4 problems, 76 points
165 minutes
1 sheet (2 pages) of notes allowed.

Good Luck!

Please write your name on each sheet of your answers.

Please write the solution of each problem on a separate sheet.
Problem 1. (12 points) Recall that for a code \( C : \mathcal{U} \rightarrow \{0, 1\}^* \), we define \( C^n : \mathcal{U}^n \rightarrow \{0, 1\}^* \) as \( C^n(u_1 \ldots u_n) = C(u_1) \ldots C(u_n) \).

(a) (4 pts) Show that if \( C \) is uniquely decodable, then for all \( n \geq 1 \), \( C^n \) is injective.

(b) (4 pts) Suppose \( C \) is not uniquely decodable. Show that there are \( u^n \) and \( v^m \) such that \( u_1 \neq v_1 \) and \( C^n(u^n) = C^m(v^m) \).

(c) (4 pts) Suppose \( C \) is not uniquely decodable. Show that there is a \( k \) such that \( C^k \) is not injective. [Hint: try \( k = n + m \).]
Problem 2. (12 points) Suppose $X_1, \ldots, X_n$ are random variables. Let

$$Y_i = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$$

denote the collection which includes all the $X$’s, except $X_i$.

(a) (4 pts) Show that $\sum_{i=1}^{n} H(X_i | Y_i) \leq H(X^n)$.

(b) (4 pts) Show that $\sum_{i=1}^{n} H(Y_i) \geq (n-1)H(X^n)$.

(c) (4 pts) What are the conditions for equality to hold in the parts above?
Problem 3. (20 points) Suppose $X_1, X_2, \ldots$ is a stochastic process with $X_i \in \{1, 2, 3, 4\}$. The process is Markov, i.e., $\Pr(X_{n+1} = x_{n+1} | X^n = x^n) = \Pr(X_{n+1} = x_{n+1} | X_n = x_n)$, and $\Pr(X_{n+1} = j | X_n = i)$ is found as the $(i, j)$ entry of the matrix

$$
P = \begin{bmatrix}
1 - \alpha & \alpha & 1/2 & 1/2 \\
\alpha & 1 - \alpha & 1/2 & 1/2 \\
1/2 & 1/2 & 1 & 0 \\
1/2 & 1/2 & 0 & 1
\end{bmatrix}.
$$

The initial state of the process $X_1$ is chosen according to the distribution $\Pr(X_1 = 1) = p, \ Pr(X_1 = 4) = 1 - p$, with $0 < p < 1$. Note that the structure of the matrix $P$ ensures that if $X_1 = 1$, then $X_n \in \{1, 2\}$ for all $n$, and if $X_1 = 4$ then $X_n \in \{3, 4\}$ for all $n$. Consequently, $\Pr(X_n \in \{1, 2\}) = p$ and $\Pr(X_n \in \{3, 4\}) = 1 - p$.

(a) (4 pts) Is the process stationary? (Not just ‘yes’ or ‘no’, explain your answer.)

(b) (4 pts) For $n \geq 1$, find $h_i = H(X_{n+1} | X_n = i)$ for $i = 1, 2, 3, 4$. Does your answer depend on $n$?

(c) (4 pts) Find $a_n = H(X_n | X^{n-1})$, $n = 1, 2, \ldots$.

(d) (4 pts) Find $b_n = H(X^n) / n$, $n = 1, 2, \ldots$.

(e) (4 pts) Does the entropy rate $H = \lim_n b_n$ exist? If so, what is $H$?
Problem 4. (32 points) Suppose $U$ is a random variable taking values in $\{1, 2, \ldots \}$. Set $L = \lfloor \log_2 U \rfloor$, that is:

<table>
<thead>
<tr>
<th>$u$</th>
<th>1 2 3 4 5 6 7 8 9 ... 16 ... 32 ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l$</td>
<td>0 1 1 2 2 2 3 3 ... 4 ... 5 ...</td>
</tr>
</tbody>
</table>

(a) (4 pts) Show that $H(U|L = j) \leq j$, $j = 0, 1, \ldots$.

(b) (4 pts) Show that $H(U|L) \leq E[L]$.

(c) (4 pts) Show that $H(U) \leq E[L] + H(L)$.

(d) (4 pts) Suppose that $\Pr(U = 1) \geq \Pr(U = 2) \geq \ldots$. Show that $1 \geq i \Pr(U = i)$.

(e) (4 pts) With $U$ as in (d), and using the result of (d), show that $E[\log_2 U] \leq H(U)$ and conclude that $E[L] \leq H(U)$.

(f) (8 pts) Suppose that $N$ is a random variable taking values in $\{0, 1, \ldots \}$ with distribution $p_N$ and $E[N] = \mu$. Let $G$ be a geometric random variable with mean $\mu$, i.e., $p_G(n) = \mu^n/(1 + \mu)^{n+1}$, $n \geq 0$.

Show that $H(G) - H(N) = D(p_N||p_G) \geq 0$, and conclude that $H(N) \leq g(\mu)$ with $g(x) = (1 + x) \log(1 + x) - x \log x$.

[Hint: Let $f(n, \mu) = -\log p_G(n) = (n + 1) \log(1 + \mu) - n \log(\mu)$. First show that $E[f(G, \mu)] = E[f(N, \mu)]$, and consequently $H(G) = \sum_n p_N(n) \log(1/p_G(n))$.]

(g) (4 pts) Show that for $U$ as in (d) and $g(x)$ as in (f),

$E[L] \geq H(U) - g(H(U))$.

[Hint: combine (f), (e), (c).]