Nuclear Power Plants (NPPs)

- **Weeks 1 & 2:** Introduction, nuclear physics basics, fission, nuclear reactors
  - Critical size, nuclear fuel cycles, NPPs *(CROCUS visit)*

- **Week 3:** Neutronics (reactor physics design) + Reactor heat transfer (fuel rod)

- **Week 4:** Reactor heat transfer (cladding, coolant) + Time-dependent reactor behaviour

- **Week 5:** Long-term reactivity changes + Principal types of nuclear power plants

- **Week 6:** Environmental aspects, nuclear safety, advanced systems
Summary, Week 2

- Nuclear fuels: U, Th... only U$^{235}$ fissile; U$^{238}$, Th$^{232}$ fertile (yield fissile Pu$^{239}$, U$^{233}$)
- Distinct characteristics of fissile, fertile nuclides
- Most fissions at low energies in a “thermal” reactor
- $\eta_c$: neutron multiplication factor for fuel (fissile/fertile mixture)
- Four-factor formula for $k_\infty$ ($\eta_c$, $f$, $\varepsilon$, $p$)
- $k_{eff}$, non-leakage probability, critical size
- Nuclear fuel cycles
- Classification of reactors according to power
- Visit to CROCUS
Neutronics (Reactor Physics)… Problems to be Treated

- Neutronics: Bridge between nuclear physics, nuclear engineering
  - Basis for the design of a nuclear reactor core

- If appropriate, energy-averaged $\sigma$’s are available, the neutrons may be considered monoenergetic, e.g. “thermal” neutrons

- Simple basis: neutron diffusion theory (1-group) for the stationary state
  - Neutron balance $\Rightarrow$ Diffusion equation

- Thereafter, neutron slowing down … provides the thermal-neutron “source”

- Finally, multiplying media
  - Steady state… criticality condition, flux (power) distribution
  - Later… kinetic behaviour and reactor control
Successive Scattering Events

- For a neutron beam incident on a slab of scattering material (moderator)
  - Exponential attenuation law for neutrons arriving at B with initial direction ("virgin" n’s)
  - Average distance for the 1st collision:
    \[ \lambda_v = \frac{1}{\Sigma_v} \] (mean free path)

- In practice, actual "flux" is much greater

- What remains valid:
  - The neutrons traverse, without collision, a distance \( x = MM' \) with probability \( e^{-\Sigma_t x} \)
  - \( \Sigma_t \): probability of a collision per cm
  - \[ \bar{\lambda}_t = \lambda_t = \frac{1}{\Sigma_t} \]
Angular Flux

- In practice, the n’s move in all directions
  - Similar to molecules of a gas

- For mono-directional neutrons
  \[ \text{Angular flux } \varphi \left( \Omega \right) \]
  
  n’s are incident on dS with a specific direction
  \[ \varphi \left( \Omega \right) d\Omega = n \left( \Omega \right) v d\Omega \]

  Corresponding reaction rate
  \[ dR = \Sigma \varphi \left( \Omega \right) d\Omega, \text{ where } \Sigma \text{ is independent of } \Omega \]

  \[ R = \int \Sigma \varphi \left( \Omega \right) d\Omega = \Sigma v \int n \left( \Omega \right) d\Omega \]

  Comment 1… A mono-directional beam \( \Rightarrow \) an angular flux
  Comment 2… For obtaining a reaction rate, one needs the scalar flux

Scalar flux
\[ \rho \left( \Omega \right) = n, \text{ total density} \]
Example: Power Density

- Macroscopic effects have to be described in the reactor core, e.g.
  - Heat generation rate at a particular location (power density, or specific power)

- One needs the scalar flux in the fuel, to obtain the thermal power density as:

\[
\omega_{sp} \text{ (W/gm)} = \frac{E_f \text{ (J)}}{\rho_c \text{ (gm/cm}^3)} \cdot \left[ R_f \text{ (cm}^{-3} \text{s}^{-1}) \right]
\]

with \( R_f = \Sigma_f \Phi = N_c \sigma_f \Phi \)

\[
\rightarrow \quad \omega_{sp} = \frac{E_f}{\rho_c} \cdot N_c \sigma_f \Phi
\]
In general, one needs to have the angular flux \( \varphi \), in order to obtain (as integral) the scalar flux \( \Phi \):

- Neutron transport theory needed (solution difficult)

An approximation, which one is often able to apply:

- Diffusion theory
- Scalar flux considered directly... implies simplified treatment of “neutron current”
Neutron Current

- Consider the neutrons moving in direction $\vec{\Omega}$ which traverse $dS$, $\perp$ fixed-axis $OX$

- Angular current, $\vec{J}(\vec{\Omega}) \Rightarrow$ no. crossing $dS$

$$J_x (\vec{r}) = \eta (\vec{r}) \cdot v \cdot \cos \theta = \phi (\vec{r}) \cdot \cos \theta \quad (cm^{-2} \cdot s^{-1})$$

