Homework 7: 1 November 2017
Traitement Quantique de l'Information

Exercise 1 *Bell states*

1) One has to show that $\langle B_{x,y}|B_{x',y'}\rangle = \delta_{x,x'}\delta_{y,y'}$. We show it explicitly for two cases:

$$\begin{aligned}\langle B_{00}|B_{00}\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle + |11\rangle)\\ &= \frac{1}{2}(\langle 00|00\rangle + \langle 00|11\rangle + \langle 11|00\rangle + \langle 11|11\rangle).\end{aligned}$$

Now we have

$$\begin{aligned}\langle 00|00\rangle = \langle 0|0\rangle \langle 0|0\rangle = 1, \langle 00|11\rangle = \langle 0|1\rangle \langle 0|1\rangle = 0,\\ \langle 11|00\rangle = \langle 1|0\rangle \langle 1|0\rangle = 0, \langle 11|11\rangle = \langle 1|1\rangle \langle 1|1\rangle = 1.\end{aligned}$$

Thus we get that $\langle B_{00}|B_{00}\rangle = \frac{1}{2}(1+0+0+1) = 1$. Now let us consider

$$\begin{aligned}\langle B_{00}|B_{01}\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)(|01\rangle + |10\rangle)\\ &= \frac{1}{2}(\langle 00|01\rangle + \langle 00|10\rangle + \langle 11|01\rangle + \langle 11|10\rangle)\\ &= \frac{1}{2}(0+0+0+0) = 0.\end{aligned}$$

2) The proof is by contradiction. Suppose there exists a_1, b_1 and a_2, b_2 such that

$$|B_{00}\rangle = (a_1 |0\rangle + b_1 |1\rangle) \otimes (a_2 |0\rangle + b_2 |1\rangle).$$

Then we have

$$\frac{1}{2}(|00\rangle + |11\rangle) = a_1 a_2 |00\rangle + a_1 b_2 |01\rangle + b_1 a_2 |10\rangle + a_2 b_2 |11\rangle .$$

Comparing the coefficients of the orthornormal basis, one has

$$\frac{1}{2} = a_1 a_2,\ \frac{1}{2} = b_1 b_2,\ a_1 b_2 = 0,\ b_1 a_2 = 0.$$

The third equality indicates that either $a_1 = 0$ or $b_2 = 0$ (or both). If $a_1 = 0$ we get a contradiction with the first equation. If on the other hand $b_2 = 0$, we get a contradiction with the second one. Therefore, there does not exist $|\psi_1\rangle$ and $|\psi_2\rangle$ such that $|B_{00}\rangle$ can be written as $|\psi_1\rangle \otimes |\psi_2\rangle$. Therefore, B_{00} is entangled.

3) We have

$$|\gamma\rangle \otimes |\gamma\rangle = (\cos(\gamma)|0\rangle + \sin(\gamma)|1\rangle) \otimes (\cos(\gamma)|0\rangle + \sin(\gamma)|1\rangle)$$
$$= \cos^2(\gamma)|00\rangle + \cos(\gamma)\sin(\gamma)|01\rangle + \sin(\gamma)\cos(\gamma)|10\rangle + \sin^2(\gamma)|11\rangle .$$

Similarly, we have

$$|\gamma_\perp\rangle \otimes |\gamma_\perp\rangle = = \sin^2(\gamma)|00\rangle - \cos(\gamma)\sin(\gamma)|01\rangle - \sin(\gamma)\cos(\gamma)|10\rangle + \cos^2(\gamma)|11\rangle .$$

Combining the two terms, we find that

$$|\gamma\rangle \otimes |\gamma\rangle + |\gamma_\perp\rangle \otimes |\gamma_\perp\rangle = (\cos^2(\gamma) + \sin^2(\gamma))|00\rangle + (\sin^2(\gamma) + \cos^2(\gamma))|11\rangle$$
$$= |00\rangle + |11\rangle$$

and thus

$$\frac{1}{\sqrt{2}}(|\gamma\rangle \otimes |\gamma\rangle + |\gamma_\perp\rangle \otimes |\gamma_\perp\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |B_{00}\rangle .$$

4) From the rule of the tensor product

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ \\ b\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix},$$

we obtain the basis states as

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$
$$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, we have

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |B_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$
$$|B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \qquad |B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Exercise 2 *Entanglement by unitary operations*

1) By definition of the tensor product:

$$(H \otimes I) |x\rangle \otimes |y\rangle = H |x\rangle \otimes I |y\rangle = H |x\rangle \otimes |y\rangle .$$

Also, one can use that $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to show that always

$$H |x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x |1\rangle).$$

Thus,

$$(H \otimes I) |x\rangle \otimes |y\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |y\rangle).$$

Now we apply CNOT. By linearity, we can apply it to each term separately. Thus,

$$\begin{aligned}(\text{CNOT})(H \otimes I) |x\rangle \otimes |y\rangle &= \frac{1}{\sqrt{2}}((CNOT) |0\rangle \otimes |y\rangle + (-1)^x (\text{CNOT}) |1\rangle \otimes |y\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |y \oplus 1\rangle) \\ &= |B_{xy}\rangle .\end{aligned}$$

2) Let us first start with $H \otimes I$. We use the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix},$$

Thus we have

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

For CNOT, we use the definition:

$$(\text{CNOT}) |x\rangle \otimes |y\rangle = |x\rangle \otimes |y \oplus x\rangle ,$$

which implies that the matrix elements are

$$\langle x'y'| \, \text{CNOT} \, |xy\rangle = \langle x', y'|x, y \otimes x\rangle = \langle x'|x\rangle \langle y'|y \oplus x\rangle = \delta_{xx'} \delta_{y \oplus x, y'} .$$

We obtain the following table with columns xy and rows $x'y'$:

	00	01	10	11
00	1	0	0	0
01	0	1	0	0
10	0	0	0	1
11	0	0	1	0

For the matrix product (CNOT)$(H \otimes I)$, we find that

$$\begin{aligned}(\text{CNOT})H \otimes I &= \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ X & -X \end{pmatrix},\end{aligned}$$

where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus,

$$(CNOT)(H \otimes I) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

One can check that for example $|B_{00}\rangle = (\text{CNOT})(H \otimes I)\,|0\rangle \otimes |0\rangle$. Finally to check the unitarity, we have to check that $UU^\dagger = U^\dagger U = I$ for $U = H \otimes I$, CNOT and (CNOT)$(H \otimes I)$. We leave this to the reader.

3) Let $U = (\text{CNOT})(H \otimes I)$. We have

$$|B_{xy}\rangle = U\,|x\rangle \otimes |y\rangle\,, \langle B_{x'y'}| = \langle x'| \otimes \langle y'|\, U^\dagger.$$

Thus,

$$\begin{aligned}\langle B_{x'y'}|B_{xy}\rangle &= \langle x'| \otimes \langle y'|\, U^\dagger U\,|x\rangle \otimes |y\rangle \\ &= \langle x'| \otimes \langle y'|\, I\,|x\rangle \otimes |y\rangle \\ &= \langle x'|x\rangle \langle y'|y\rangle = \delta_{xx'}\delta_{yy'}.\end{aligned}$$