Solution Sketches for Exercise Set 7 of Homotopical Algebra

Some preliminary abstract nonsense

As we will see below, many properties of sSet are actually properties of so-called presheaf categories, i.e. categories of the form $\mathrm{Fun}(I^{op}, \mathrm{Set})$ for some small category I. (And many others are not.)

We will solve many of the exercises using these formal properties. The exercises may have been thought to be solved by "explicit calculations", but this approach will hopefully give a more conceptual understanding of the statements.

From now on we fix a small category I (and will later specialize to $I = \Delta$.)

Recall the Yoneda embedding, i.e. the fully faithful functor

$$y : I \longrightarrow \mathrm{Fun}(I^{op}, \mathrm{Set})$$
$$i \longmapsto \mathrm{Hom}_I(-, i).$$

Now let a functor $F : I \to \mathcal{C}$ be given, where $\mathcal{C}$ is a cocomplete category. Define a functor

$$\mathrm{Sing}_F : \mathcal{C} \longrightarrow \mathrm{Fun}(I^{op}, \mathrm{Set})$$
$$C \longmapsto \begin{cases} i \in \mathrm{ob}\, I \longmapsto \mathrm{Hom}_{\mathcal{C}}(F(i), C) \\ (f : i \to j) \in \mathrm{mor}\, I \longmapsto \begin{array}{c} \mathrm{Hom}_{\mathcal{C}}(F(i), C) \ni \varphi \circ F(f) \\ \uparrow \qquad\qquad \uparrow \\ \mathrm{Hom}_{\mathcal{C}}(F(j), C) \ni \varphi \end{array} \end{cases}$$

<u>Claim:</u> A left adjoint of Sing_F can be constructed as

$$\hat{F}(X) := \mathrm{Coeq}\left(\coprod_{f : i \to j \in \mathrm{mor}\, I} \ \coprod_{x \in X_j} F(i) \overset{(F(i)_{f,x} \xrightarrow{F(f)} F(j)_x)_{f,x}}{\underset{(F(i)_{f,x} \xrightarrow{id} F(i)_{X(f)(x)})_{f,x}}{\rightrightarrows}} \coprod_{k \in \mathrm{ob}\, I} \ \coprod_{x \in X_k} F(k) \right).$$

> We denote $X(i)$, $i \in \mathrm{ob}\, I$ also by X_i to be compatible with the simplicial notation.

<u>Proof:</u> Functoriality is a straightforward check. We sketch the adjunction isomorphism:

$$\mathrm{Hom}_{\mathcal{C}}(\hat{F}(X), C) \overset{\text{Univ. prop. of coeq.}}{\cong} \left\{ \left(F(k) \xrightarrow{\varphi_{k,x}} C \right)_{\substack{k \in I \\ x \in X_k}} \;\middle|\; \text{for all } f : i \to j,\ x \in X_j \quad \begin{array}{ccc} F(i) & \xrightarrow{F(f)} & F(j) \\ \| & \circlearrowleft & \downarrow \varphi_{j,x} \\ F(i) & \xrightarrow[\varphi_{i, X(f)(x)}]{} & C \end{array} \rightsquigarrow \boxed{\begin{array}{c} \varphi_{j,x} \circ F(f) \\ \| \\ \varphi_{i, X(f)(x)} \end{array}} \right\}$$

[Cont. on the next page.]

$$\mathrm{Hom}_{\mathcal{C}}(\hat{F}(X), C) \cong \left\{ X_k \xrightarrow{\varphi_{k,-}} \mathrm{Hom}_{\mathcal{C}}(F(k), C) \;\middle|\; \text{for all } f: i \to j \text{ in } I,\ x \in X_j:\ \varphi_{j,x} \circ F(f) = \varphi_{i, X(f)(x)},\ \text{i.e.} \right.$$

$$\begin{tikzcd} X_j \arrow[r, "\varphi_{k,-}"] \arrow[d, "X(f)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}(F(j), C) \arrow[d, "- \circ F(f)"] \\ X_i \arrow[r, "\varphi_{i,-}"'] & \mathrm{Hom}(F(i), C) \end{tikzcd} \Bigg\}$$

This is a naturality square!

$$\underset{\substack{\text{Def. of} \\ \text{nat transf.}}}{\cong} \mathrm{Nat}_{I^{op} \to \mathrm{Set}}\big(X, \mathrm{Hom}(F(-), C)\big)$$

$$\underset{\text{Def of } \mathrm{Sing}_F}{=} \mathrm{Hom}_{\mathrm{Fun}(I^{op}, \mathrm{Set})}(X, \mathrm{Sing}_F C). \quad \square$$

<u>Exercise 1</u>: Let $I = \Delta$. Then $\mathrm{Fun}(I^{op}, \mathrm{Set}) = \mathrm{sSet}$.

