Exercise 1 *Orthonormal basis and measurement principle*

1) It involves the following checking:

\[ \langle \alpha | \alpha \rangle = \left( \cos \alpha \langle x \rangle + \sin \alpha \langle y \rangle \right) \left( \cos \alpha \langle x \rangle + \sin \alpha \langle y \rangle \right) \]

\[ = \cos^2 \alpha + \sin^2 \alpha = 1 \]

\[ \langle \alpha_\perp | \alpha_\perp \rangle = \left( - \sin \alpha \langle x \rangle + \cos \alpha \langle y \rangle \right) \left( - \sin \alpha \langle x \rangle + \cos \alpha \langle y \rangle \right) \]

\[ = \cos^2 \alpha + \sin^2 \alpha = 1 \]

\[ \langle \alpha_\perp | \alpha \rangle = \left( - \sin \alpha \langle x \rangle + \cos \alpha \langle y \rangle \right) \left( \cos \alpha \langle x \rangle + \sin \alpha \langle y \rangle \right) \]

\[ = - \sin \alpha \cos \alpha + \cos \alpha \sin \alpha = 0 \]

\[ \langle R | R \rangle = \frac{1}{2} \left( \langle x \rangle - i \langle y \rangle \right) \left( \langle x \rangle + i \langle y \rangle \right) = \frac{1}{2} \left( 1 + (-i) \right) = 1 \]

\[ \langle L | L \rangle = \frac{1}{2} \left( \langle x \rangle + i \langle y \rangle \right) \left( \langle x \rangle - i \langle y \rangle \right) = \frac{1}{2} \left( 1 + i(-i) \right) = 1 \]

\[ \langle R | L \rangle = \frac{1}{2} \left( \langle x \rangle - i \langle y \rangle \right) \left( \langle x \rangle - i \langle y \rangle \right) = \frac{1}{2} \left( 1 + (-i)(-i) \right) = 0 \]

2) For each experiment, the possible states just after the measurement would be the corresponding measurement basis with the following probabilities:

\[ \text{Prob}(|x\rangle) = |\langle x | \psi \rangle|^2 = \cos^2 \theta \]

\[ \text{Prob}(|y\rangle) = |\langle y | \psi \rangle|^2 = |(\sin \theta) e^{i \varphi}|^2 = \sin^2 \theta \]

where we use \( e^{i \varphi} = \cos \varphi + i \sin \varphi \) so that \( |e^{i \varphi}|^2 = \cos^2 \varphi + \sin^2 \varphi = 1 \). For the other probabilities we have:

\[ \text{Prob}(|R\rangle) = |\langle R | \psi \rangle|^2 \]

\[ = \left| \frac{1}{\sqrt{2}} \left( \langle x \rangle - i \langle y \rangle \right) \left( \cos \theta \langle x \rangle + (\sin \theta) e^{i \varphi} \langle y \rangle \right) \right|^2 \]

\[ = \left| \frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} i \sin \theta e^{i \varphi} \right|^2 \]

\[ = \left| \frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta \sin \varphi - \frac{1}{\sqrt{2}} i \sin \theta \cos \varphi \right|^2 \]

\[ = \left( \frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta \sin \varphi \right)^2 + \left( \frac{1}{\sqrt{2}} \sin \theta \cos \varphi \right)^2 \]

\[ = \frac{1}{2} + \cos \theta \sin \theta \sin \varphi \]
\[
\text{Prob}(\langle L \rangle) = |\langle L|\psi\rangle|^2 = \frac{1}{\sqrt{2}} \left| (\langle x \rangle + i \langle y \rangle) (\cos \theta |x\rangle + (\sin \theta)e^{i\varphi} |y\rangle) \right|^2 \\
= \frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} i \sin \theta e^{i\varphi} \\
= \frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta \sin \varphi + \frac{1}{\sqrt{2}} i \sin \theta \cos \varphi \\
= \left( \frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta \sin \varphi \right)^2 + \left( \frac{1}{\sqrt{2}} \sin \theta \cos \varphi \right)^2 \\
= \frac{1}{2} - \cos \theta \sin \theta \sin \varphi
\]