- Total current (from left to right):

$$J_x^+ = \int_{\Omega_x > 0} J_x (\vec{r}) \, d\Omega$$

$$= \int_{\Omega_x > 0} \phi (\vec{r}) \cdot s_x \, d\Omega$$

with $s_x = \cos \theta$

- In the other direction:

$$J_x^- = -\int_{\Omega_x < 0} J_x (\vec{r}) \, d\Omega = -\int_{\Omega_x < 0} \phi (\vec{r}) \cdot s_x \, d\Omega$$
Net Current

Net current traversing \( dS \) (s\(^{-1}\)) is given by

\[
J_x = J_x^+ + J_x^- \quad \text{with both} \quad J_x^+, J_x^- \quad \text{taken to be positive}
\]

Effectively,

\[
\mathbf{J}_x = \int_{4\pi} \varphi(\hat{\mathbf{r}}) \cdot \hat{\mathbf{e}}_x \, d\Omega
\]

- Expresses the neutron balance across \( dS \), does not depend on \( \Omega \) (i.e. is an integral quantity) and has the dimensions of cm\(^2\)s\(^{-1}\) (like scalar flux), but can be either +ive or -ive

Similarly,

\[
J_y = \int_{4\pi} \varphi(\hat{\mathbf{r}}) \cdot \hat{\mathbf{e}}_y \, d\Omega
\]

\[
J_z = \int_{4\pi} \varphi(\hat{\mathbf{r}}) \cdot \hat{\mathbf{e}}_z \, d\Omega
\]

\[
\Rightarrow \mathbf{J} = \mathbf{J}_x \cdot \hat{\mathbf{i}} + \mathbf{J}_y \cdot \hat{\mathbf{j}} + \mathbf{J}_z \cdot \hat{\mathbf{k}}
\]

(\( \mathbf{J} \) is thus a vector)
Neutron Balance Equation

If one considers a volume element $\Delta V$ (with a uniform source, $Q \, n/cm^3.s$)

Leakage + Absorptions = Productions (for steady-state conditions)

$\Delta L + \Sigma_a \Phi \Delta V = Q \Delta V$

where the leakage $\Delta L$ depends on the net currents across the different faces...
Leakage as a Function of Net Current

For the direction OX, net no. of n’s entering from face ABCD = \( J_x(x) \Delta y. \Delta z \)

No. Leaving from A'B'C'D' = \( J_x(x+\Delta x) \Delta y. \Delta z \)

Thus, losses along OX = \( \{ J_x(x+\Delta x) - J_x(x) \} \Delta y. \Delta z \)

Similarly, losses along OY = \( \frac{\partial J_y}{\partial y} \Delta y \Delta V \) and losses along OZ = \( \frac{\partial J_z}{\partial z} \Delta V \)

Thus, \( \Delta L = \left[ \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right] \Delta V = (\text{div} \vec{J}) \Delta V \)

(all 6 faces)
**Neutron Balance Equation**

One obtained

\[ \Delta L + \Sigma_a \Phi \Delta V = Q \Delta V \]

with

\[ \Delta L = (\text{div} \, \vec{J}) \cdot \Delta V \]

Thus,

\[ (\text{div} \, \vec{J}) + \Sigma_a \Phi = Q \]

*(Neutron Balance Equation)*

However, there are 2 unknowns: \( \vec{J} \) and \( \Phi \)

\[ \Rightarrow \] A 2\(^{nd}\) equation is needed…
Fick’s Law

- On the basis of certain assumptions (diffusion theory), e.g.
  \[ \Sigma_a << \Sigma_s \quad (\Sigma_t \approx \Sigma_s) \]

\[
\vec{J} = -D \cdot \text{grad } n = -D \cdot \text{grad } \Phi \quad (\text{Fick’s Law})
\]

with \( D \approx \frac{\lambda}{3} \approx \frac{1}{3\Sigma_t} \) (Diffusion coefficient)

- Similar to other physical laws
  - Heat conduction, mixing of inhomogeneous solutions, etc.
Diffusion Equation

1. Neutron Balance…
   \[ \text{div} \ \vec{J} + \Sigma_a \phi = Q \]

2. Fick’s Law…
   \[ \vec{J} = -D \ \text{grad} \ \phi \]
   \[ \Rightarrow \ \text{div} \left[ D(\vec{r}) \ \text{grad} \ \phi(\vec{r}) \right] - \Sigma_a(\vec{r}) \ \phi(\vec{r}) + Q(\vec{r}) = 0 \]

If the system is homog. \((\Sigma_a, D \rightarrow \text{constants})\),

\[ D \ \Delta \phi - \Sigma_a \phi + Q = 0 \]

\[ \Rightarrow \text{Diffusion Equation} \]
Laplace Equation

\[ \nabla^2 \phi = \text{div} (\text{grad} \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \]

**One-dimensional cases**

- Planar geometry .................................

- Cylindrical geometry (axis of symmetry) ....