Let $K_\bullet \in \mathrm{sSet}$.

Note that we have

$$\tilde{K}_\bullet := \coprod_{n \geq 0} K_n \times \Delta[n] \Big/ \sim \;\cong \coprod_{[n] \in \Delta} \coprod_{x \in K_n} \mathrm{Hom}(-, [n]) \Big/ \sim$$

Moreover, by Exercise 3, every morphism in Δ is a composition of δ's and σ's, so the equivalence relation generated by relations of the form $(x, \delta^i \xi) \sim (d_i x, \xi)$ and $(x, \sigma^j \xi) \sim (s_j x, \xi)$

$$\boxed{\begin{matrix} \| & \| & \| & \| \\ (x, \mathrm{Hom}([\ell], \delta^i)(\xi)) & (X(\delta^i)(x), \xi) & (x, \mathrm{Hom}([\ell], \sigma^j)(\xi)) & (X(\sigma^j)(x), \xi) \end{matrix}}$$

coincides with the equivalence relation generated by the relations $(x, \mathrm{Hom}([\ell], f)(\xi)) \sim (X(f)(x), \xi)$ for all $f: [n] \to [p]$, $x \in X_p$, $\xi \in \mathrm{Hom}([\ell], [n]) = \Delta[n]_\ell$.

Thus, $\tilde{K}_\bullet$ is actually isomorphic to $\hat{y}(K_\bullet)$:

$$\begin{tikzcd} \Delta \arrow[r, "F = y"] \arrow[d, "y"'] & \mathcal{C} = \mathrm{sSet} \arrow[dl, bend left, "\mathrm{Sing}_y"] \\ \mathrm{sSet} \arrow[ur, "\hat{y}"] & \end{tikzcd} \qquad K_\bullet \mapsto \tilde{K}_\bullet$$

(†): Any two left adjoints of a functor are isomorphic. See page 4 for a reference.

Now note that we have a natural isomorphism $\mathrm{Sing}_y X = \mathrm{Hom}_{\mathrm{sSet}}(y(-), X) \underset{\text{Yoneda Lemma}}{\cong} X$. Hence $\mathrm{id}_{\mathrm{sSet}}$ is another left adjoint of $\mathrm{Sing}_y \cong \mathrm{id}_{\mathrm{sSet}}$ and uniqueness of adjoints(†) implies that $\hat{y} \cong \mathrm{id}_{\mathrm{sSet}}$. Considering this for a specific $K_\bullet \in \mathrm{sSet}$, we see that $\tilde{K}_\bullet \cong K_\bullet$.

<u>Remark</u>. An analogous argument shows that $\hat{y} \cong \mathrm{id}_{\mathrm{Fun}(I^{op}, \mathrm{Set})}$ for the Yoneda embedding $y: I \to \mathrm{Fun}(I^{op}, \mathrm{Set})$ of any small cat. I.

Interlude with more abstract nonsense

(We stick to $I = \Delta$ here, but analogous statements hold for every presheaf category.)

Let $F: \Delta \to \mathcal{C}$ be as before. We will now interpret $\hat{F}: sSet \to \mathcal{C}$ as "the cocontinuous extension of F to sSet".

$$\begin{tikzcd} F: I \arrow[r] \arrow[d, hook, "y"'] & \mathcal{C} \\ Fun(I^{op}, Set) \arrow[ur, dashed, "\hat{F}"'] & \end{tikzcd}$$

Claim: Let $\tilde{F}: sSet \to \mathcal{C}$ be a functor s.t.

(I) $\tilde{F}$ preserves colimits.

(II) $\tilde{F} \circ y \cong F$.

Then $\tilde{F}$ is left adjoint to $Sing_F$.