\[
\text{Prob}(\langle \alpha \rangle) = |\langle \alpha|\psi\rangle|^2 = \left| (\cos \alpha \langle x \rangle + \sin \alpha \langle y \rangle) (\cos \theta |x\rangle + (\sin \theta)e^{i\varphi} |y\rangle) \right|^2 \\
= |\cos \alpha \cos \theta + \sin \alpha \sin \theta e^{i\varphi}|^2 \\
= |\cos \alpha \cos \theta + \sin \alpha \sin \theta \cos \varphi + i \sin \alpha \sin \theta \sin \varphi|^2 \\
= (\cos \alpha \cos \theta + \sin \alpha \cos \theta \cos \varphi)^2 + (\sin \alpha \sin \theta \sin \varphi)^2 \\
= \cos^2 \alpha \cos^2 \theta + 2 \cos \alpha \sin \alpha \cos \theta \sin \varphi + \sin^2 \alpha \sin^2 \theta \\
\text{Prob}(\langle \alpha_\perp \rangle) = |\langle \alpha_\perp|\psi\rangle|^2 = \left| (\sin \alpha \langle x \rangle + \cos \alpha \langle y \rangle) (\cos \theta |x\rangle + (\sin \theta)e^{i\varphi} |y\rangle) \right|^2 \\
= |\sin \alpha \cos \theta + \cos \alpha \sin \theta e^{i\varphi}|^2 \\
= |\sin \alpha \cos \theta + \cos \alpha \sin \theta \cos \varphi + i \cos \alpha \sin \theta \sin \varphi|^2 \\
= (\sin \alpha \cos \theta - \cos \alpha \sin \theta \cos \varphi)^2 + (\cos \alpha \sin \theta \sin \varphi)^2 \\
= \sin^2 \alpha \cos^2 \theta - 2 \cos \alpha \sin \alpha \cos \theta \sin \varphi + \cos^2 \alpha \sin^2 \theta
\]

One can verify that these probabilities are normalized to one,

\[
\text{Prob}(\langle x \rangle) + \text{Prob}(\langle y \rangle) = \text{Prob}(\langle R \rangle) + \text{Prob}(\langle L \rangle) = \text{Prob}(\langle \alpha \rangle) + \text{Prob}(\langle \alpha_\perp \rangle) = 1.
\]

**Exercise 2 Matrices in Dirac’s notation**

1) As \{\langle x \rangle, \langle y \rangle\} is a set of orthonormal basis, we have

\[
(\gamma^* \langle x \rangle + \delta^* \langle y \rangle) (\alpha \langle x \rangle + \beta \langle y \rangle) = \gamma^* \alpha \langle x|x\rangle + \gamma^* \beta \langle x|y\rangle + \delta^* \alpha \langle y|x\rangle + \delta^* \beta \langle y|y\rangle \\
= \gamma^* \alpha + \delta^* \beta.
\]
2) \[
(\alpha |x\rangle + \beta |y\rangle) (\gamma^* \langle x| + \delta^* \langle y|) \\
= \alpha \gamma^* |x\rangle \langle x| + \alpha \delta^* |x\rangle \langle y| + \beta \gamma^* |y\rangle \langle x| + \beta \delta^* |y\rangle \langle y| \\
= \alpha \gamma^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \ 0) + \alpha \delta^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} (0 \ 1) + \beta \gamma^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 \ 0) + \beta \delta^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \ 1) \\
= \begin{pmatrix} \alpha \gamma^* \\ \beta \gamma^* \end{pmatrix} \begin{pmatrix} \alpha \delta^* \\ \beta \delta^* \end{pmatrix}
\]

3) 
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\
= a_{11} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \ 0) + a_{12} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (0 \ 1) + a_{21} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 \ 0) + a_{22} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \ 1) \\
= a_{11} |x\rangle \langle x| + a_{12} |x\rangle \langle y| + a_{21} |y\rangle \langle x| + a_{22} |y\rangle \langle y| 
\]

