Final Exam: 17 January 2017
Traitement Quantique de l'Information

Exercise 1 *Interferomètre double (35 points)*

(a) (5 points) The state before the detectors is

$$\begin{aligned}|s\rangle = HXHXH\,|h\rangle &= \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = \frac{1}{\sqrt{2}}(|h\rangle - |v\rangle)\end{aligned}$$

(b) (10 points) $P(D_1) = |\,\langle h|s\rangle\,|^2 = \frac{1}{2}$ and $P(D_2) = |\,\langle v|s\rangle\,|^2 = \frac{1}{2}$.

(c) (10 points) The matrix that models the dephaser is $P = \begin{pmatrix}e^{i\varphi} & 0\\ 0 & 1\end{pmatrix}$.

(d) (10points) The state exited from the interferometer is

$$\begin{aligned}|s_P\rangle &= HXPHXH\,|h\rangle\\ &= \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}e^{i\varphi} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &= \frac{e^{i\varphi}}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = \frac{e^{i\varphi}}{\sqrt{2}}(|h\rangle - |v\rangle)\end{aligned}$$

and therefore the new probabilities $P(D_1) = |\,\langle h|s_P\rangle\,|^2 = \frac{1}{2}$ and $P(D_2) = |\,\langle v|s_P\rangle\,|^2 = \frac{1}{2}$ remain unchanged.

Exercise 2 *Dynamique du spin (30 points)*

(a) (10 points) The Hamiltonian $\mathcal{H}$ is given by $-\hbar J\sigma_1^x\otimes\sigma_2^z = -\hbar JA$, where $A = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}$

and $A^2 = I$. Thus, we have

$$\begin{aligned}
U_t \equiv \exp\left(-\frac{it}{\hbar}\mathcal{H}\right) &= \exp(itJA) \\
&= \sum_k \frac{(itJA)^k}{k!} \\
&= \sum_{k \text{ even}} \frac{(itJA)^k}{k!} + \sum_{k \text{ odd}} \frac{(itJA)^k}{k!} \\
&= \sum_{k \text{ even}} \frac{(itJ)^k}{k!} I + \sum_{k \text{ odd}} \frac{(itJ)^k}{k!} A \\
&= \cos(Jt)I + i\sin(Jt)A.
\end{aligned}$$

(b) (10 points) With $|\Psi_0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$, we have

$$\begin{aligned}
|\Psi_t\rangle &= U_t |\Psi_0\rangle \\
&= \cos(Jt)|\Psi_0\rangle + i\sin(Jt)A|\Psi_0\rangle \\
&= \frac{\cos(Jt)}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{i\sin(Jt)}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}\cos(Jt)\\ \cos(Jt)\\ i\sin(Jt)\\ -i\sin(Jt)\end{pmatrix}.
\end{aligned}$$

(c) (10 points) The probability of observing the first spin in state $|\uparrow\rangle$ is

$$\begin{aligned}
\mathrm{Prob}(|\uparrow\rangle) &= |(\langle\uparrow| \otimes \langle\uparrow|)|\Psi_t\rangle|^2 + |(\langle\uparrow| \otimes \langle\downarrow|)|\Psi_t\rangle|^2 \\
&= \left(\frac{1}{\sqrt{2}}\cos(Jt)\right)^2 + \left(\frac{1}{\sqrt{2}}\cos(Jt)\right)^2 \\
&= (\cos(Jt))^2.
\end{aligned}$$

Exercise 3 *Intrication (35 points)*

(a) (10 points) We prove by showing the contrary contradictory. Suppose $|\Psi_{\mathrm{GHZ}}\rangle = (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle) \otimes (c_0|0\rangle + c_1|1\rangle)$. Expanding the products gives

$$a_0b_0c_0|000\rangle + a_0b_0c_1|001\rangle + \cdots + a_1b_1c_1|111\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

This implies

$$a_0 b_0 c_0 = \frac{1}{\sqrt{2}}, \tag{1}$$
$$a_0 b_0 c_1 = 0, \tag{2}$$
$$a_1 b_1 c_1 = \frac{1}{\sqrt{2}}. \tag{3}$$

Equation (1) requires a_0, b_0, c_0 non-zero. This forces the only possibility of (2) is having $c_1 = 0$. However, this would imply $a_1 b_1 c_1 = 0$ and contradict (3).