Proof: Let $K_\bullet \in sSet$, $C \in \mathcal{C}$. Then we have natural iso's

$$\begin{aligned}
Hom_{\mathcal{C}}(\tilde{F}(K_\bullet), C) &\overset{\text{Exc. 1}}{\cong} Hom_{\mathcal{C}}\Big(\tilde{F}\Big(coeq\Big(\coprod_{f:[i]\to[j]} \coprod_{x \in K_j} \Delta[i] \rightrightarrows \coprod_{[n]} \coprod_{x \in K_n} \Delta[n]\Big)\Big), C\Big) \\
\text{$\tilde{F}$ commutes with colimits} \quad &\overset{(A)}{\cong} Hom_{\mathcal{C}}\Big(coeq\Big(\coprod_{f:[i]\to[j]} \coprod_{x \in K_j} \tilde{F}(\Delta[i]) \rightrightarrows \coprod_{[n]} \coprod_{x \in K_n} \tilde{F}(\Delta[n])\Big), C\Big) \\
\tilde{F}\circ y \cong F,\ y([n]) = \Delta[n] \quad &\cong Hom_{\mathcal{C}}\Big(coeq\Big(\coprod_{f:[i]\to[j]} \coprod_{x \in K_j} F([i]) \rightrightarrows \coprod_{[n]} \coprod_{x \in K_n} F([n])\Big), C\Big) \\
\text{universal property of colimits} \quad &\overset{(B)}{\cong} eq\Big(\prod_{f:[i]\to[j]} \prod_{x \in K_j} \underbrace{Hom_{\mathcal{C}}(F[i], C)}_{\cong Hom_{sSet}(\Delta[i], Hom_{\mathcal{C}}(F(-), C))} \leftleftarrows \prod_{[n]} \prod_{x \in K_n} \underbrace{Hom_{\mathcal{C}}(F[n], C)}_{\cong Hom_{sSet}(\Delta[n], Hom_{\mathcal{C}}(F(-), C))}\Big) \\
&\qquad \text{Yoneda Lemma} \\
\text{def. of } Sing_F \quad &\cong eq\Big(\prod_{f:[i]\to[j]} \prod_{x \in K_j} Hom_{sSet}(\Delta[i], Sing_F(C)) \leftleftarrows \prod_{[n]} \prod_{x \in K_n} Hom_{sSet}(\Delta[n], Sing_F(C))\Big) \\
\text{univ. prop of colim.} \quad &\overset{(B')}{\cong} Hom_{sSet}\Big(coeq\Big(\coprod_{f:[i]\to[j]} \coprod_{x \in K_j} \Delta[i] \rightrightarrows \coprod_{[n]} \coprod_{x \in K_n} \Delta[n]\Big), Sing_F(C)\Big) \\
\text{Exercise 1} \quad &\cong Hom_{sSet}(K_\bullet, Sing_F(C)).
\end{aligned}$$

In the intermediate steps the naturality on the sSet-variable might not look obvious, but one can check that any map $K_\bullet \to L_\bullet$ of sSet's induces maps between the corr. limits/colimits which are compatible with the given identifications

Thus $\tilde{F} \dashv Sing_F$.

Remark: The crucial ingredient of the proof is the fact that one can express every simplicial set (= presheaf on Δ) in a natural manner as a colimit of $\Delta[k]$'s ($\cong$ representable presheaves) so that the steps (A), (B) and (B') work, so how the colimit actually looks like (e.g. the complicated expression with the coequalizer and nested coproducts) is not relevant.

Remark: A straightforward calculation shows that the converse of the last claim is also true, i.e. any left adjoint of Sing_F has the properties (I) and (II).

Moreover, the following fact implies that given F, the left adjoint of F or equivalently "the cocontinuous extension of F to sSet" is essentially unique:

Fact (Corollary IV.1.1 in "Categories for the Working Mathematician"): Any two left adjoints to a given functors are isomorphic.

Remark: There is yet another description of $\hat{F}$, namely it is the left Kan extension of F along y.