4) As 
\[
|\alpha\rangle = \cos \alpha |x\rangle + \sin \alpha |y\rangle, \\
|\alpha_\perp\rangle = -\sin \alpha |x\rangle + \cos \alpha |y\rangle,
\]
we find \( |x\rangle, |y\rangle \) by linear combination of the above two equations:
\[
\cos \alpha |\alpha\rangle - \sin \alpha |\alpha_\perp\rangle = \left(\cos^2 \alpha + \sin^2 \alpha \right) |x\rangle = |x\rangle, \\
\sin \alpha |\alpha\rangle + \cos \alpha |\alpha_\perp\rangle = \left(\sin^2 \alpha + \cos^2 \alpha \right) |y\rangle = |y\rangle.
\]

Then we have
\[
|x\rangle \langle x| = \left(\cos \alpha |\alpha\rangle - \sin \alpha |\alpha_\perp\rangle\right) \left(\cos \alpha \langle \alpha| - \sin \alpha \langle \alpha_\perp|\right) \\
= \cos^2 \alpha |\alpha\rangle \langle \alpha| - \sin \alpha \cos \alpha |\alpha\rangle \langle \alpha_\perp| - \sin \alpha \cos \alpha |\alpha_\perp\rangle \langle \alpha| + \sin^2 \alpha |\alpha_\perp\rangle \langle \alpha_\perp| \\
|x\rangle \langle y| = \left(\cos \alpha |\alpha\rangle - \sin \alpha |\alpha_\perp\rangle\right) \left(\sin \alpha \langle \alpha| + \cos \alpha \langle \alpha_\perp|\right) \\
= \sin \alpha \cos \alpha |\alpha\rangle \langle \alpha| + \cos^2 \alpha |\alpha\rangle \langle \alpha_\perp| - \sin^2 \alpha |\alpha_\perp\rangle \langle \alpha| - \sin \alpha \cos \alpha |\alpha_\perp\rangle \langle \alpha_\perp| \\
|y\rangle \langle x| = \left(\sin \alpha |\alpha\rangle + \cos \alpha |\alpha_\perp\rangle\right) \left(\sin \alpha \langle \alpha| - \sin \alpha \langle \alpha_\perp|\right) \\
= \sin \alpha \cos \alpha |\alpha\rangle \langle \alpha| - \sin^2 \alpha |\alpha\rangle \langle \alpha_\perp| + \cos^2 \alpha |\alpha_\perp\rangle \langle \alpha| - \sin \alpha \cos \alpha |\alpha_\perp\rangle \langle \alpha_\perp| \\
|y\rangle \langle y| = \left(\sin \alpha |\alpha\rangle + \cos \alpha |\alpha_\perp\rangle\right) \left(\sin \alpha \langle \alpha| + \cos \alpha \langle \alpha_\perp|\right) \\
= \sin^2 \alpha |\alpha\rangle \langle \alpha| + \sin \alpha \cos \alpha |\alpha_\perp\rangle \langle \alpha_\perp| + \sin \alpha \cos \alpha |\alpha_\perp\rangle \langle \alpha| + \cos^2 \alpha |\alpha_\perp\rangle \langle \alpha_\perp|.
\]

Substituting all these into \( A \), we have
\[
A = a_{11} |x\rangle \langle x| + a_{12} |x\rangle \langle y| + a_{21} |y\rangle \langle x| + a_{22} |y\rangle \langle y| \\
= \left(a_{11} \cos^2 \alpha + a_{12} \sin \alpha \cos \alpha + a_{21} \sin \alpha \cos \alpha + a_{22} \sin^2 \alpha \right) |\alpha\rangle \langle \alpha| \\
+ \left(-a_{11} \sin \alpha \cos \alpha + a_{12} \cos^2 \alpha - a_{21} \sin^2 \alpha + a_{22} \sin \alpha \cos \alpha \right) |\alpha\rangle \langle \alpha_\perp| \\
+ \left(-a_{11} \sin \alpha \cos \alpha - a_{12} \sin^2 \alpha + a_{21} \cos^2 \alpha + a_{22} \sin \alpha \cos \alpha \right) |\alpha_\perp\rangle \langle \alpha| \\
+ \left(a_{11} \sin^2 \alpha - a_{12} \sin \alpha \cos \alpha - a_{21} \sin \alpha \cos \alpha + a_{22} \cos^2 \alpha \right) |\alpha_\perp\rangle \langle \alpha_\perp|.
\]
Therefore, we have
\[
\begin{align*}
\tilde{a}_{11} &= a_{11} \cos^2 \alpha + a_{12} \sin \alpha \cos \alpha + a_{21} \sin \alpha \cos \alpha + a_{22} \sin^2 \alpha, \\
\tilde{a}_{12} &= -a_{11} \sin \alpha \cos \alpha + a_{12} \cos^2 \alpha - a_{21} \sin^2 \alpha + a_{22} \sin \alpha \cos \alpha, \\
\tilde{a}_{21} &= a_{11} \sin \alpha \cos \alpha - a_{12} \sin^2 \alpha + a_{21} \cos^2 \alpha + a_{22} \sin \alpha \cos \alpha, \\
\tilde{a}_{22} &= a_{11} \sin^2 \alpha - a_{12} \sin \alpha \cos \alpha - a_{21} \sin \alpha \cos \alpha + a_{22} \cos^2 \alpha.
\end{align*}
\]

