1) The “Ket” and the associated Dirac or usual vector notations are:

- \(|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(\langle H | = (1 \ 0)\)
- \(|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\) and \(\langle H | = (0 \ 1)\)
- \(\alpha |H\rangle + \beta |V\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\) and \(\alpha^* \langle H | + \beta^* \langle V | = (\alpha^* \ \beta^*)\)

2) In Dirac notation:

\[
\begin{pmatrix} \gamma^* \langle H | + \delta^* \langle V | \end{pmatrix} \begin{pmatrix} \alpha |H\rangle + \beta |V\rangle \\
\gamma^* \alpha \langle H|H\rangle + \gamma^* \beta \langle H|V\rangle + \delta^* \alpha \langle V|H\rangle + \delta^* \beta \langle V|V\rangle = \gamma^* \alpha + \delta^* \beta
\]

because \(\langle H|V\rangle = \langle V|H\rangle = 0\) and \(\langle H|H\rangle = \langle V|V\rangle = 1\).

The equivalent vector notation is

\[
\begin{pmatrix} \gamma^* & \delta^* \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \gamma^* \alpha + \delta^* \beta.
\]

3) We have \(R^\dagger = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\) and \(R^\dagger = R^\dagger^* = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\). Thus

\[
RR^\dagger = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
R^\dagger R = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Matrices satisfying \(MM^\dagger = M^\dagger M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) are called unitary matrices.

Let us compute \(R (\alpha |H\rangle + \beta |V\rangle)\) in Dirac notation. By linearity of matrix operations,

\[
R (\alpha |H\rangle + \beta |V\rangle) = \alpha R |H\rangle + \beta R |V\rangle = \alpha i |V\rangle + \beta i |H\rangle = i (\alpha |V\rangle + \beta |H\rangle).
\]
\[ \alpha |H\rangle + \beta |V\rangle \] incoming
\[ i(\beta |H\rangle + \alpha |V\rangle) \] outgoing

4) We have
\[
S|H\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} (|H\rangle + i|V\rangle),
\]
\[
S|V\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (i|H\rangle + |V\rangle),
\]
\[
S(\alpha|H\rangle + \beta|V\rangle) = \alpha S|H\rangle + \beta S|V\rangle
\]
\[
= \frac{\alpha}{\sqrt{2}} (|H\rangle + i|V\rangle) + \frac{\beta}{\sqrt{2}} (i|H\rangle + |V\rangle)
\]
\[
= \frac{\alpha + i\beta}{\sqrt{2}} |H\rangle + \frac{i\alpha + \beta}{\sqrt{2}} |V\rangle.
\]

Refer to question 5 for the picture of the operations.

5) The semi-transparent mirror prepares photons in state \( S(\alpha|H\rangle + \beta|V\rangle) \). It is then measured by the detector \( D \) which detects photons in state \( |V\rangle \). Therefore, the probability of finding a photon in \( D \) is the probability of finding a photon in state \( |V\rangle \) given that photons in state \( S(\alpha|H\rangle + \beta|V\rangle) \) are produced. By measurement postulate (which will be formally introduced in Chapter 3 of the lecture note) we have
\[
\text{Prob}(D) = |\langle V | S(\alpha|H\rangle + \beta|V\rangle) |^2.
\]

From the previous question we have
\[
S(\alpha|H\rangle + \beta|V\rangle) = \frac{\alpha + i\beta}{\sqrt{2}} |H\rangle + \frac{i\alpha + \beta}{\sqrt{2}} |V\rangle,
\]
\[
\langle V | S(\alpha|H\rangle + \beta|V\rangle) = \frac{\alpha + i\beta}{\sqrt{2}} \langle V | H\rangle + \frac{i\alpha + \beta}{\sqrt{2}} \langle V | V\rangle
\]
\[
= \frac{i\alpha + \beta}{\sqrt{2}}.
\]

So we find
\[
\text{Prob}(D) = \left| \frac{i\alpha + \beta}{\sqrt{2}} \right|^2 = \frac{1}{2} |i\alpha + \beta|^2 = \frac{1}{2} (\alpha^2 + \beta^2) = \frac{1}{2}.
\]
6) The state after $S$ is

$$ S |H\rangle = \frac{1}{\sqrt{2}} (|H\rangle + i |V\rangle) $$

The state after $R$ is

$$ RS |H\rangle = \frac{1}{\sqrt{2}} (R |H\rangle + iR |V\rangle) $$

$$ = \frac{i}{\sqrt{2}} (|V\rangle + i |H\rangle). $$

The state after the second $S$ is

$$ SRS |H\rangle = \frac{i}{\sqrt{2}} (S |V\rangle + iS |H\rangle) $$

$$ = \frac{i}{\sqrt{2}} \left( \frac{i |H\rangle + |V\rangle}{\sqrt{2}} + i \cdot \frac{|H\rangle + i |V\rangle}{\sqrt{2}} \right) $$

$$ = - |H\rangle. $$

Thus

$$ \text{Prob}(D_1) = |\langle V| H \rangle|^2 = 0 $$

$$ \text{Prob}(D_2) = |\langle H| H \rangle|^2 = 1. $$

All photons so in detection $D_2$! For “classical balls” we would expect a split between $D_1$ and $D_2$. For example, if $S$ act as half–half splitters we would expect $\text{Prob}(D_1) = \text{Prob}(D_2) = 1/2$. The quantum behavior is completely different!