Homotopical Algebra
Solution Sketches for Exercise Set 1

2018-03-05

Exercise 1

a) Let $C \subseteq I$ be compact and $W \subseteq X \times I$ open. We want to show that $\eta_X^{-1}(\mathscr{P}_{C,W}) \subseteq X$ is open.
Let $x \in \eta_X^{-1}(\mathscr{P}_{C,W})$, i. e. for all $t \in C$, $(x,t) \in W$. Thus, for all $t \in C$ we can find a neighborhood of the form $U_t \times V_t$ of (x,t) which is contained in W where $U_t \subset X$ is an open neighborhood of x and $V_t \subseteq I$ is an open neighborhood of t. Now $\{V_t \cap C\}_{t \in C}$ is an open cover of C, so by compactness there exist $t_1, \ldots, t_n \in C$ s. t. $C \subseteq \bigcup_{i=1}^n V_{t_i}$.
Now consider a point $x' \in \bigcap_{i=1}^n U_{t_i}$. For $s \in C$, let i_s be such that $s \in C_{i_s}$. Then $(x', s) \in U_{t_{i_s}} \times V_{t_{i_s}} \subseteq W$. Hence $x' \in \eta_X^{-1}(\mathscr{P}_{C,W})$. This means that $\bigcap_{i=1}^n U_{t_i}$ is a neighborhood of x contained in $\eta_X^{-1}(\mathscr{P}_{C,W})$. Thus $\eta_X^{-1}(\mathscr{P}_{C,W})$ is open.

b) Let $U \subseteq X$ be open. We want to show that $\mathrm{ev}_X^{-1}(U) \subseteq PX \times I$ is open.
Let $(\lambda, t) \in \mathrm{ev}_X^{-1}(U) \subseteq PX \times I$, i. e. $\lambda(t) \in U$. Then, since λ is continuous, there exists an open neighborhood $V \subseteq I$ of t s. t. $\lambda(\overline{V}) \subseteq U$.[1] Note that $\overline{V}$ is compact, so $\mathscr{P}_{\overline{V},U} \times V \subseteq PX \times I$ is an open neighborhood of (λ, t). Moreover, if $(\mu, s) \in \mathscr{P}_{\overline{V},U} \times V$, then $\mathrm{ev}_X(\mu, s) = \mu(s) \subseteq U$, so $\mathscr{P}_{\overline{V},U} \subseteq \mathrm{ev}_X^{-1}(U)$. Thus $\mathrm{ev}_X^{-1}(U)$ is open.

These two statement yield the following important

Corollary 1. *Let X, Y be topological spaces. Then we have a bijection*

$$\begin{aligned} P(-) \circ \eta_X \colon \mathscr{Top}(X \times I, Y) &\leftrightarrow \mathscr{Top}(X, PY) \colon \mathrm{ev}_Y \circ (- \times I). \\ H &\mapsto (x \mapsto (t \mapsto H(x,t))) \\ ((x,t) \mapsto h(x)(t)) &\leftarrow h \end{aligned}$$

[1] Note that here we crucially use that I is *(strongly) locally compact*. In fact, many statements we have seen about the path space fail for the space of continuous functions between two arbitrary spaces.

This, or at least an equivalent statement, was mentioned in the first lecture, but I am including it because it will be crucial in the solution of Exercise 2.

The last statement of the exercise can be proven as follows:

c) γ is the composition

$$X \xrightarrow{\eta_X} P(X \times I) \xrightarrow{P(\mathrm{pr}_1 \colon X \times I \to X)} PX$$

and hence continuous.

Exercise 2

Let $X, Y, W, Z, f, g, i, j, r, s$ be as in the exercise. Assume that $u \colon Z \to W$ is a homotopy inverse to $g \colon W \to Z$ witnessed by homotopies $H \colon h \circ g \sim \mathrm{Id}_W$ and $K \colon g \circ h \sim \mathrm{Id}_Z$. Then the composition

$$Y \xrightarrow{j} Z \xrightarrow{u} W \xrightarrow{r} X$$

is a homotopy inverse to $f \colon X \to Y$ witnessed by the homotopies

$$X \times I \xrightarrow{i \times I} W \times I \xrightarrow{H} W \xrightarrow{r} X$$

and

$$Y \times I \xrightarrow{j \times I} Z \times I \xrightarrow{K} Z \xrightarrow{s} Y.$$

Exercise 3

a) The image of the continuous map $\mathrm{ev}_0 \times P(f) \colon PX \to X \times PY$ lies in $P_f \subseteq X \times PY$ and considering it as a map to the subspace P_f we obtain q_f, so q_f is continuous. (Or one can show continuity by calculating that $q_f^{-1}((U \times \mathscr{P}_{C,V}) \cap \mathscr{P}_f) = \mathscr{P}_{\{0\},U} \cap \mathscr{P}_{C,f^{-1}(V)}$.)