An alternative approach to find \(\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{21}, \tilde{a}_{22}\):
Suppose we replace the matrix notations by \(|\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(|\alpha_\perp\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\) such that
\(
|x\rangle = \cos \alpha \, |\alpha\rangle - \sin \alpha \, |\alpha_\perp\rangle = \begin{pmatrix} \cos \alpha \\ -\sin \alpha \end{pmatrix}
\) and \(|y\rangle = \sin \alpha \, |\alpha\rangle + \cos \alpha \, |\alpha_\perp\rangle = \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} \). Consider the equation
\[
\tilde{a}_{11} \langle \alpha | + \tilde{a}_{12} \langle \alpha_\perp | + \tilde{a}_{21} \langle \alpha_\perp | + \tilde{a}_{22} \langle \alpha_\perp | \langle \alpha_\perp |
= a_{11} \langle x | + a_{12} \langle x | + a_{21} \langle y | + a_{22} \langle y | \langle y |.
\]

In the \(|\alpha, \alpha_\perp\rangle\) basis, the L.H.S. equals \(\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix}\) and the R.H.S. equals
\[
\begin{pmatrix} a_{11} \cos \alpha & a_{11} \sin \alpha \\ -a_{21} \sin \alpha & a_{21} \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + \begin{pmatrix} a_{12} \sin \alpha & a_{12} \cos \alpha \\ a_{22} \cos \alpha & a_{22} \sin \alpha \end{pmatrix} \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix}.
\]

\[
= \begin{pmatrix} a_{11} \cos \alpha & a_{12} \sin \alpha \\ -a_{21} \sin \alpha & a_{22} \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}
= P \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} P^{-1}
\]

where \(P = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}\). Therefore we find that the usual basis transformation rule
\[
\begin{pmatrix} \tilde{a}_{11} \\ \tilde{a}_{12} \\ \tilde{a}_{21} \\ \tilde{a}_{22} \end{pmatrix} = P \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} P^{-1}
\]

Exercise 3 Interferometer revisited
1) Using Exercise 2.3, we have
\[
S = |H\rangle \langle H| + i |H\rangle \langle V| + |V\rangle \langle H| + |V\rangle \langle V| \\
R = i |H\rangle \langle V| + i |V\rangle \langle H|.
\]

2) The computation in Dirac’s notation is
\[
RS = \frac{1}{\sqrt{2}} (- |H\rangle \langle H| + i |H\rangle \langle V| + i |V\rangle \langle H| - |V\rangle \langle V|),
\]
\[
SRS = - |H\rangle \langle H| - |V\rangle \langle V|.
\]
The computation in usual matrix notation is

\[ SRS = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \]

3) \[ SRS |H\rangle = \left( -|H\rangle \langle H| - |V\rangle \langle V| \right) |H\rangle = - |H\rangle \]

\[ |\langle H| SRS |H\rangle|^2 = | - \langle H|H\rangle |^2 = 1 \]

\[ |\langle V| SRS |H\rangle|^2 = | - \langle V|H\rangle |^2 = 0 \]

The experimental set-up:

![Experimental set-up diagram]