Exercise 2

(a) Note that we have
$$\mathrm{Map}(K_\bullet, L_\bullet)_{(-)} := \mathrm{Hom}_{sSet}(K_\bullet \times \Delta[-], L_\bullet)$$
is just $\mathrm{Sing}_{K_\bullet \times \Delta[-]}(L_\bullet) = \mathrm{Sing}_{K_\bullet \times y(-)}(L_\bullet)$ where $K_\bullet \times \Delta[-] : \Delta \to sSet$,

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d, "y"] & sSet \arrow[dl, "\mathrm{Map}(K_\bullet,-)"] \\ sSet & \end{tikzcd}$$

i.e. face and deg maps are induced by precomposition with δ^i_s and σ^{j}_s.

(b) Note that for a diagram $D : B \to sSet$ and $n \in \mathbb{N}$ we have a bijection
$$\Big(\operatorname*{colim}_{b\in B}(D(b)_\bullet \times K_\bullet)\Big)_n \overset{\text{pointwise limits}}{=} \operatorname*{colim}_{b\in B}(D(b)_\bullet \times K_\bullet)_n$$
$$\overset{\text{pointwise colimits}}{=} \operatorname*{colim}_{b\in B} D(b)_n \times K_n$$
$$\cong \Big(\operatorname*{colim}_{b\in B} D(b)_n\Big) \times K_n$$
$$\overset{\text{pw. colim.}}{=} \Big(\operatorname*{colim}_{b\in B} D(b)\Big)_n \times K_n$$
$$\overset{\text{p.w. lim.}}{=} \Big(\Big(\operatorname*{colim}_{b\in B} D(b)\Big) \times K\Big)_n$$

$\boxed{-\times K_n \text{ commutes with colimits in Set b/c it's left adj. to } \mathrm{Hom}_{Set}(K_n,-)}$

which is actually induced by the natural map
$$\operatorname*{colim}_{b\in B}(D(b) \times K) \to \Big(\operatorname*{colim}_{b\in B} D(b)\Big) \times K.$$

Thus $-\times K_\bullet$ preserves colimits in sSet. Moreover, we tautologically have $(-\times K_\bullet)\circ y \cong K_\bullet \times y(-) : \Delta^{op} \to sSet$. Thus, by the the characterization in the abstract nonsense part, it must be left adjoint to
$$\mathrm{Sing}_{K_\bullet \times y(-)} \overset{\text{Part (a)}}{=} \mathrm{Map}(K_\bullet, -).$$

(c) We have natural (in $[n]\in\Delta$ and $J_\bullet, K_\bullet, L_\bullet \in sSet$) isomorphisms:
$$\mathrm{Hom}_{sSet}(\Delta[n], \mathrm{Map}(J_\bullet, \mathrm{Map}(K_\bullet, L_\bullet))) \overset{\text{Yoneda}}{\cong} \mathrm{Map}(J_\bullet, \mathrm{Map}(K_\bullet, L_\bullet))_n$$
$$\overset{\text{Def.}}{=} \mathrm{Hom}_{sSet}(J_\bullet \times \Delta[n], \mathrm{Map}(K_\bullet, L_\bullet))$$
$$\overset{(b)}{\cong} \mathrm{Hom}_{sSet}(J_\bullet \times \Delta[n] \times K_\bullet, L_\bullet)$$
$$\cong \mathrm{Hom}_{sSet}(J_\bullet \times K_\bullet \times \Delta[n], L_\bullet)$$
$$\overset{\text{Def.}}{=} \mathrm{Map}(J_\bullet \times K_\bullet, L_\bullet)_n$$
$$\overset{\text{Yoneda}}{\cong} \mathrm{Hom}_{sSet}(\Delta[n], \mathrm{Map}(J_\bullet \times K_\bullet, L_\bullet)).$$

Thus $\mathrm{Map}(J_\bullet, \mathrm{Map}(K_\bullet, L_\bullet)) \cong \mathrm{Map}(J_\bullet \times K_\bullet, L_\bullet)$.

Exercise 3

Idea: First factor f as a surjection $[n] \to [n-k] \hookrightarrow [m]$ followed by an injection. Then find i's by considering which elements are "missed" in the image and j's by looking at preimages of elements of $[n-k]$.