- Spherical geometry (centre of symmetry) …
Domain of application of the diffusion equation, very wide
- Describes behaviour of the scalar flux (not just the attenuation of a beam)

Involves partial derivatives of 2\textsuperscript{nd} order (equation type: elliptic)
- Requires provision of an appropriate boundary condition at external surface

For an isolated system (no neutrons entering from the exterior)... \( \Phi(S_e) = 0 \), where \( S_e \) is the “extrapolated” surface
- Distance of extrapolation \( \ll \) physical dimensions \( (d \approx 0.71 \lambda_t) \)
One may write the diffusion equation as:

\[ D \nabla^2 \phi - \Sigma_a \phi + Q = 0 \]

with \( L^2 = \frac{D}{\Sigma_a} \) (diffusion area)

\[ \nabla^2 \phi - \frac{\phi}{L^2} = \frac{Q}{\rho} \]

L\(^2\)… related to the average square of the distance at which neutron is absorbed

i.e. \( L^2 \approx < \rho^2 > \)

Role of L much more important than that of \( \lambda_t \)
Till now, we have discussed the behaviour of monoenergetic neutrons
  • E.g. thermal neutrons, with appropriately averaged cross-sections…

A thermal reactor, however, has n’s between ~ 2 MeV and ~ 0.01 eV
  • One needs to study how the average $E$ changes from ~ 2 MeV to $3/2 \, kT$
  • Slowing down process determines the “thermal -neutron source”

In the case of a fast reactor, there is also slowing down
  • Average $E$ changes from ~ 2 MeV to ~ 100 keV

→ *In any case, one needs to determine the neutron energy spectrum* $\Phi(\vec{r}, E)$
   *for evaluating the different reaction rates.*
Study of an Elastic Collision

- Most important slowing-down mechanism: elastic scattering by moderator nuclei
  - Inelastic scattering also plays a role, but only for fast neutrons (E \geq 1 \text{ MeV})
  - Consider the most common situation
    - Nucleus at rest, of mass \( A \) (rel. to the neutron mass)

\[\begin{align*}
\text{L - System} \\
\text{C - System}
\end{align*}\]

- Advantageous to consider the C- System
  - A single parameter, \( \theta_c \), characterises the collision (instead of 2, in the L - System)
Study of an Elastic Collision (contd.)

CM velocity

For the neutron:
\[ v'_c = v'_n - v_g = \frac{A v'_n}{A+1} \]

For the nucleus:
\[ v'_c = -v'_g = -\frac{v'_n}{A+1} \]

Considering conservation of momentum and energy in the C-System, and then coming back to the L-System:

\[ \frac{E}{E'} = \frac{A^2 + 2A \cos \theta_c + 1}{(A+1)^2} \]
Study of an Elastic Collision (contd.2)

- For $\theta_c = 0$, $E = E'$ (no loss of energy)
- For $\theta_c = \pi$, $E = \alpha E'$ (maximal energy loss) with $\alpha = \left(\frac{A-1}{A+1}\right)^2$

- The energy loss depends on $\theta_c$, but also strongly on $A$
  
  - E.g. For $H^1$, $A = 1$, $\alpha = 0$
    \[ \rightarrow \text{A loss of 100% is possible in a single collision} \]

  For $H^2$, $A = 2$, $\alpha = 1/9$
  \[ \rightarrow \text{Max. loss possible in a single collision } \sim 89\% \]

  etc…
Average Energy Loss

- **Average energy loss:**
  \[
  <\Delta E> = \int_0^\infty (E' - E) \cdot P(E' \rightarrow E) \, dE
  \]
  \[
  = \int_{\xi E'}^{E'} \frac{E' - E}{(1 - \xi) E'} \, dE = \frac{1 - \alpha}{2} \cdot E' \quad \Rightarrow \text{Result depends on energy}
  \]

- **Average logarithmic energy loss:**
  \[
  \xi = <\ln \left( \frac{E'}{E} \right) > = \int_0^\infty \ln \left( \frac{E'}{E} \right) \cdot P(E' \rightarrow E) \, dE
  \]
  \[
  = \int_{\xi E'}^{E'} \ln \left( \frac{E'}{E} \right) \cdot \frac{dE}{(1 - \xi) E'}
  \]
  With
  \[
  \kappa = \frac{E}{E'}, \quad \xi = -\int_\kappa^1 \ln \kappa \cdot \frac{d\kappa}{1 - \kappa} = -\frac{1}{1 - \kappa} \left[ \kappa \ln \kappa - \kappa \right]_\kappa = 1 - \frac{\alpha}{1 - \alpha} \ln \left( \frac{1}{\alpha} \right)
  \]
  With
  \[
  \kappa = \left( \frac{A - 1}{A + 1} \right)^2, \quad \xi = 1 - \frac{(A-1)^2}{2A} \ln \left( \frac{A+1}{A-1} \right)
  \]
  \[
  \Rightarrow \xi \text{ not dependent on energy, only on } A \quad (\text{For } A > 10, \xi \approx \frac{2}{A - \frac{1}{3}})
  \]
Moderator Characteristics

- Average logarithmic energy loss, $\xi$
- Macroscopic Slowing-down Power $\sim \xi \Sigma_s$
- Moderating Ratio $\sim \frac{\xi \Sigma_s}{\Sigma_a}$

*(performance criteria)*
Equation for the Critical Reactor

- One uses the 1-group diffusion equation for the stationary case (critical reactor):
  \[ \nabla^2 \Phi - \Sigma_a \Phi + \mathcal{Q} = 0 \]

- Thereby:
  \[ \mathcal{Q} = \mathcal{Q}_f = \vec{v} \cdot (S_f \Lambda) = \vec{v} \cdot R_f = \vec{v} \cdot \Sigma_f \Phi \]

- Thus,
  \[ \nabla^2 \Phi + (\Sigma_f - \Sigma_a) \Phi = 0 \]

- Can be solved for a given geometry.
- Condition to be satisfied can be identified.