4) We have

\[ SRDS = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -e^{i\varphi_1} - e^{i\varphi_2} & -ie^{i\varphi_1} + ie^{i\varphi_2} \\ ie^{i\varphi_1} - ie^{i\varphi_2} & -e^{i\varphi_1} - e^{i\varphi_2} \end{pmatrix} \]

and

\[ SRDS |H\rangle = \frac{1}{2} \begin{pmatrix} -e^{i\varphi_1} - e^{i\varphi_2} \\ ie^{i\varphi_1} - ie^{i\varphi_2} \end{pmatrix}, \]

which in Dirac notation is

\[ SRDS |H\rangle = -\frac{e^{i\varphi_1} + e^{i\varphi_2}}{2} |H\rangle + \frac{ie^{i\varphi_1} - ie^{i\varphi_2}}{2} |V\rangle. \]

Then we have

\[ |\langle H| SRDS |H\rangle|^2 = \left| \frac{e^{i\varphi_1} + e^{i\varphi_2}}{2} \right|^2 \]

\[ = \frac{1}{4} |\cos \varphi_1 + i \sin \varphi_1 + \cos \varphi_1 + i \sin \varphi_2|^2 \]

\[ = \frac{1}{4} \left( (\cos \varphi_1 + \cos \varphi_2)^2 + (\sin \varphi_1 + \sin \varphi_2)^2 \right) \]

\[ = \frac{1}{4} \left( 2 + 2 \cos \varphi_1 \cos \varphi_2 + 2 \sin \varphi_1 \sin \varphi_2 \right) \]

\[ = \frac{1}{2} (1 + \cos(\varphi_1 - \varphi_2)) \]

\[ = \cos^2 \left( \frac{\varphi_1 - \varphi_2}{2} \right) \]
and

\[ |\langle V \rangle SRDS |H\rangle|^2 = \left| \frac{i e^{i\varphi_1} - i e^{i\varphi_2}}{2} \right|^2 \]
\[ = \frac{1}{4} \left| -\sin \varphi_1 + i \cos \varphi_1 + \sin \varphi_1 - i \cos \varphi_1 \right|^2 \]
\[ = \frac{1}{4} \left( (\sin \varphi_1 - \sin \varphi_2)^2 + (\cos \varphi_1 - \cos \varphi_2)^2 \right) \]
\[ = \frac{1}{4} \left( 2 - 2 \cos \varphi_1 \cos \varphi_2 - 2 \sin \varphi_1 \sin \varphi_2 \right) \]
\[ = \frac{1}{2} \left( 1 - \cos(\varphi_1 - \varphi_2) \right) \]
\[ = \sin^2 \left( \frac{\varphi_1 - \varphi_2}{2} \right) \]

The experimental set-up:

A proof that \( SRDS \) is unitary: Recall the notation \( A^\dagger = A^{\top,*} \). We have checked that \( SS^\dagger = S^\dagger S = I \) and \( RR^\dagger = R^\dagger R = I \) in Homework 2. It is also easy to check \( DD^\dagger = D^\dagger D = I \). The product of unitary matrices is unitary, indeed

\[ (U_1 U_2)(U_1 U_2)^\dagger = U_1 U_2 U_2^\dagger U_1^\dagger = U_1 U_1^\dagger = I. \]

Remarks: The matrix elements are \( \begin{pmatrix} \langle H \rangle SRDS |H\rangle & \langle H \rangle SRDS |V\rangle \\ \langle V \rangle SRDS |H\rangle & \langle V \rangle SRDS |V\rangle \end{pmatrix} \).

• We saw that for a unitary matrix the sum of the modulus squares of rows or columns equal 1. For example for the first column we have \( |\langle H \rangle SRDS |H\rangle|^2 + |\langle V \rangle SRDS |H\rangle|^2 = 1 \). This expresses the fact that the two probabilities of finding the photon in state \( \langle H \rangle \) or \( \langle V \rangle \) after the measurement is 1.

• Similarly if we would do an experiment with a photon coming in state \( |V\rangle \) when it enters the interferometer, the probabilities of finding it in state \( \langle H \rangle \) or \( \langle V \rangle \) at the detectors should sum to 1. This means \( |\langle H \rangle SRDS |V\rangle|^2 + |\langle V \rangle SRDS |V\rangle|^2 = 1 \) which is the sum of the modulus squares of the second column.

• In fact for each of the sum of columns (or rows) that sums to 1, there is an experimental interpretation.