Exercise 4

(a) Set:

- $ob(\tau(P)) = P$
- $Hom_{\tau(P)}(p,q) \begin{cases} \{*_{p,q}\} & p \le q \\ \varphi & o.w. \end{cases}$
- Identities: reflexivity of $\le$.
- Composition: transitivity of $\le$.
- $\varphi: P \to Q$ order preserving (i.e. $p \le p' \Rightarrow \varphi(p) \le \varphi(p')$), then

define $\tau(\varphi)$ as:

- on objects: φ
- on morphisms: $\tau(\varphi)(*_{p,q}) = *_{\varphi(p),\varphi(q)}$

This is indeed a functor because φ preserves the order.

Seeing that τ respects identities and composition is straightforward

$\rightsquigarrow$ Functor τ: Poset $\to$ Cat.

This is faithful because if two functors are equal, then they are equal on objects which recovers the original order preserving map.

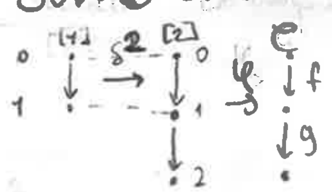
In fact, it is also full: If $F: \tau(P) \to \tau(Q)$ is a functor and $p \le q$, then $F(p) \le F(q)$ is witnessed by $F(*_{p,q})$, so $F^{ob}: ob(\tau(P)) = P \to Q = ob(\tau(Q))$ is a map of posets with $\tau F^{ob} = F$.

(b) • d^0/d^n: take the last / first $n-1$ arrows.

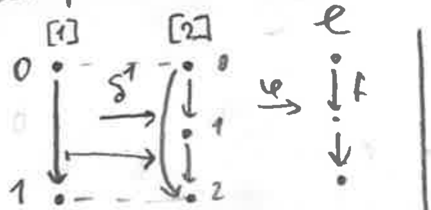
• other d^i: compose the two arrows which come before & after the vertex i.

• s^i: insert an identity at vertex i.

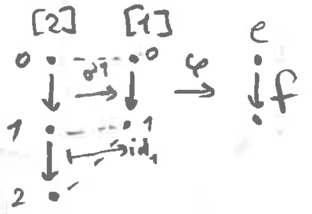
Some small examples:



$\rightsquigarrow d^2\varphi = \varphi \circ \tau(\delta^2) = f$



$\rightsquigarrow d^1\varphi = \varphi \circ \tau(\delta^1) = g \circ f$



$\rightsquigarrow s^1\varphi = \varphi \circ \tau(\sigma^1) = \cdot \xrightarrow{f} \cdot \xrightarrow{id} \cdot$

• Effect of a functor $F : \mathcal{C} \to \mathcal{D}$:

$$(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} C_n) \longmapsto (F(C_0) \xrightarrow{F(f_1)} F(C_1) \xrightarrow{F(f_2)} \dots \xrightarrow{F(f_n)} C_n)$$

Note that this is compatible with s^i's and d^i's because F respects identities and compositions.

• Degenerate simplices are, by the description of s^i's given above, those composable arrows $C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} C_n$ for which at least one f_i is an identity morphism. Thus non-deg. simplices are those composable arrows $C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} C_n$ with $f_i = \mathrm{id}_{C_i}$ for all i.

(c) We have the foll. iso.s which are nat. in $[n], [m] \in \Delta$:

$$\begin{aligned} \mathrm{Hom}_{sSet}(\Delta[m], N_\bullet \tau[n]) &\overset{\text{Yoneda}}{\cong} (N_\bullet \tau[n])_m \\ &= \mathrm{Hom}_{Cat}(\tau[m], \tau[n]) \\ &\overset{\tau \text{ is fully faithful}}{\cong} \mathrm{Hom}_{Poset}([m],[n]) \\ &= \mathrm{Hom}_{\Delta}([m],[n]) = (\Delta[n])_m . \end{aligned}$$

Remark: ~~In~~ the language above, $N_\bullet$ is just Sing_τ.

Thus we have $N_\bullet \tau[n] \cong \Delta[n]$.

(d) Simplices of $B_\bullet(\mathbb{Z}/2)$ correspond to sequences of elements in $\mathbb{Z}/2$ (since all arrows in $B\mathbb{Z}/2$ are composable).

Face maps correspond to leaving out the first (d^n) or the last element (d^0), or multiplying two adjacent elements (other d^i's).

Degeneracy maps correspond to adding neutral elements to a sequence.

$(\bar{1}, \dots, \bar{1})$ (n-times) is the unique non-degenerate n-simplex (since all other sequences of length n include a neutral element).

