ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 15 Information Theory and Coding
Midterm exam Oct. 30, 2018

4 problems, 76 points
165 minutes
1 sheet (2 pages) of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

PROBLEM 1. (12 points) Recall that for a code $\mathcal{C} : \mathcal{U} \to \{0,1\}^*$, we define $\mathcal{C}^n : \mathcal{U}^n \to \{0,1\}^*$ as $\mathcal{C}^n(u_1 \dots u_n) = \mathcal{C}(u_1)\dots\mathcal{C}(u_n)$.

(a) (4 pts) Show that if $\mathcal{C}$ is uniquely decodable, then for all $n \geq 1$, $\mathcal{C}^n$ is injective.

(b) (4 pts) Suppose $\mathcal{C}$ is not uniquely decodable. Show that there are u^n and v^m such that $u_1 \neq v_1$ and $\mathcal{C}^n(u^n) = \mathcal{C}^m(v^m)$.

(c) (4 pts) Suppose $\mathcal{C}$ is not uniquely decodable. Show that there is a k such that $\mathcal{C}^k$ is not injective. [Hint: try $k = n + m$.]

Problem 2. (12 points) Suppose $X_1, \ldots, X_n$ are random variables. Let

$$Y_i = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$$

denote the collection which includes all the X's, except X_i.

(a) (4 pts) Show that $\sum_{i=1}^{n} H(X_i|Y_i) \leq H(X^n)$.

(b) (4 pts) Show that $\sum_{i=1}^{n} H(Y_i) \geq (n-1)H(X^n)$.

(c) (4 pts) What are the conditions for equality to hold in the parts above?

PROBLEM 3. (20 points) Suppose $X_1, X_2, \ldots$ is a stochastic process with $X_i \in \{1,2,3,4\}$. The process is Markov, i.e., $\Pr(X_{n+1} = x_{n+1} | X^n = x^n) = \Pr(X_{n+1} = x_{n+1} | X_n = x_n)$, and $\Pr(X_{n+1} = j | X_n = i)$ is found as the (i,j) entry of the matrix

$$P = \begin{bmatrix} 1-\alpha & \alpha & & \\ \alpha & 1-\alpha & & \\ & & 1/2 & 1/2 \\ & & 1/2 & 1/2 \end{bmatrix}.$$

The initial state of the process X_1 is chosen according to the distribution

$$\Pr(X_1 = 1) = p, \quad \Pr(X_1 = 4) = 1 - p,$$

with $0 < p < 1$. Note that the structure of the matrix P ensures that if $X_1 = 1$, then $X_n \in \{1,2\}$ for all n, and if $X_1 = 4$ then $X_n \in \{3,4\}$ for all n. Consequently, $\Pr(X_n \in \{1,2\}) = p$ and $\Pr(X_n \in \{3,4\}) = 1 - p$.

(a) (4 pts) Is the process stationary? (Not just 'yes' or 'no', explain your answer.)

(b) (4 pts) For $n \geq 1$, find $h_i = H(X_{n+1} | X_n = i)$ for $i = 1,2,3,4$. Does your answer depend on n?

(c) (4 pts) Find $a_n = H(X_n | X^{n-1})$, $n = 1, 2, \ldots$.

(d) (4 pts) Find $b_n = H(X^n)/n$, $n = 1, 2, \ldots$.

(e) (4 pts) Does the entropy rate $H = \lim_n b_n$ exist? If so, what is H?

Problem 4. (32 points) Suppose U is a random variable taking values in $\{1, 2, \dots\}$. Set $L = \lfloor \log_2 U \rfloor$, that is:

u	1	2	3	4	5	6	7	8	9	$\dots$	16	$\dots$	32	$\dots$
l	0	1	1	2	2	2	2	3	3	$\dots$	4	$\dots$	5	$\dots$

(a) (4 pts) Show that $H(U|L = j) \leq j$, $j = 0, 1, \dots$.

(b) (4 pts) Show that $H(U|L) \leq E[L]$.

(c) (4 pts) Show that $H(U) \leq E[L] + H(L)$.

(d) (4 pts) Suppose that $\Pr(U = 1) \geq \Pr(U = 2) \geq \dots$. Show that $1 \geq i \Pr(U = i)$.

(e) (4 pts) With U as in (d), and using the result of (d), show that $E[\log_2 U] \leq H(U)$ and conclude that $E[L] \leq H(U)$.

(f) (8 pts) Suppose that N is a random variable taking values in $\{0, 1, \dots\}$ with distribution p_N and $E[N] = \mu$. Let G be a geometric random variable with mean μ, i.e., $p_G(n) = \mu^n/(1 + \mu)^{1+n}$, $n \geq 0$.

Show that $H(G) - H(N) = D(p_N \| p_G) \geq 0$, and conclude that $H(N) \leq g(\mu)$ with $g(x) = (1 + x) \log(1 + x) - x \log x$.

[Hint: Let $f(n, \mu) = -\log p_G(n) = (n + 1) \log(1 + \mu) - n \log(\mu)$. First show that $E[f(G, \mu)] = E[f(N, \mu)]$, and consequently $H(G) = \sum_n p_N(n) \log(1/p_G(n))$.]

(g) (4 pts) Show that for U as in (d) and $g(x)$ as in (f),

$$E[L] \geq H(U) - g(H(U)).$$

[Hint: combine (f), (e), (c).]