in $\mathbf{M}$ since acyclic cofibrations are preserved by pushouts, so the corresponding morphism in $B/\mathbf{M}$ is an acyclic cofibration, in particular a weak equivalence in $B/\mathbf{M}$. This means that $(-) \amalg_p B\colon Z/\mathbf{M} \to B/\mathbf{M}$ maps acyclic cofibrations between cofibrant objects to weak equivalences, so by Ken Brown's lemma it maps weak equivalences between cofibrant objects to weak equivalences.

Now, applying this result to the cofibrant objects $Z \overset{h}{\rightarrowtail} T$, $Z \overset{i'}{\rightarrowtail} X'$ and the weak equivalence given by $f\colon T \xrightarrow{\sim} X'$, we obtain that $f' = f \amalg_p B$ is a weak equivalence (in $B/\mathbf{M}$ and thus in $\mathbf{M}$).

b) This should be straightforward using that V preserves pushouts and *creates* cofibrations and weak equivalences.
c) Formally dual to the previous part.
d) First we observe that the desired statement follows easily from the statement in the hint: If fibrations in the injective model structure are in particular pointwise

fibrations, then, since pullbacks in $\mathbf{M}^J$ are computed pointwise, pullbacks along fibrations in the injective model structure are at every object $j \in J$ pullbacks (in $\mathbf{M}$) along fibrations (in $\mathbf{M}$), so they preserve pointwise weak equivalences (i. e. weak equivalences in $\mathbf{M}^J$) as $\mathbf{M}$ is right proper.
To show that fibrations in the injective model structure are in particular pointwise fibrations, let $\alpha\colon F \to G$ be an injective fibration in $\mathbf{M}^J$, $k\colon A \to B$ an acyclic cofibration in $\mathbf{M}$, $j \in J$, and consider a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "k"'] & F(j) \arrow[d, "\alpha_j"] \\ B \arrow[r] \arrow[ur, dashed] & G(j) \end{tikzcd} . \tag{1}$$

Let $\iota_j\colon * \to J$ be the functor from the terminal category which "picks the object $j \in J$". Note that under the identification $\mathbf{M} \cong \mathbf{M}^*$, ι_j^* corresponds to "evaluation at j". Thus, under the adjunction $(\iota_j)_! \dashv \iota_j^*$, the lifting problem (1) transposes to

$$\begin{tikzcd} (\iota_j)_!(A) \arrow[r] \arrow[d, "k"'] & F \arrow[d, "\alpha"] \\ (\iota_j)_!(B) \arrow[r] \arrow[ur, dashed] & G \end{tikzcd} \tag{2}$$

where we identify objects of $\mathbf{M}$ with the corresponding functors $* \to \mathbf{M}$.
Now we will describe Kan extensions of the form $(\iota_j)_!(X)$ for $X \in \mathbf{M}$ more explicitly in order to solve this transposed lifting problem. For this, we have to identify the category ι_j/j' for $j' \in J$ which has a description similar to i/t_0 from Exercise 3a): It has objects of the form $(*, f\colon \iota_j(*) = j \to j')$ and no non-identity morphisms (since $*$ has no non-identity morphisms). Thus ι_j/j' is a discrete category whose objects can be identified with $\mathrm{Hom}_J(j, j')$ and $(\iota_j)_!(X)(j')$ is given by the coproduct $\coprod_{\mathrm{Hom}_J(j,j')} X$. Similarly, one can check that given a morphism $k\colon A \to B$ in $\mathbf{M}$, the induced map $(\iota_j)_!(k)_{j'}\colon (\iota_j)_!(A)(j') \cong \coprod_{\mathrm{Hom}_J(j,j')} A \to \coprod_{\mathrm{Hom}_J(j,j')} \cong (\iota_j)_!(B)(j')$ is given by "a coproduct of k's".
Hence, since acyclic cofibrations are stable under coproducts, $(\iota_j)_!(k)$ is a pointwise acyclic cofibration, thus an acyclic cofibration in the injective model structure if k is an acyclic cofibration in $\mathbf{M}$. Thus, since α is a fibration in the injective model structure, the lifting problem (2) and hence the lifting problem (1) can be solved, which implies that α is a pointwise fibration.

e) Formally dual to the previous part.

Exercise 5

a) Note that since $V \dashv G$ is a Quillen adjunction, we have $\mathrm{Cof}_{\mathbf{M}} \subseteq V^{-1}(\mathrm{Cof}_{\mathbf{N}})$ and thus $\mathrm{RLP}(V^{-1}(\mathrm{Cof}_{\mathbf{N}})) \subseteq \mathrm{RLP}(\mathrm{Cof}_{\mathbf{M}}) = \mathrm{WE}_{\mathbf{M}} \cap \mathrm{Fib}_{\mathbf{M}} \subseteq \mathrm{WE}_{\mathbf{M}}$. Now $V(\mathrm{WE}_{\mathbf{M}}) \subseteq \mathrm{WE}_{\mathbf{N}}$ by assumption, so we indeed have $\mathrm{RLP}(V^{-1}(\mathrm{Cof}_{\mathbf{N}})) \subseteq V^{-1}(\mathrm{WE}_{\mathbf{N}})$, i. e. the acyclicity condition is satisfied. Thus, again by the main result of [HKRS17], there is a model structure on $\mathbf{M}$ left induced by $V \dashv G$.

Moreover, note again that $\mathrm{Cof}_{\mathbf{M}} \subseteq V^{-1}(\mathrm{Cof}_{\mathbf{N}}) = \mathrm{Cof}_{\mathbf{M}_{\mathrm{left}}}$ (i.e. $\mathrm{Id}_{\mathbf{M}} \colon \mathbf{M} \to \mathbf{M}_{\mathrm{left}}$ preserves cofibrations) and that $\mathrm{WE}_{\mathbf{M}} \subseteq V^{-1}(\mathrm{WE}_{\mathbf{N}}) = \mathrm{WE}_{\mathbf{M}_{\mathrm{left}}}$ (i.e. $\mathrm{Id}_{\mathbf{M}} \colon \mathbf{M} \to \mathbf{M}_{\mathrm{left}}$ preserves weak equivalences and thus acyclic cofibrations), so we obtain a sequence of Quillen pairs as in the exercise, where the adjunction on the right is a Quillen pair by the definition of a left induced model structure.

b) Formally dual to the previous part.

Exercise 6

a) In that case, we have

$$\mathrm{RLP}(V^{-1}(\mathrm{Cof}')) \subseteq \mathrm{RLP}(V^{-1}(\mathrm{Cof})) \subseteq V^{-1}(\mathrm{WE}) \subseteq V^{-1}(\mathrm{WE}')$$

where the second inclusion holds because the model structure left induced from $(\mathrm{WE}, \mathrm{Cof}, \mathrm{Fib})$ exists. Thus, once again by [HKRS17], the model structure left induced from $(\mathrm{WE}', \mathrm{Cof}', \mathrm{Fib}')$ exists.

b) Formally dual to the previous part.

References

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