Homotopical Algebra
Solution Sketches for Exercise Set 2

2018-03-30

Exercises 1 & 2

Note that Hurewicz fibrations are precisely the class

$$\mathrm{RLP}\left(\left\{X \xrightarrow{\mathrm{Id}_X \times 0} X \times I \mid X \in \mathbf{Top}\right\}\right).$$

and that Hurewicz cofibrations are precisely the class

$$\mathrm{LLP}\left(\left\{PY \xrightarrow{\mathrm{ev}_0} Y \mid Y \in \mathbf{Top}\right\}\right).$$

Hence these exercises are special cases of Exercises 5(a) & 5(b) in Exercise Set 3 (and the proofs in the general case are easy generalizations of the proofs in these special cases), so we refer to the upcoming solution sketches for these exercises.

Exercise 3

Note that since $A \subseteq X$ is closed, $A \times Z \subseteq X \times Z$ is also closed, so we can use the retraction criterion discussed in the lecture. Now, given a retraction $r\colon X \times I \to (A \times I) \cup (X \times \{0\})$ witnessing the fact that $j\colon A \hookrightarrow X$ is a Hurewicz cofibration, we define

$$\begin{gathered} X \times Z \times I \to (A \times Z \times I) \cup (X \times Z \times \{0\}) \\ (x, z, t) \mapsto (\mathrm{pr}_1(r(x,t)), z, \mathrm{pr}_2(r(x,t))) \end{gathered}$$

which can be checked to be well-defined, continuous and a retraction of

$$(A \times Z \times I) \cup (X \times Z \times \{0\}) \subseteq X \times Z \times I.$$

Exercise 4

Let $X := \bigcup_{n=0}^{\infty} A_n$. Let $i\colon A_0 \hookrightarrow X$ be the inclusion. Since i is the inclusion of a closed subspace, it is enough to show that $(A_0 \times I) \cup (X \times \{0\}) \subset X \times I$ admits a retraction. We want to define such a map by defining continuous maps $r_n\colon A_n \times I \to A_0 \times I \cup X \times \{0\}$ and then "gluing" them, for which we need the following statements whose proofs crucially use the special property of the topology on X:[1]

Lemma 1. *A map $f\colon X \to Y$ is continuous iff $f|_{A_n}$ is continuous for all $n \in \mathbb{N}$.*

Proof. Note that

$$\begin{aligned}
&f\colon X \to Y \text{ is continuous}\\
\Leftrightarrow\ &\text{For all closed } Z \subseteq Y,\ f^{-1}(Z) \subseteq X \text{ is closed.}\\
\Leftrightarrow\ &\text{For all closed } Z \subseteq Y, \text{ for all } n \in \mathbb{N},\ f^{-1}(Z) \cap A_n \subseteq A_n \text{ is closed.}\\
\Leftrightarrow\ &\text{For all } n \in \mathbb{N}, \text{ for all closed } Z \subseteq Y,\ f^{-1}(Z) \cap A_n = f|_{A_n}^{-1}(Z) \subseteq A_n \text{ is closed.}\\
\Leftrightarrow\ &\text{For all } n \in \mathbb{N},\ f|_{A_n}^{-1}(Z)\colon A_n \to Y \text{ is continuous.}
\end{aligned}$$

□

Corollary 2. *Let Y be a topological space. A map $g\colon X \times I \to Y$ is continuous iff $g|_{A_n \times I}$ is continuous for all $n \in \mathbb{N}$.*

Proof. Note that

$$\begin{aligned}
&g\colon X \times I \to Y \text{ is continuous}\\
\Leftrightarrow\ &\widehat{g}\colon X \to PY,\ \widehat{g}(x) := g(x,-) \text{ is continuous.}\\
\Leftrightarrow\ &\text{For all } n \in \mathbb{N},\ \widehat{g}_n := \widehat{g}|_{A_n}\colon A_n \to PY \text{ is continuous.}\\
\Leftrightarrow\ &\text{For all } n \in \mathbb{N},\ \widehat{g}'_n\colon A_n \times I \to Y,\ \widehat{g}'_n(x,t) := \widehat{g}_n(x)(t) = g|_{A_n \times I}(x,t), \text{ is continuous.}\\
\Leftrightarrow\ &\text{For all } n \in \mathbb{N},\ g|_{A_n \times I} \text{ is continuous.}
\end{aligned}$$