- Simplest systems... homogeneous spherical reactor, slab reactor.

One-group Reactor Equation

\[ \nabla^2 \Phi + \left[ \frac{(k_{\infty} - 1)}{L^2} \right] \Phi = 0 \]

with
\[ B_m^2 = \frac{k_{\infty} - 1}{L^2} \]

Material Buckling (depends only on material properties)
Spherical Reactor

One has

\[ \frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{d\phi}{d\rho} \right) + B_m^2 \phi = 0 \]

with \( B_m^2 = \frac{k_{en}}{\rho} \)

Using

\[ \phi(\rho) = \frac{X(\rho)}{\rho} \]

\[ \frac{d^2X}{d\rho^2} + B_m^2 X = 0 \]

\[ X(\rho) = A \sin B_m \rho + C \cos B_m \rho \]

, i.e.

\[ \phi(\rho) = A \cdot \frac{\sin B_m \rho}{\rho} + C \cdot \frac{\cos B_m \rho}{\rho} \]

For the finite system:

1. \( \Phi \neq \infty \) at \( \rho = 0 \) \( \implies \) \( C = 0 \)

2. \( \Phi = 0 \) at \( \rho = R + d = R + 0.71\lambda_t \) , i.e. \( \sin \{B_m (R + d)\} = 0 \)
Spherical Reactor (contd.)

- From the condition \( \sin \{B_m(R+d)\} = 0 \):
  \[ B_m = B_i = \frac{i\pi}{R + d} \]
  for \( i = 1, 2, \ldots \)

- For the critical reactor, only \( i = 1 \) is valid
  - Smallest eigenvalue
    \( \Rightarrow \) Fundamental Mode

- Other solutions: higher harmonics
  - Exist only in a subcritical system
  - E.g. near the external source

- Critical condition for the spherical reactor is thus:
  \[ B_m^2 = B^2 = \left( \frac{\pi}{R+d} \right)^2 \]

- Geometrical Buckling
  (depends only on system dimensions)
Spherical Reactor (contd.2)

The flux distribution is:

$$\phi(\xi) = \frac{A}{\xi} \sin \left( \frac{\pi \xi}{R+d} \right)$$

For a given medium (specific values of $B_m^2$, $d$), $R$ is determined by the criticality condition:

- Critical radius:
  $$R_c = \left( \frac{\pi}{B_m} \right) - d$$

- Critical mass:
  $$M_c = \frac{4}{3} \pi R_c^3 \rho$$

Conversely, if the size ($R$) is fixed, the material properties need to be identified which yield the appropriate $B_m^2$ (material buckling)...

- E.g. adjust the enrichment, i.e. $k_\infty$
Comments

- For a bare homogeneous reactor, the criticality condition demands an eigenvalue search for the reactor equation: \( \nabla^2 \phi + B_n^2 \phi = 0 \)
  - Eigenvalues need to go to zero at the extrapolated outer surface

- The square of the smallest eigenvalue: \( B^2 \) (geometrical buckling)

- Criticality condition: \( B_m^2 = B^2 \)

- Flux distribution: proportional to the eigenfunction corresponding to \( B^2 \)
  - Absolute level of the flux not yet known
  - A constant (\( A \), in the example considered) remains undetermined, depends on the power (determined, in turn, by technological constraints…)

*Neutronics allows us to determine the criticality condition and the spatial distribution of the flux, but does not fix the latter’s absolute value…*
Slab Reactor

- System infinite in the $X, Y$ directions
- Height $H$
- Flux, function only of $z$

- Reactor Equation: 
  \[
  \frac{d^2\phi}{dz^2} + B^2 \phi = 0
  \]

- General solution: 
  \[
  \phi(z) = A \cos Bz + C \sin Bz
  \]

- Flux, symmetric relative to plane $z = 0$ \( \frac{d\Phi}{dz} = 0 \) at $z = 0$ \( \Rightarrow C = 0 \)
Slab Reactor (contd.)

- Thus, \( \Phi(z) = A \cos Bz \)

- Condition \( \Phi = 0 \) at \( \lfloor \frac{H}{2} + d \rfloor \) yields as eigenvalues:

  \[ \lambda_i = (2i + 1) \cdot \left( \frac{\pi}{2 \left( \frac{H}{2} + d \right)} \right) \]

  with \( i = 0, 1, 2, \ldots \)

- Square of the smallest eigenvalue (\( i = 0 \))

  \[ B^2 = \left( \frac{\pi}{H + 2d} \right)^2 \] (geometrical buckling)

- Flux distribution:

  \[ \Phi(z) = A \cos \left( \frac{\pi z}{H + 2d} \right) \]
Cylindrical Reactor

- For a cylindrical reactor of height $H$ and radius $a$, the critical reactor equation is:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} + B^2 \phi = 0$$

- With the assumption $\phi(r, z) = R(r) \cdot \bar{Z}(z)$,

$$\frac{1}{R} \left[ \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right] + \frac{1}{\bar{Z}} \frac{d^2 \bar{Z}}{dz^2} = -B^2$$

- $f_n.$ of $r$ tends to $-B^2$ and $f_n.$ of $z$ tends to constant $-\alpha^2$. 
Cylindrical Reactor (contd.)