(e) I don't know how to show this by "applying the Yoneda lemma", but the standard proof goes as follows:

- <u>Injectivity</u>: If two functors induce the same morphisms on nerves, they in particular induce the same maps on 1-simplices which correspond to morphisms in the original categories, so they must agree.
- <u>Surjectivity</u>: Given a morphism $f: N_\bullet\mathcal{C} \to N_\bullet\mathcal{D}$, define a functor F via: $\mathrm{ob}\,\mathcal{C} \cong N_0\mathcal{C} \xrightarrow{f_0} N_0\mathcal{D} \cong \mathrm{ob}\,\mathcal{D}$

$$\mathrm{mor}\,\mathcal{D} \cong N_1\mathcal{C} \xrightarrow{f_1} N_1\mathcal{D} \cong \mathrm{mor}\,\mathcal{D}.$$

This is compatible with identities because f is comp. with degeneracies and compatible with composition because f is compatible with face maps.

Now we want to to show that $N_\bullet F = f$. This boils down to showing that higher simplices can be recovered from $N_{\leq 1}\mathcal{C}$ (resp. $N_{\leq 1}\mathcal{D}$) as follows:

Let $[n] \in \Delta$. Let $f: [1] \to [n]$, $0 \mapsto n-1$, $1 \mapsto n$, $g: [n-1] \to [n]$, $i \mapsto i$ which are both composites of some δ^i's. These yield a comm. diagram

$$\begin{tikzcd}
\text{``n comp arrows''} \arrow[dd, mapsto] \arrow[rrr, bend left, mapsto] & N_n\mathcal{C} \arrow[r, "N_g\mathcal{C}"] \arrow[d, "N_f\mathcal{C}"'] & N_{n-1}\mathcal{C} \arrow[d, "N_{(0\mapsto n-1)}\mathcal{C}"] & \text{``first n-1 arrows''} \arrow[dd, mapsto] \\
 & N_1\mathcal{C} \arrow[r, "N_{(0\mapsto 0)}\mathcal{C}"'] & N_0\mathcal{C} & \\
\text{``last arrow''} \arrow[rrr, bend right, mapsto] & & & \text{``(n-1)-th object''}
\end{tikzcd}$$

which in fact equips $N_n\mathcal{C}$ as a pullback of the diagram without it, i.e. $N_n\mathcal{C} \cong N_1\mathcal{C} \times_{N_0\mathcal{C}} N_{n-1}\mathcal{C}$. Iterating this, we get a bijection $N_n\mathcal{C} \cong N_1\mathcal{C} \times_{N_0\mathcal{C}} \dots \times_{N_0\mathcal{C}} N_1\mathcal{C}$ (<u>Warning</u>: each "$\times_{N_0\mathcal{C}}$" here is a pullback by different maps.)

<u>Rk</u>: This is usually called the "Segal condition".

Since everything is natural (in n and in $\mathcal{C}$), this implies that a simplicial morphism btw. nerves is indeed determined by what it does on 0- and 1-simplices.

Exercise 5:

Note that standard topological simplices with face inclusions and degeneracy collapses define a functor $\hat{\Delta}^{(-)}: \Delta \to \mathrm{Top}$. In fact, we have $|-|_\Delta = \hat{\Delta}^{(-)}$ by the definition of both sides as certain quotient spaces (+ the fact that all morph. in Δ are comp. of δ^i's and σ^j's as used in Exercise 1).

Thus we have $|\Delta[n]| = |y([n])| = \hat{\Delta}^{(-)}(y[n]) \cong \Delta^n$

Defining properties of $\hat{\Delta}$.

To identify $|\partial\Delta[n]|$ we note that $\partial\Delta[n] = \bigcup_{0\leq i\leq n} d^i\Delta[n-1]$ and ~~that~~ the intersection of $d^i\Delta[n-1]$'s are isomorphic to some unions of simplices of dim. $< n-1$. Thus, $\partial\Delta[n]$ can be described as an iterated pushout involving unions (which are also iterated pushouts) of lower dimensional simplices. Now, using induction and that $|-| = \hat{\Delta}^{(-)}$ preserves colimits and thus pushouts, one can deduce that $|\partial\Delta[n]| \cong \partial\Delta^n$.