ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 6 Information Theory and Coding

Homework 3 Oct. 02, 2018

PROBLEM 1. The following problem concerns a technique known as run length coding. Along with being a useful technique, it should make you look carefully into the sense in which Huffman coding is optimal. A source produces a sequence of independent binary digits with probabilities $P(\mathbf{0}) = 0.9$ and $P(\mathbf{1}) = 0.1$. We shall encode this sequence in two stages, first counting the number of 0's between successive 1's in the source output, and then encoding these counts into binary code words. The first stage of encoding maps source sequences into intermediate digits by the following rule:

Source Sequence	Intermediate Digits (# of zeros)
1	0
01	1
001	2
0001	3
$\vdots$	$\vdots$
00000001	7
00000000	8

Thus the following sequence is encoded as follows:

1	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**1**
0,			2,								8,			2,	0,					4

The final stage of encoding assigns a code word of length 1 to the intermediate digit 8 and codewords of length 4 to the other intermediate digits.

(a) Justify that the overall code is uniquely decodable.

(b) Find the average number $\bar{N}$ of source digits per intermediate digit.

(c) Find the average number $\bar{M}$ of encoded binary digits per intermediate digit.

(d) Show, by appeal to the law of large numbers, that for a very long source sequence of source digits, the ratio of the number of encoded binary digits to the number of source digits will with high probability be close to $\bar{M}/\bar{N}$. Compare this ratio to the average number of code letters per source letter for a Huffman code encoding four source digits at a time.

PROBLEM 2. Suppose X,Y and Z are random variables.

(a) Show that $H(X) + H(Y) + H(Z) \geq \frac{1}{2}\big[H(XY) + H(YZ) + H(ZX)\big]$.

(b) Show that $H(XY) + H(YZ) \geq H(XYZ) + H(Y)$.

(c) Show that

$$2\big[H(XY) + H(YZ) + H(ZX)\big] \geq 3H(XYZ) + H(X) + H(Y) + H(Z).$$

(d) Show that $H(XY) + H(YZ) + H(ZX) \geq 2H(XYZ)$.

(e) Suppose n points in three dimensions are arranged so that their their projections to the xy, yz and zx planes give n_{xy}, n_{yz} and n_{zx} points. Clearly $n_{xy} \leq n$, $n_{yz} \leq n$, $n_{zx} \leq n$. Use part (d) show that

$$n_{xy} n_{yz} n_{zx} \geq n^2.$$

PROBLEM 3. Let X be a random variable taking values in M points $a_1, \ldots, a_M$, and let $P_X(a_M) = \alpha$. Show that

$$H(X) = \alpha \log \frac{1}{\alpha} + (1-\alpha) \log \frac{1}{1-\alpha} + (1-\alpha) H(Y)$$

where Y is a random variable taking values in $M-1$ points $a_1, \ldots, a_{M-1}$ with probabilities $P_Y(a_j) = P_X(a_j)/(1-\alpha)$; $1 \leq j \leq M-1$. Show that

$$H(X) \leq \alpha \log \frac{1}{\alpha} + (1-\alpha) \log \frac{1}{1-\alpha} + (1-\alpha) \log(M-1)$$

and determine the condition for equality.

PROBLEM 4. Let X, Y, Z be discrete random variables. Prove the validity of the following inequalities and find the conditions for equality:

(a) $I(X, Y; Z) \geq I(X; Z)$.

(b) $H(X, Y|Z) \geq H(X|Z)$.

(c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.

(d) $I(X; Z|Y) \geq I(Z; Y|X) - I(Z; Y) + I(X; Z)$.

PROBLEM 5. For a stationary process $X_1, X_2, \ldots$, show that

(a) $\frac{1}{n} H(X_1, \ldots, X_n) \geq H(X_n | X_{n-1}, \ldots, X_1)$.

(b) $\frac{1}{n} H(X_1, \ldots, X_n) \leq \frac{1}{n-1} H(X_1, \ldots, X_{n-1})$.

PROBLEM 6. Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process. Prove that

$$H(X_0 | X_{-1}, \ldots, X_{-n}) = H(X_0 | X_1, \ldots, X_n).$$

That is: the conditional entropy of the present given the past is equal to the conditional entropy of the present given the future.

PROBLEM 7. Let $X \multimap Y \multimap (Z, W)$ form a Markov chain. Show that

$$I(X; Z) + I(X; W) \leq I(X; Y) + I(Z; W)$$