Thus,

\[
\frac{d^2Z}{dz^2} + \alpha^2 Z = 0
\]

...(1)

\[
\frac{d^2R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \beta^2 R = 0
\]

...(2)

\[\begin{align*}
\alpha^2 & \rightarrow \text{axial buckling} \\
\beta^2 & \rightarrow \text{radial buckling}
\end{align*}\]

From (1), \( Z(z) = A' \cos \alpha z \), and with \( \phi = 0 \) at \( z_1 = \frac{H}{2} + d \),

\[\alpha = \frac{\pi}{H+2d}\]

(smallst eigenvalue)

From (2), \( R(r) = A'' [J_0(\beta r)] \) (not applicable... \( \beta \neq -\infty \)

and with \( \phi = 0 \) at \( r = a+d \),

\[\beta = \frac{2.405}{a+d}\]

(smallst eigenvalue)
Comments on Criticality Condition - 1

(a) Rewriting above equation: \[ k_\infty = 1 + L^2 B^2 \quad \ldots \ (1) \]

Previously, one had: \[ k_{\text{eff}} = k_\infty \cdot P_{\text{NF}} - \frac{\text{Abs.}}{\text{Abs. + Leakage}} \quad \ldots \ (2) \]

From (1) (criticality condition): \[ k_{\text{eff}} = \frac{k_\infty}{1 + L^2 B^2} = 1 \quad \ldots \ (3) \]

Comparing (1), (3): \[ P_{\text{NF}} = \frac{1}{1 + L^2 B^2} \quad \text{(Non-leakage Probability)} \]
Comments on Criticality Condition - 2

- Assumption of simple 1-group equation: all neutrons are “born” thermal
  - $L^2$: diffusion area… measure of displacement of neutrons during diffusion as slow neutrons

- Displacement of neutrons also occurs during slowing down
  - Corresponding “area”: $\tau_{th}$, slowing down area

- Better measure of total displacement given by “migration area”, $M^2$
  - $M^2 = L^2 + \tau_{th}$

- One may then write the modified 1-group reactor equation:

$$\nabla^2 \phi_{th} + \left[ \frac{k_{\infty} - 1}{M^2} \right] \phi_{th} = 0$$

with the critical condition:

$$B_m^2 = \frac{k_{\infty} - 1}{M^2} = \beta^2$$

- Non-leakage probability given by:

$$P_{NF} \approx \frac{1}{1 + M^2 B^2}$$
Ex. 5

- **Find a relationship between the coefficient** $A$, **which determines the absolute value of the neutronic flux**, and the **thermal power** $P$ of a reactor of the following shape:
  
  (a) spherical,
  
  (b) cylindrical,
  
  (c) parallelepiped.

  *You may neglect the extrapolation distance.*
Ex. 5… Solution

(a) For the spherical reactor, neglecting $d$:

$$\phi(r) \approx \frac{A \cdot \sin \left( \frac{\pi \cdot r}{R} \right)}{r}$$

with $R \sim R_e = R + d$

The constant $A$ and the power $P$ are related by:

$$P = \int_0^R E_f \cdot \Sigma_f \cdot \Phi(r) \cdot 4\pi \cdot r^2 \cdot dr$$

where $E_f$ is the energy released per fission (in joules) and $\Sigma_f$ is the macroscopic fission cross-section of the multiplying medium.

Thus,

$$P = 4\pi \cdot A \cdot E_f \cdot \Sigma_f \cdot \left[ \frac{\rho^2 \sin \left( \frac{\pi \cdot \rho}{R} \right)}{\rho} \right] \cdot dr = 4\pi \cdot A \cdot E_f \cdot \Sigma_f \cdot \frac{R^2}{\pi}$$

$$\Rightarrow \quad A = \frac{P}{4 \cdot R^2 \cdot E_f \cdot \Sigma_f}$$
(b) For the cylindrical reactor (radius $R$, height $H$), in an analogous manner:

$$\Phi(r,z) \approx A \cdot J_0\left(\frac{2.405 \cdot r}{R}\right) \cdot \cos\left(\frac{\pi \cdot z}{H}\right)$$

and

$$P = \int_0^R \int_{-\frac{H}{2}}^{+\frac{H}{2}} E_f \cdot \Sigma_f \cdot \Phi(r,z) \cdot (2\pi \cdot r \cdot dr) \cdot dz$$

Assuming separation of variables and carrying out the double integration,

$$A = \frac{2.405 \cdot \pi \cdot P}{4 \cdot V \cdot E_f \cdot \Sigma_f \cdot J_1(2.405)} = \frac{3.64 \cdot P}{V \cdot E_f \cdot \Sigma_f}$$
Ex. 5... Solution (contd.2)