b) ("$\Rightarrow$"): Consider the diagram

$$\begin{array}{ccccc} P_p & & \xrightarrow{\mathrm{pr}_1} & & E \\ \downarrow{\scriptstyle P_p \times \{0\}} & & & & \downarrow{\scriptstyle p} \\ P_p \times I & \xrightarrow[\mathrm{pr}_2 \times I]{} & PB \times I & \xrightarrow[\mathrm{ev}_B]{} & B \end{array}$$

which commutes by a straightforward calculation and where the arrows are all continuous by the way they are expressed in terms of continuous maps. Since p is a Hurewicz fibration, there is a lift $\phi \colon P_p \times I \to E$ which makes both triangles that arise in the above diagram commute. Using these commutativity relations, one can check that the continuous map

$$\begin{aligned} P(\phi) \circ \eta_{P_p} \colon P_p &\to PE \\ (x, \mu) &\mapsto (t \mapsto \phi((x, \mu), t)) \end{aligned}$$

(cf. Corollary 1) is indeed a section of q_p.
("$\Leftarrow$"): Let a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\ g\ } & E \\ {\scriptstyle A\times\{0\}}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ A\times I & \xrightarrow[H]{} & B \end{array}$$

be given. Then a straightforward calculation shows that the (continuous!) map $(A, P(H)\circ\eta_A)\colon A \to A \times PE$ (cf. Corollary 1) has its image in P_p which yields a map $u\colon A \to P_p$. Let ψ be given by the composition

$$A \xrightarrow{u} P_p \xrightarrow{s} PE$$

where s is a section of $q_p\colon PE \to P_p$, and set $\phi := \mathrm{ev}_E \circ (\psi \times I)\colon A \times I \to E$. Now, using that $s \circ q_p = \mathrm{Id}_{P_p}$, one can check that ϕ solves the lifting problem above.

c) Finding a section $s\colon P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \to P(P(X))$ is, by Corollary 1, equivalent to finding a map $\hat{s}\colon P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times I \times I \to X$ with certain properties. Indeed, a straightforward calculation shows that such a map yields a section if and only if

$$\hat{s}(\lambda, \mu_0, \mu_1, s, t) = \begin{cases} \lambda(t) & s = 0 \\ \mu_0(s) & t = 0 \\ \mu_1(s) & t = 1 \end{cases}.$$

This means that the value of $\hat{s}$ on $P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times J \subseteq P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times I \times I$, where $J := (\{0\} \times I) \cup (I \times \{0\}) \cup (I \times \{1\})$, is determined, and we need to extend the given function to $P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times I \times I$.
More precisely, we have a cover by finitely many closed sets

$$P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times J = (P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times \{0\} \times I) \cup (P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times I \times \{0\}) \cup (P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times I \times \{1\})$$

and functions

$$\begin{aligned} P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times \{0\} \times I &\to X \\ (\lambda, \mu_0, \mu_1, 0, t) &\mapsto \lambda(t), \end{aligned}$$

$$\begin{aligned} P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times I \times \{0\} &\to X \\ (\lambda, \mu_0, \mu_1, s, 0) &\mapsto \mu_0(s), \end{aligned}$$

$$\begin{aligned} P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times I \times \{1\} &\to X \\ (\lambda, \mu_0, \mu_1, s, 1) &\mapsto \mu_1(s) \end{aligned}$$

which (are all continuous because they are expressed in terms of projections and evaluation maps and) agree on the intersections. Thus, by the gluing lemma, we obtain a continuous map $\widetilde{s}\colon P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times J \to X$.

Now, if we can find a continuous map $r\colon I \times I \to J$ s. t. $r|_J = \mathrm{Id}_J$ and define $\hat{s}$ as the composition

$$P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times I \times I \xrightarrow{P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times r} P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times J \xrightarrow{\tilde{s}} X,$$

then $\hat{s}$ will agree with $\tilde{s}$ on $P_{(\mathrm{ev}_0,\mathrm{ev}_1)} \times J$, so will yield a section as desired. Such a map can be given by

$$r(s,t) = \begin{cases} (0, \frac{s-4t}{2s-4}) & \frac{s}{4} \leqslant t \leqslant 1 - \frac{s}{4} \\ (\frac{s-4t}{1-2t}, 0) & t \leqslant \frac{s}{4} \\ (\frac{s+4t-4}{2t-1}, 1) & t \geqslant 1 - \frac{s}{4} \end{cases}$$

(which can be shown to be continuous by the gluing lemma). “Geometrically”, this can be thought of as sending (s,t) to the unique intersection point of the line which goes through $(2, \frac{1}{2})$ and (s,t) with J. (I wish I had a picture for this.)