Homotopical Algebra
~~Solution Sketches~~ References for Exercise Set 8

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I don't think I will have time to type solution sketches for this set soon, so I am posting some references to help you in the upcoming exercises which use results from this set.

Exercise 1

This should be straightforward check (and as usual, one essentially has to do only the half of it because $A/\mathbf{M} \cong (\mathbf{M}^{\mathrm{op}}/A)^{\mathrm{op}}$). A proof sketch (which is formulated in a slightly different context[1], but still applies to ours) can be found in [Hov99, Proposition 1.1.8].

Exercise 2

Remark 1. As pointed out in the exercise session, in the definition of a transfinite composition, one also needs to require that for all limit ordinals $\gamma < \lambda$, the morphisms $(F(\alpha) \xrightarrow{F(\alpha<\gamma)} F(\gamma))_{\alpha<\gamma}$ endow $F(\gamma)$ with the structure of a colimit of the system $(F(\alpha))_{\alpha<\gamma}$.

Part (a) should be straightforward check using transfinite induction on $\beta < \lambda$, using the LLP in the successor step and the colimit property in the limit case.

Part (b) is doing essentially the same thing in the opposite category and at this point, I will refrain from spelling this out.

[1]For example, Hovey assumes that the factorizations of (M4) are in a certain sense "functorial".

Exercise 3

This originally appeared in [Col06]. See also [MP12, Theorem 17.3.1] for a more "concise" treatment.

Exercise 4

See [Cam] for an account of *all* model structures[2] on **Set**.

Exercise 5

See [Rez] for a treatment of this. Fibrations will turn out to be the so-called *isofibrations*:

Definition 2. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is called an *isofibration* if for all $c \in \mathbf{C}$ and every isomorphism $\phi\colon F(c) \xrightarrow{\cong} d$ in $\mathbf{D}$, there exists an isomorphism $\tilde{\phi}\colon c \xrightarrow{\cong} c'$ in $\mathbf{C}$ with $F(\tilde{\phi}) = \phi$.

Remark 3. Note that being an isofibration is equivalent to having RLP w. r. t. the following functor G:

Let $\ast$ be the category with only one object and only the identity morphism. Let I be the category with two objects x, y and only two non-identity morphisms $f\colon x \to y$, $g\colon y \to x$ with $gf = \mathrm{Id}_x$ and $fg = \mathrm{Id}_y$. Set $G\colon \ast \to I$ to be the functor which sends the only object of $\ast$ to $x \in \operatorname{ob} I$ (and the identity to the identity).

References

[Cam] Omar Antolín Camarena. The nine model category structures on the category of sets. Available on http://www.matem.unam.mx/omar/notes/modelcatsets.html. 2

[Col06] Michael Cole. Mixing model structures. *Topology Appl.*, 153(7):1016–1032, 2006. 2

[Hov99] Mark Hovey. *Model categories.*, volume 63. Providence, RI: American Mathematical Society, 1999. 1

[MP12] J. P. May and K. Ponto. *More concise algebraic topology. Localization, completion, and model categories.* Chicago, IL: University of Chicago Press, 2012. 2

[Rez] Charles Rezk. A model category for categories. Available on https://faculty.math.illinois.edu/~rezk/cat-ho.dvi. 2

[2] Camarena uses a slightly different axiomatization of model categories which is, however, equivalent to ours.