(a) For the parallelepiped reactor of dimensions $a$, $b$, $c$:

$$\Phi(x, y, z) \equiv A \cdot \cos\left(\frac{\pi \cdot x}{a}\right) \cdot \cos\left(\frac{\pi \cdot y}{b}\right) \cdot \cos\left(\frac{\pi \cdot z}{c}\right)$$

and

$$P = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} E_f \cdot \Sigma_f \cdot \Phi(x, y, z) \cdot dx \cdot dy \cdot dz$$

Assuming separation of variables and carrying out the triple integration,

$$A = \frac{\pi^3 \cdot P}{8 \cdot V \cdot E_f \cdot \Sigma_f} \equiv \frac{3.88 \cdot P}{V \cdot E_f \cdot \Sigma_f}$$
Ex. 6

- Calculate the values of $\Phi_{\text{max}} / \Phi$ for the bare homogeneous reactors of Ex. 5.
Ex. 6... Solution

For a bare homogeneous reactor, the maximum value of the flux $\Phi_{\text{max}}$ is to be found at the centre of the system. The average flux value $\overline{\Phi}$ corresponds to:

$$\overline{\Phi} \approx \frac{1}{V} \int_{V} \Phi \cdot dV = \frac{P}{V \cdot E_f \cdot \Sigma_f}$$

where $V$ is the volume of the reactor.

Thus, for each of the 3 geometries in Ex. 5,

$$\frac{\Phi_{\text{max}}}{\overline{\Phi}} = \frac{\text{Flux value at centre}}{P / (V \cdot E_f \cdot \Sigma_f)}$$

(a) For the spherical reactor,

$$\frac{\Phi_{\text{max}}}{\overline{\Phi}} = \lim_{\rho \to 0} \frac{A \cdot \sin \left( \frac{\pi \cdot \rho}{R} \right)}{P / (V \cdot E_f \cdot \Sigma_f)} = \frac{A \cdot \left( \frac{\pi}{R} \right)}{P \left( \frac{4 \cdot R^2 \cdot E_f \cdot \Sigma_f}{V \cdot E_f \cdot \Sigma_f} \right)} = \frac{\pi}{3} \approx 3.29$$
Ex. 6… Solution (contd.)

(b) For the cylindrical reactor,

\[
\frac{\Phi_{\text{max}}}{\Phi} = \lim_{r \to 0, \ z \to 0} \left[ A \cdot J_0\left(\frac{2.405 \cdot r}{R}\right) \cdot \cos\left(\frac{\pi \cdot z}{H}\right) \right] = \frac{A}{P/(V \cdot E_f \cdot \Sigma_f)} = \frac{3.64 \cdot P}{V \cdot E_f \cdot \Sigma_f} = 3.64
\]

(c) For the parallelepiped reactor of dimensions \(a\), \(b\), \(c\):

\[
\frac{\Phi_{\text{max}}}{\Phi} = \lim_{x \to 0, \ y \to 0, \ z \to 0} \left[ A \cdot \cos\left(\frac{\pi \cdot x}{a}\right) \cdot \cos\left(\frac{\pi \cdot y}{b}\right) \cdot \cos\left(\frac{\pi \cdot z}{c}\right) \right] = \frac{A}{P/(V \cdot E_f \cdot \Sigma_f)}
\]

\[
= \frac{\pi^3 \cdot P}{8 \cdot V \cdot E_f \cdot \Sigma_f} = \frac{\pi^3}{8} \approx 3.88
\]
A bare spherical reactor of 50 cm radius is operated at a thermal power 100 MW. If $\Sigma_f = 0.0047 \text{ cm}^{-1}$, what are the maximum and average values of the neutron flux in the reactor?

(Take $E_f = 3.2 \cdot 10^{-11} \text{ J/fission}$)

Note: This exercise concerns the simplified modelling of a high-flux research reactor (small, high-leakage core).
One has:
\[
\Phi_{\text{max}} = \frac{\pi \cdot P}{4 \cdot R^3 \cdot E_f \cdot \Sigma_f} = \frac{\pi \cdot 10^8}{4 \cdot (50)^3 \cdot (3.2 \cdot 10^{-11}) \cdot (0.0047)} = 4.18 \cdot 10^{15} \text{ n cm}^{-2} \cdot \text{s}^{-1}
\]

and
\[
\overline{\Phi} \approx \frac{\Phi_{\text{max}}}{3.29} = 1.27 \cdot 10^{15} \text{ n cm}^{-2} \cdot \text{s}^{-1}
\]
Reactor Core Thermalhydraulics

- One has seen that neutronics...
  - Provides the condition for reactor criticality
  - Allows determination of power distributions in the reactor core
  - Does not, however, fix the neutron flux level, i.e. the absolute power of the NPP

- It is the technological limits which do that...
  - Melting points, thermal stresses, etc. at “hot spot” (most demanding for materials)

- Analysis needed of “hottest cell” (that with maximum power density)

- Fixes all absolute values, for a given margin of safety…
Radial temperature distribution within fuel rod

