Exercise 1 Interferomètre double (35 points)

(a) (5 points) The state before the detectors is

\[ |s\rangle = HXHXXH |h\rangle = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (|h\rangle - |v\rangle) \]

(b) (10 points) \( P(D_1) = |\langle h|s\rangle|^2 = \frac{1}{2} \) and \( P(D_2) = |\langle v|s\rangle|^2 = \frac{1}{2} \).

(c) (10 points) The matrix that models the dephaser is \( P = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix} \).

(d) (10 points) The state exited from the interferometer is

\[ |s_P\rangle = HXPHXH |h\rangle = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} = \frac{e^{i\varphi}}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{e^{i\varphi}}{\sqrt{2}} (|h\rangle - |v\rangle) \]

and therefore the new probabilities \( P(D_1) = |\langle h|s_P\rangle|^2 = \frac{1}{2} \) and \( P(D_2) = |\langle v|s_P\rangle|^2 = \frac{1}{2} \) remain unchanged.

Exercise 2 Dynamique du spin (30 points)

(a) (10 points) The Hamiltonian \( \mathcal{H} \) is given by \(-\hbar J \sigma_1^x \otimes \sigma_2^z = -\hbar J A\), where \( A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \)
and \(A^2 = I\). Thus, we have

\[
U_t \equiv \exp\left(\frac{-it}{\hbar} \mathcal{H}\right) = \exp(itJA)
\]

\[
= \sum_k \frac{(itJA)^k}{k!}
\]

\[
= \sum_{k \text{ even}} \frac{(itJ)^k}{k!} I + \sum_{k \text{ odd}} \frac{(itJ)^k}{k!} A
\]

\[
= \cos(Jt) I + i \sin(Jt) A.
\]

(b) (10 points) With \(|\Psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\), we have

\[
|\Psi_t\rangle = U_t |\Psi_0\rangle
\]

\[
= \cos(Jt) |\Psi_0\rangle + i \sin(Jt) A |\Psi_0\rangle
\]

\[
= \frac{\cos(Jt)}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{i \sin(Jt)}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(Jt) \\ \cos(Jt) \\ i \sin(Jt) \\ -i \sin(Jt) \end{pmatrix}.
\]

(c) (10 points) The probability of observing the first spin in state \(|\uparrow\rangle\) is

\[
\text{Prob}(\uparrow) = |\langle \uparrow | \otimes \langle \uparrow | |\Psi_t\rangle|^2 + |\langle \uparrow | \otimes \langle \downarrow | |\Psi_t\rangle|^2
\]

\[
= \left( \frac{1}{\sqrt{2}} \cos(Jt) \right)^2 + \left( \frac{1}{\sqrt{2}} \cos(Jt) \right)^2
\]

\[
= (\cos(Jt))^2.
\]

**Exercise 3 Intrication (35 points)**

(a) (10 points) We prove by showing the contrary contradictory. Suppose \(|\Psi_{\text{GHZ}}\rangle = (a_0 |0\rangle + a_1 |1\rangle) \otimes (b_0 |0\rangle + b_1 |1\rangle) \otimes (c_0 |0\rangle + c_1 |1\rangle)\). Expanding the products gives

\[
a_0b_0c_0 |000\rangle + a_0b_0c_1 |001\rangle + \cdots + a_1b_1c_1 |111\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle).
\]
This implies
\[ a_0b_0c_0 = \frac{1}{\sqrt{2}}, \]  
\[ a_0b_0c_1 = 0, \]  
\[ a_1b_1c_1 = \frac{1}{\sqrt{2}}. \]

Equation (1) requires \( a_0, b_0, c_0 \) non-zero. This forces the only possibility of (2) is having \( c_1 = 0 \). However, this would imply \( a_1b_1c_1 = 0 \) and contradict (3).

There is also another possibility (up to permutations of qubits) that has to be checked, namely the factorization

\[ (a|0⟩ + b|b⟩) ⊗ (a₀₀|00⟩ + a₀₁|01⟩ + a₁₀|10⟩ + a₁₁|11⟩) \]

Expanding gives in particular \( aa₀₀ = 1/\sqrt{2} \) so \( a ≠ 0 \) and \( a₀₀ ≠ 0 \), also \( aa₀₁ = 0 \) so \( a₀₁ = 0 \). Similarly we find \( b ≠ 0 \) and \( a₁₀ = 0 \) and \( a₁₁ ≠ 0 \). So the factorization is reduced to

\[ (a|0⟩ + b|b⟩) ⊗ (a₀₀|00⟩ + a₁₁|11⟩) \]

with all coefficients non-zero. But expanding this gives more terms than in \( |Ψ⟩_{GHZ} \). So there is again a contradiction.