There is also aother possibility (up to permutations of qubits) that has to be checked, namely the factorization

$$(a|0\rangle + b|b\rangle) \otimes (a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle)$$

Expanding gives in particular $aa_{00} = 1/\sqrt{2}$ so $a \neq 0$ and $a_{00} \neq 0$, also $aa_{01} = 0$ so $a_{01} = 0$. Similarly we find $b \neq 0$ and $a_{10} = 0$ and $a_{11} \neq 0$. So the factorization is reduced to

$$(a|0\rangle + b|b\rangle) \otimes (a_{00}|00\rangle + a_{11}|11\rangle)$$

with all coefficients non-zero. But expanding this gives more terms than in $|\Psi\rangle_{\mathrm{GHZ}}$. So there is again a contradiction.

(b) (10 points) We have $|\Psi_1\rangle \neq |\Psi_{\mathrm{GHZ}}\rangle$ because

$$\begin{aligned}
|\Psi_1\rangle &= H \otimes H \otimes H \,|\Psi_{\mathrm{GHZ}}\rangle \\
&= \frac{1}{\sqrt{2}} \left(H\,|0\rangle \otimes H\,|0\rangle \otimes H\,|0\rangle + H\,|1\rangle \otimes H\,|1\rangle \otimes H\,|1\rangle\right) \\
&= \frac{1}{4} \left((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) + (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle)\right) \\
&= \frac{1}{2} \left(|000\rangle + |011\rangle + |110\rangle + |101\rangle\right).
\end{aligned}$$

However for the Bell state $|B_{00}\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we have $H \otimes H\,|B_{00}\rangle = |B_{00}\rangle$.

(c) (10 points) The projection on the basis $\{|+\rangle, |-\rangle\} \equiv \left\{\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right\}$ is

$$(|+\rangle\langle+| + |-\rangle\langle-|) \otimes I \otimes I\,|\Psi_{\mathrm{GHZ}}\rangle = \frac{1}{2}\,|+\rangle \otimes (|00\rangle + |11\rangle) + \frac{1}{2}\,|-\rangle \otimes (|00\rangle - |11\rangle).$$

Therefore, the possible states after measurement is either $|+\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ or $|-\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. If we observe the first qubit is in state $|+\rangle$, then the two other qubits are in state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, which is the entangled Bell state $|B_{00}\rangle$.

(d) (5 points) From (c) we see that $|\Psi_{\mathrm{GHZ}}\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes |B_{00}\rangle + |-\rangle \otimes |B_{10}\rangle\right)$. So an orthonormal basis can be

$$\begin{aligned}
\{&|+\rangle \otimes |B_{00}\rangle, |-\rangle \otimes |B_{10}\rangle, \\
&|+\rangle \otimes |B_{01}\rangle, |-\rangle \otimes |B_{11}\rangle, \\
&|+\rangle \otimes |B_{10}\rangle, |-\rangle \otimes |B_{00}\rangle, \\
&|+\rangle \otimes |B_{11}\rangle, |-\rangle \otimes |B_{01}\rangle\}
\end{aligned}$$

such that the first qubit is spanned by $\{|+\rangle, |-\rangle\}$ and the last two qubits are spanned by $\{|B_{00}\rangle, |B_{01}\rangle, |B_{10}\rangle, |B_{11}\rangle\}$. To set $|\Psi_{\text{GHZ}}\rangle$ as an element of the basis, we can transform the above basis and have

$$\Big\{\frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{00}\rangle + |-\rangle \otimes |B_{10}\rangle), \frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{00}\rangle - |-\rangle \otimes |B_{10}\rangle),$$
$$\frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{01}\rangle + |-\rangle \otimes |B_{11}\rangle), \frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{01}\rangle - |-\rangle \otimes |B_{11}\rangle),$$
$$\frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{10}\rangle + |-\rangle \otimes |B_{00}\rangle), \frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{10}\rangle - |-\rangle \otimes |B_{00}\rangle),$$
$$\frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{11}\rangle + |-\rangle \otimes |B_{01}\rangle), \frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{11}\rangle - |-\rangle \otimes |B_{01}\rangle)\Big\}.$$

One can check that this is equivalent to

$$\Big\{\frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle), \frac{1}{\sqrt{2}}(|001\rangle \pm |110\rangle), \frac{1}{\sqrt{2}}(|010\rangle \pm |101\rangle), \frac{1}{\sqrt{2}}(|100\rangle \pm |011\rangle)\Big\}.$$