- Thermal conduction in fuel determined by
  - Fourier’s Law: \( \vec{\psi}(\vec{r}) = -\lambda \nabla T(\vec{r}) \) \( \quad \ldots (1) \)
  - Thermal Balance: \( \text{div} \vec{\psi} = \omega_c \) \( \quad \ldots (2) \)

- From (1) and (2),
  - Steady-state Thermal Conduction Equation: \( \lambda \nabla^2 T + \omega_c = 0 \) \( \quad \ldots (3) \)

Analogy with neutronics
- (1) is like Fick’s Law, (2) like Neutron Balance, (3) like Diffusion Equation
Cylindrical Fuel Rod

- Most usual case… “thin” rods, with axial conduction ≈ 0 \( (dT/dz \approx 0) \)
  - One obtains:
    \[
    \lambda \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) \right] + \bar{\omega}_c = 0 \quad \ldots (4)
    \]
  - Assuming \( \omega_c \) to be constant at a given height \( z \) (\( \Sigma_f \Phi \) does not depend on \( r \) in rod), one obtains, by integration of (4):
    \[
    r \frac{dT}{dr} = -\frac{\bar{\omega}_c r^2}{2\lambda} + A
    \]
    \[
    \Rightarrow \quad T(r) = -\frac{\bar{\omega}_c r^2}{4\lambda} + A \ln r + B
    \]
  - \( T \) is finite at \( r = 0 \), which makes \( A = 0 \)
  - At \( r = a \) (periphery), \( T = T(a) \)
    \[
    \Rightarrow \quad B = T(a) + \frac{\bar{\omega}_c a^2}{4\lambda}
    \]
  - Thus,
    \[
    \lambda \left[ T(r) - T(a) \right] = \frac{\bar{\omega}_c}{4} (a^2 - r^2)
    \]
    \[
    \ldots (5)
    \]
Cylindrical Fuel Rod (contd.)

- Maximum temperature \((r = 0)\):
  \[
  T_0 - T_g = \frac{\bar{w}_e a^2}{4\lambda} = \frac{\bar{w}_e}{4\pi\lambda} \tag{6}
  \]
  with linear power \((W/cm)\)

- Analogy with Ohm’s Law
  - \(T\) is like potential difference, \(\bar{w}_e\) like current, \(\frac{1}{4\pi\lambda}\) like resistance

- In reality, \(T_g < T(a)\) because of:
  - Rapid drop at interface between fuel and cladding (imperfect contact / gas gap)
  - Finite thermal conductivity of cladding
Cladding/Coolant Heat Exchange

- For the liquid coolant, one has the Convection Law:
  \[ \psi(a') = h(T_g - T_c) \quad \ldots (7) \]
  \( a' = a + b \)  
  (in the book, one assumes \( a' = a \))

- \( h \) depends on properties of the liquid, geometry of the “cell”, as also the flow
  - One has empirical correlations (forced convection)

- In steady state,
  \[ 2\pi a \cdot \psi(a) = \bar{w} \quad \ldots (8) \]
  heat exchanged  
  heat produced

- From (7) and (8),
  \[ T_g - T_c = \frac{\bar{w}}{2\pi a h} \quad \ldots (9) \]
  Analogy with Ohm’s Law:
  is thermal “resistance” of coolant
Empirical Relations for $h$

For $h$, an often used correlation for single-phase water is that of Coburn:

$$h = 0.02 \frac{\lambda}{D_H} \cdot \left( \frac{G}{D_H} \right)^{0.8} \cdot \left( \frac{C_p \mu}{\lambda} \right)^{0.4}$$

- **$G$** is the unitary mass flow rate
- **$D_H$**, the hydraulic diameter
- **$\lambda$, $\mu$, $C_p$** ≈ constants for a given liquid (at some average temperature)
- Total mass flow
- Flow cross-section
- Wetted surface area

Effectively,

$$h \sim a \cdot \frac{W^{0.8}}{V_m}$$
Conductivity Integral

- One has $T_0 > T_g > T_c$ ... The first difference is much greater than the second
  - Conclusions regarding the fuel do not depend much on the coolant

- While considering $T_0 - T_g = \frac{\bar{\omega}_2}{4\pi \lambda}$ ... (6), $\lambda$ was assumed constant
  - Not so in reality, e.g. for UO$_2$...

The more general form of Eq. (6) is:

$$\overline{I}(T_0) = \int_{T_g}^{T_0} \lambda(\tau) \, d\tau = \frac{\bar{\omega}_2}{4\pi}$$  ... (10)
Characteristics of UO$_2$, UC

<table>
<thead>
<tr>
<th>Fuel</th>
<th>$T_{\text{melt}}$ (°C)</th>
<th>$\lambda_{\text{avg}}$ (W/cm°C)</th>
<th>$I_M$ (W/cm)</th>
<th>$\bar{w}_I = 4\pi I_M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UO$_2$</td>
<td>2700</td>
<td>0.025</td>
<td>55</td>
<td>690</td>
</tr>
<tr>
<td>UC</td>
<td>2400</td>
<td>0.17</td>
<td>330</td>
<td>4100</td>
</tr>
</tbody>
</table>