□

Now since each inclusion $A_{n-1} \subseteq A_n$ is a Hurewicz cofibration, we have retractions

$$s'_n\colon A_n \times I \to (A_{n-1} \times I) \cup (A_n \times \{0\}).$$

Using the gluing lemma and the fact that s'_n is a retraction, one can show that this extends to a map

$$s_n\colon (A_n \times I) \cup (X \times \{0\}) \xrightarrow{s'_n \cup \mathrm{Id}_{X \times \{0\}}} (A_{n-1} \times I) \times (X \times \{0\}).$$

[1]Since this is coming out so late, I may as well use the language of category theory to explain these statements: Lemma 1 is saying that $\bigcup_{n=0}^{\infty} A_n$ is the colimit of the diagram $(A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_n \hookrightarrow \ldots)$ in **Top** and Corollary 2 is a special case of the fact that $(-) \times I$ commutes with colimits because it is left adjoint to $P(-)$.

Next, for $n \in \mathbb{N}$ we define the continuous map

$$r_n := (s_1 \circ \cdots \circ s_n)|_{A_n \times I} : A_n \times I \to (A_0 \times I) \cup (X \times \{0\}),$$

which is basically "applying s_i's until one lands in $(A_0 \times I) \cup (X \times \{0\})$". Using the retraction properties of the s_i''s one can check that $r_n|_{A_{n-1}\times I} = r_{n-1}$, so by Corollary 2, these glue to a map $r\colon X \times I \to (A_0 \times I) \cup (X \times \{0\})$, which can, by observing that $r_0 = \mathrm{Id}_{(A_0 \times I)\cup(X\times\{0\})}$, be shown to have the desired property.

Exercise 5

Remark 3. In the following, we omit the discussion of the continuity of the maps defined which follows from the "usual tricks" such as
- composing continuous maps,
- defining a map into a product space by defining continuous components,
- showing the continuity of the corresponding map $Z \times I \to W$ instead of a map $Z \to PW$,
- the gluing lemma.

Given a map $f\colon X \to Y$, following the hint, we consider the factorization[2]

$$f\colon X \xrightarrow{\mathrm{Id}_X \times c_{(-)}} P_f \xrightarrow{\mathrm{ev}_1 \circ \mathrm{pr}_2} Y.$$

It is not hard to show that the first map is a homotopy equivalence and the second map is a Hurewicz fibration, but unfortunately, showing that $j := \mathrm{Id}_X \times c_{(-)}\colon X \to P_f$ is a Hurewicz cofibration doesn't quite work. We need to "thicken up P_f" and factor j further[3] as

$$j\colon X \xrightarrow{\mathrm{Id}_X \times c_{(-)} \times \{0\}} (j(X) \times \{0\}) \cup (P_f \times (0,1]) \xrightarrow{\mathrm{pr}_1} P_f$$

where we consider $E := (j(X) \times \{0\}) \cup (P_f \times (0,1])$ as a subspace of $P_f \times I$.

Now we will show that the map

$$\begin{aligned} i\colon X &\xrightarrow{\mathrm{Id}_X \times c_{(-)} \times \{0\}} E \\ x &\mapsto (x, c_x, 0) \end{aligned}$$

is an acyclic Hurewicz cofibration and that the map

$$\begin{aligned} q\colon E &\xrightarrow{\mathrm{ev}_1 \circ \mathrm{pr}_2 \circ \mathrm{pr}_1} Y \\ (x, \lambda, t) &\mapsto \lambda(1) \end{aligned}$$

is a Hurewicz fibration which will yield the desired result since $f = q \circ i$.

[2] More precisely, $\mathrm{Id}_X \times c_{(-)}$ and $\mathrm{ev}_1 \circ \mathrm{pr}_2$ are maps to resp. from $X \times PY$ and we consider their (co)restriction to P_f.

[3] See also https://math.stackexchange.com/a/1179143 for a discussion of this.

We start by showing that q is a Hurewicz fibration. Let a lifting problem

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\mathrm{Id}_A \times 0"'] & E \arrow[d, "q"] \\ A \times I \arrow[r, "H"'] \arrow[ur, dashed, "?"] & Y \end{tikzcd}$$

be given. Let $g_1 \colon A \to X$, $g_2 \colon A \to PY$, $g_3 \colon A \to I$ be the components of g (after composing it with the inclusion $E \subseteq X \times PY \times I$). We will define a lift $\widetilde{H} \colon A \times I \to E$ componentwise by analyzing the conditions the desired lifting properties enforce on it.

We set $\widetilde{H}_1(a, s) := g_1(a)$ for $(a, s) \in A \times I$ which ensures that the upper triangle in the resulting diagram will commute in the first coordinate.