(b) (10 points) We have \( |Ψ₁⟩ \neq |Ψ_{GHZ}⟩ \) because

\[ |Ψ₁⟩ = H ⊗ H ⊗ H |Ψ_{GHZ}⟩ \]

\[ = \frac{1}{\sqrt{2}} (H |0⟩ ⊗ H |0⟩ ⊗ H |0⟩ + H |1⟩ ⊗ H |1⟩ ⊗ H |1⟩) \]

\[ = \frac{1}{4} (|0⟩ + |1⟩) ⊗ (|0⟩ + |1⟩) ⊗ (|0⟩ + |1⟩) + (|0⟩ − |1⟩) ⊗ (|0⟩ − |1⟩) ⊗ (|0⟩ − |1⟩) \]

\[ = \frac{1}{2} (|000⟩ + |011⟩ + |110⟩ + |101⟩). \]

However for the Bell state \( |B₀₀⟩ ≡ \frac{1}{\sqrt{2}}(|00⟩ + |11⟩) \), we have \( H ⊗ H |B₀₀⟩ = |B₀₀⟩ \).

(c) (10 points) The projection on the basis \{+⟩, −⟩\} \[ \equiv \left\{ \frac{|0⟩ + |1⟩}{\sqrt{2}}, \frac{|0⟩ − |1⟩}{\sqrt{2}} \right\} \]

is

\[ (|+⟩ ⟨+| + ⟨−| −⟩) ⊗ I ⊗ I |Ψ_{GHZ}⟩ = \frac{1}{2} |+⟩ ⊗ (|00⟩ + |11⟩) + \frac{1}{2} −⟩ ⊗ (|00⟩ − |11⟩). \]

Therefore, the possible states after measurement is either \( |+⟩ ⊗ \frac{1}{\sqrt{2}} (|00⟩ + |11⟩) \) or \( −⟩ ⊗ \frac{1}{\sqrt{2}} (|00⟩ − |11⟩) \). If we observe the first qubit is in state \( |+⟩ \), then the two other qubits are in state \( \frac{1}{\sqrt{2}}(|00⟩ + |11⟩) \), which is the entangled Bell state \( |B₀₀⟩ \).

(d) (5 points) From (c) we see that \( |Ψ_{GHZ}⟩ = \frac{1}{\sqrt{2}} (|+⟩ ⊗ |B₀₀⟩ + −⟩ ⊗ |B₁₀⟩) \). So an orthonormal basis can be

\[ \{ |+⟩ ⊗ |B₀₀⟩, −⟩ ⊗ |B₁₀⟩, |+⟩ ⊗ |B₀₁⟩, −⟩ ⊗ |B₁₁⟩, |+⟩ ⊗ |B₁₀⟩, −⟩ ⊗ |B₀₀⟩, |+⟩ ⊗ |B₁₁⟩, −⟩ ⊗ |B₀₁⟩ \} \]
such that the first qubit is spanned by \(|+\rangle, |-\rangle\) and the last two qubits are spanned by \(|B_{00}\rangle, |B_{01}\rangle, |B_{10}\rangle, |B_{11}\rangle\). To set \(|\Psi_{\text{GHZ}}\rangle\) as an element of the basis, we can transform the above basis and have

\[
\left\{ \frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{00}\rangle + |\rangle \otimes |B_{10}\rangle), \frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{00}\rangle - |\rangle \otimes |B_{10}\rangle), \frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{01}\rangle + |\rangle \otimes |B_{11}\rangle), \frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{01}\rangle - |\rangle \otimes |B_{11}\rangle), \frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{10}\rangle + |\rangle \otimes |B_{00}\rangle), \frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{10}\rangle - |\rangle \otimes |B_{00}\rangle), \frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{11}\rangle + |\rangle \otimes |B_{01}\rangle), \frac{1}{\sqrt{2}}(|+\rangle \otimes |B_{11}\rangle - |\rangle \otimes |B_{01}\rangle) \right\}.
\]

One can check that this is equivalent to

\[
\left\{ \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle), \frac{1}{\sqrt{2}}(|001\rangle \pm |110\rangle), \frac{1}{\sqrt{2}}(|010\rangle \pm |101\rangle), \frac{1}{\sqrt{2}}(|100\rangle \pm |011\rangle) \right\}.
\]