- $I_M$ calculated above as $I_M = \lambda_{\text{avg}} (T_{\text{melt}} - T_g)$, with $T_g = 480°C$ (arbitrary)
- For steel as cladding, $(T_g)_{\text{max}} \approx 620°C$ → $I_M$ is reduced only by about 7%
  - Rather weak dependence... uncertainties, and hence safety margin needed, much larger
  - In practice, e.g. for UO$_2$, $\bar{w}_I < 450 \text{ W/cm}$
- UO$_2$ ... most common fuel material (LWRs: UO$_2$, PuO$_2$/UO$_2$)
- UC ... $\omega_I$ can be much higher, but UC not compatible with H$_2$O (chemical reaction)
- $U_{\text{met}}$ ... $\lambda$ much higher, but $I_M$ low because of limit on $(T_0)_{\text{max}}$ ... phase change $\approx 600°C$
Technological Limit for the Fuel

- $T_g$ is often known... in any case, $T_0$ and $\lambda$ determine $I$, and hence $\bar{\omega}_I$.
- $(T_0)_{\text{max}}$ is limited (melting temp., thermal stresses imposed by temp. gradient, etc.)
- The corresponding limit for $I(I_M)$ gives a 1st technological constraint for $(\bar{\omega}_c)_{\text{max}}$

$$Limiting \ the \ linear \ power \ at \ the \ "hot \ spot", \ in \ turn, \ limits \ the \ reactor \ power$$

- Above formulation $(I_M)$, very useful...
  - Does not depend on fuel rod geometry
  - Two different fuels (λ, $T_{\text{melt}}$, ...) may have the same $I_M$, and hence the same limit
  - For a given fuel rod,

$$\bar{\omega}_M (W/gm) = \frac{\bar{\omega}_c}{\rho_c} = \frac{\bar{\omega}_c}{\pi a^2 \rho_c} = \frac{4I}{a^2 \rho_c}$$

> indicates fuel mass needed for a given total power
Ex. 8

- The core of a Pressurized Water Reactor (PWR) operates at a thermal power of 3100 MW. It is made up of 203 fuel assemblies, each consisting of 193 individual fuel rods. These rods are of 1.05 cm diameter and 360 cm length, and contain UO$_2$ pellets of 0.92 cm diameter.

Determine:

(a) $\bar{\omega}_e$, the average linear power

(b) $\bar{\Psi}$, the average heat flux on the surface of the fuel rods

(c) $\bar{\omega}_c$, the average power density in the fuel
Ex. 8... Solution

(a) Total length of fuel rods = 193.203 \cdot (3.60\text{m}) = 141044 \text{ m}

Thus,
\[ \bar{\omega}_c = \frac{3100 \cdot 1000}{141044} = 22 \text{ kW/m} \]

(b) \[ \bar{\Psi} = \frac{\bar{\omega}_c}{(100 \text{ cm/m}) \cdot 2\pi \cdot a'} = \frac{22000}{100 \cdot \pi \cdot 1.05} = 66.7 \text{ W/cm}^2 \text{ (2a': diam. of rod)} \]

(c) \[ \bar{\omega}_c = \frac{\bar{\omega}_c}{(100 \text{ cm/m}) \cdot \pi \cdot a^2} = \frac{22000}{100 \cdot \pi \cdot (0.92)^2} = 331 \text{ W/cm}^3 \text{ (2a: diam. of pellet)} \]
Ex. 9

A fuel rod in a PWR has a linear power of 51 kW/m. The UO$_2$ pellets have a diameter of 0.964 cm and a thermal conductivity value of 0.028 W/cm°C. Determine the temperature difference between the centre and the outer surface of the fuel rod, assuming that the cladding material is a perfect conductor (zero thermal resistance) and that there is no gap between the fuel and the clad.
Ex. 9... Solution

The temperature difference across the UO$_2$ is given by:

$$T_0 - T_g = \frac{\bar{\omega}_c \cdot a^2}{4 \cdot \lambda_c} = \frac{\bar{\omega}_p}{4 \cdot \pi \cdot \lambda_c} = \frac{(51000 \text{ W/100 cm})}{4 \cdot \pi \cdot (0.028 \text{ W/cm}^\circ \text{C})} = 1450^\circ \text{C}$$

This value corresponds to the temperature difference across the rod, because of the assumptions that have been made regarding the cladding material and the absence of a gas gap between the clad and the pellet.

N.B.: In reality, the temperature difference across the rod would be several hundreds of degrees higher, due to the thermal resistance of the cladding and, even more important, of the gas gap.
Summary, Week 3

- Neutron propagation, angular flux, scalar flux, neutron current
- Neutron balance for a volume element; Fick’s Law; Diffusion Equation
- Slowing down, moderator characteristics ($\xi$, etc.)
- Multiplying media, multiplication factors; One-group Reactor Equation
- Material and geometrical buckling; Bare homogeneous reactors (sphere, slab, cylinder, etc.)
- Absolute flux and reactor power; Maximum- to-average flux ratio

- Technological (heat transfer) constraints on maximum power
- Thermal conduction in fuel rod
  - Temperature much higher at centre
  - 1st constraint: “conductivity integral” of fuel