

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 13

Homework 6

Information Theory and Coding

Oct. 23, 2018

PROBLEM 1. Suppose $L : \mathbb{R}^K \rightarrow \mathbb{R}^N$ is a linear function and $g : \mathbb{R}^N \rightarrow \mathbb{R}$ is a concave function. Show that $f : \mathbb{R}^K \rightarrow \mathbb{R}$ defined as $f(x) = g(L(x))$ is concave.

PROBLEM 2. From the notes on Lempel-Ziv algorithm, we know that the total length n of c distinct binary strings satisfies

$$n > c \log_2(c/8)$$

The same technique, when applied to the non-binary strings yields

$$n > c \log_K(c/K^3)$$

where K is the size of the alphabet the letters of the string belong to. This inequality lower bounds n in terms of c . We will now show that n can also be upper bounded in terms of c .

- (a) Show that, if $n \geq \frac{1}{2}m(m-1)$, then $c \geq m$.
- (b) Find a sequence for which the bound in (a) is met with equality.
- (c) Show now that $n < \frac{1}{2}c(c+1)$.

PROBLEM 3. Let X be the channel input. Assume that the channel output Y is passed through a data processor in such a way that no information is lost. That is,

$$I(X; Y) = I(X; Z)$$

where Z is the processor output. Find an example where $H(Y) > H(Z)$ and find an example where $H(Y) < H(Z)$.

Hint: The data processor does not have to be deterministic

PROBLEM 4. A “ K -ary erasure channel with erasure probability p ” is described as follows: the input U belongs to the alphabet $\{1, \dots, K\}$, the output V belongs to the alphabet $\{1, \dots, K\} \cup \{?\}$, and if u is the input, the output V equals u with probability $1 - p$, and equals $?$ with probability p . Note that $\Pr(V = ?) = p$ regardless of the input distribution.

- (a) Show that $\Pr(U = u | V = ?) = p_U(u)$.
- (b) Show that $I(U; V) = (1 - p)H(U)$.
- (c) Find the capacity of this channel and the input distribution that maximizes the mutual information.

PROBLEM 5. We are given a memoryless stationary binary symmetric channel $\text{BSC}(\epsilon)$. Namely, if $X_1, \dots, X_n \in \{0, 1\}$ are the input of this channel and $Y_1, \dots, Y_n \in \{0, 1\}$ are the output, we have:

$$P(Y_i | X_i, X^{i-1}, Y^{i-1}) = P(Y_i | X_i) = \begin{cases} 1 - \epsilon & \text{if } Y_i = X_i, \\ \epsilon & \text{otherwise.} \end{cases}$$

Let W be a random variable that is uniform in $\{0, 1\}$ and consider a communication system with feedback which transmits the value of W to the receiver as follows:

- At time $t = 1$, the transmitter sends $X_1 = W$ through the channel.
 - At time $t = i + 1 \leq n$, the transmitter gets the value of Y_i from the feedback and sends $X_{i+1} = Y_i$ through the channel.
- (a) Give the capacity C of the channel in terms of ϵ , and show that $C = 0$ when $\epsilon = \frac{1}{2}$.
- (b) Show that if $\epsilon = \frac{1}{2}$, $I(X^n; Y^n) = n - 1$. This means that $I(X^n; Y^n) \leq nC$ does not hold for this system.
- (c) Show that although $I(X^n; Y^n) > nC$ when $\epsilon = \frac{1}{2}$, we still have $I(W; Y^n) \leq nC$.

Note that since W is the useful information that is being transmitted, it is the value of $I(W; Y^n)$ that we are interested in when we want to compute the amount of information that is shared with the receiver.