Next, we want to define $\widetilde{H}_2 \colon A \times I \to PY$ which is equivalent to defining a map $\Psi \colon A \times I \times I \to Y$. The commutativity of the upper triangle in the second coordinate will enforce that $\widetilde{H}_2(a, 0)(t) = \Psi(a, 0, t) = g_2(a)(t)$ for all $a \in A$ and $t \in I$, whereas the commutativity of the lower triangle will mean that $\widetilde{H}_2(a, s)(1) = \Psi(a, s, 1) = H(a, s)$ must hold for all $a \in A$ and $s \in I$. Moreover, the condition that the image of $\widetilde{H}$ must lie in $E \subseteq X \times PY \times I$ will require that $\widetilde{H}_2(a, s)(0) = \Psi(a, s, 0) = f(\widetilde{H}_1(a, s)) = f(g_1(a))$ for all $(a, s) \in A \times I$. Since we have $g_2(a)(1) = q(g(a)) = H(a, 0)$ and $g_2(a)(0) = f(g_1(a))$ for all $a \in A$, these conditions can be encoded in a map $\Psi' \colon A \times ((\{0\} \times I) \cup (I \times \{0\}) \cup (I \times \{1\})) \to Y$ given by $\Psi'(a, 0, t) := g_2(a)(t)$, $\Psi'(a, s, 1) := H(a, s)$ and $\Psi'(a, s, 0) = f(g_1(a))$. Now, like in Exercise 3 of the previous set, one can find a retraction $r \colon A \times I \times I \to A \times ((\{0\} \times I) \cup (I \times \{0\}) \cup (I \times \{1\}))$ and define Ψ to be $\Psi' \circ r$.

Finally, we set $\widetilde{H}_3(a, s) := s + (1 - s) \cdot g_3(a)$ for $(a, s) \in A \times I$ which ensures that the upper triangle will commute in the third coordinate and that the image of $\widetilde{H}$ will land in $E \subseteq X \times PY \times I$.

Now, using the properties indicated in the above paragraphs, it is a straightforward verification to check that $\widetilde{H}$ is indeed a lift with the desired properties.

Next, we show that $i \colon X \to E$ is a homotopy equivalence. Consider $h \colon E \to X$ given by $h(x, \lambda, t) := x$. Then we have $h \circ i = \mathrm{Id}_X$. Moreover, $i \circ h$ is homotopic to Id_E via the homotopy $H \colon E \times I \to H$ given by $H(x, \lambda, s, t) := (x, \lambda(t \cdot (-)), t \cdot s)$.

To show that i is a cofibration, we will use an alternative description of closed Hurewicz cofibrations[4] which is given by the following

Fact 4 ([Bre93, Theorem VII.1.5]). *Let $A \subseteq Z$ be a closed subspace. Then the inclusion $A \hookrightarrow Z$ is a Hurewicz cofibration if and only if there exists a neighborhood U of A and maps $\phi \colon Z \to I$, $H \colon U \times I \to Z$ such that*

- $A = \phi^{-1}(0)$,
- $\phi(Z \backslash U) = \{1\}$,

[4] One could check the retraction criterion for i by hand, but that would include some tedious continuity checks which are "blackboxed" in the proof of Fact 4 in our approach. Moreover, i is in a sense "engineered" to make this new criterion apply to it, so this approach also explains how one could come up with it.

- *$H(a,t) = a$ for all $(a,t) \in A \times I$,*
- *$H(u,0) = u$ and $H(u,1) \in A$ for all $u \in U$.*

We omit the proof which can be found in the reference. See also [Str] (and related notes) for a more "modern" approach with a similar criterion.

Now i is a homeomorphism into its image (whose inverse is given by $h|_{i(X)}$). Moreover, $U \subseteq E$, $\phi\colon E \to I$ and $H\colon U \times I \to E$ satisfying the conditions of Fact 4 for $Z = E$ and $A = i(X)$ can be defined as follows:

- $U := \{((x,\lambda,s) \in E \mid s < 1\}$,
- $\phi(x,\lambda,s) := s$ for all $(x,\lambda,s) \in E$,
- $H(x,\lambda,s,t) := (x, \lambda((1-t)\cdot(-)), (1-t)\cdot s)$ for all $(x,\lambda,s,t) \in U \times I$.

Thus i is indeed a cofibration.

References

[Bre93] Glen E. Bredon. *Topology and geometry.* New York: Springer-Verlag, 1993. 4

[Str] Neil P. Strickland. Fibrations and cofibrations. Available on https://neil-strickland.staff.shef.ac.uk/courses/homotopy/. 5