Homework 3: 5 October 2017
Traitement Quantique de l'Information

Exercise 1 *Orthonormal basis and measurement principle*

1) It involves the following checking:

$$\begin{aligned}
\langle\alpha|\alpha\rangle &= \big(\cos\alpha\,\langle x| + \sin\alpha\,\langle y|\,\big)\big(\cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle\,\big) \\
&= \cos^2\alpha + \sin^2\alpha = 1 \\
\langle\alpha_\perp|\alpha_\perp\rangle &= \big(-\sin\alpha\,\langle x| + \cos\alpha\,\langle y|\,\big)\big(-\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle\,\big) \\
&= \cos^2\alpha + \sin^2\alpha = 1 \\
\langle\alpha_\perp|\alpha\rangle &= \big(-\sin\alpha\,\langle x| + \cos\alpha\,\langle y|\,\big)\big(\cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle\,\big) \\
&= -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha = 0
\end{aligned}$$

$$\begin{aligned}
\langle R|R\rangle &= \frac{1}{2}\big(\,\langle x| - i\,\langle y|\,\big)\big(\,|x\rangle + i\,|y\rangle\,\big) = \frac{1}{2}(1^2 + (-i)i) = 1 \\
\langle L|L\rangle &= \frac{1}{2}\big(\,\langle x| + i\,\langle y|\,\big)\big(\,|x\rangle - i\,|y\rangle\,\big) = \frac{1}{2}(1^2 + i(-i)) = 1 \\
\langle R|L\rangle &= \frac{1}{2}\big(\,\langle x| - i\,\langle y|\,\big)\big(\,|x\rangle - i\,|y\rangle\,\big) = \frac{1}{2}(1^2 + (-i)(-i)) = 0
\end{aligned}$$

2) For each experiment, the possible states just after the measurement would be the corresponding measurement basis with the following probabilities:

$$\begin{aligned}
\mathrm{Prob}(|x\rangle) &= |\langle x|\psi\rangle|^2 = \cos^2\theta \\
\mathrm{Prob}(|y\rangle) &= |\langle y|\psi\rangle|^2 = |(\sin\theta)e^{i\varphi}|^2 = \sin^2\theta
\end{aligned}$$

where we use $e^{i\varphi} = \cos\varphi + i\sin\varphi$ so that $|e^{i\varphi}|^2 = \cos^2\varphi + \sin^2\varphi = 1$. For the other probabilities we have:

$$\begin{aligned}
\mathrm{Prob}(|R\rangle) &= |\langle R|\psi\rangle|^2 \\
&= \left|\frac{1}{\sqrt{2}}\left(\langle x| - i\,\langle y|\right)\left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}i\sin\theta e^{i\varphi}\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\sin\varphi - \frac{1}{\sqrt{2}}i\sin\theta\cos\varphi\right|^2 \\
&= \left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\sin\varphi\right)^2 + \left(\frac{1}{\sqrt{2}}\sin\theta\cos\varphi\right)^2 \\
&= \frac{1}{2} + \cos\theta\sin\theta\sin\varphi
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Prob}(|L\rangle) &= |\langle L|\psi\rangle|^2 \\
&= \left|\frac{1}{\sqrt{2}}\left(\langle x| + i\,\langle y|\right)\left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}i\sin\theta e^{i\varphi}\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\sin\varphi + \frac{1}{\sqrt{2}}i\sin\theta\cos\varphi\right|^2 \\
&= \left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\sin\varphi\right)^2 + \left(\frac{1}{\sqrt{2}}\sin\theta\cos\varphi\right)^2 \\
&= \frac{1}{2} - \cos\theta\sin\theta\sin\varphi
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Prob}(|\alpha\rangle) &= |\langle \alpha|\psi\rangle|^2 \\
&= \left|\left(\cos\alpha\,\langle x| + \sin\alpha\,\langle y|\right)\left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle\right)\right|^2 \\
&= \left|\cos\alpha\cos\theta + \sin\alpha\sin\theta e^{i\varphi}\right|^2 \\
&= |\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\varphi + i\sin\alpha\sin\theta\sin\varphi|^2 \\
&= (\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\varphi)^2 + (\sin\alpha\sin\theta\sin\varphi)^2 \\
&= \cos^2\alpha\cos^2\theta + 2\cos\alpha\sin\alpha\cos\theta\sin\theta\cos\varphi + \sin^2\alpha\sin^2\theta \\
\mathrm{Prob}(|\alpha_\perp\rangle) &= |\langle \alpha_\perp|\psi\rangle|^2 \\
&= \left|\left(\sin\alpha\,\langle x| + \cos\alpha\,\langle y|\right)\left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle\right)\right|^2 \\
&= \left|-\sin\alpha\cos\theta + \cos\alpha\sin\theta e^{i\varphi}\right|^2 \\
&= |-\sin\alpha\cos\theta + \cos\alpha\sin\theta\cos\varphi + i\cos\alpha\sin\theta\sin\varphi|^2 \\
&= (\sin\alpha\cos\theta - \cos\alpha\sin\theta\cos\varphi)^2 + (\cos\alpha\sin\theta\sin\varphi)^2 \\
&= \sin^2\alpha\cos^2\theta - 2\cos\alpha\sin\alpha\cos\theta\sin\theta\cos\varphi + \cos^2\alpha\sin^2\theta
\end{aligned}
$$

One can verify that these probabilities are normalized to one,

$$
\mathrm{Prob}(|x\rangle) + \mathrm{Prob}(|y\rangle) = \mathrm{Prob}(|R\rangle) + \mathrm{Prob}(|L\rangle) = \mathrm{Prob}(|\alpha\rangle) + \mathrm{Prob}(|\alpha_\perp\rangle) = 1.
$$

Exercise 2 *Matrices in Dirac's notation*

1) As $\{|x\rangle, |y\rangle\}$ is a set of orthonormal basis, we have

$$
\begin{aligned}
\left(\gamma^*\,\langle x| + \delta^*\,\langle y|\right)\left(\alpha\,|x\rangle + \beta\,|y\rangle\right) &= \gamma^*\alpha\,\langle x|x\rangle + \gamma^*\beta\,\langle x|y\rangle + \delta^*\alpha\,\langle y|x\rangle + \delta^*\beta\,\langle y|y\rangle \\
&= \gamma^*\alpha + \delta^*\beta.
\end{aligned}
$$

2)

$$\begin{aligned}
&(\alpha\,|x\rangle + \beta\,|y\rangle)\,(\gamma^*\,\langle x| + \delta^*\,\langle y|)\\
&\quad= \alpha\gamma^*\,|x\rangle\,\langle x| + \alpha\delta^*\,|x\rangle\,\langle y| + \beta\gamma^*\,|y\rangle\,\langle x| + \beta\delta^*\,|y\rangle\,\langle y|\\
&\quad= \alpha\gamma^*\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \alpha\delta^*\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} + \beta\gamma^*\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \beta\delta^*\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\\
&\quad= \begin{pmatrix}\alpha\gamma^* & \alpha\delta^*\\ \beta\gamma^* & \beta\delta^*\end{pmatrix}
\end{aligned}$$

3)

$$\begin{aligned}
A &= \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\\
&= a_{11}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + a_{12}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} + a_{21}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + a_{22}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\\
&= a_{11}\,|x\rangle\,\langle x| + a_{12}\,|x\rangle\,\langle y| + a_{21}\,|y\rangle\,\langle x| + a_{22}\,|y\rangle\,\langle y|
\end{aligned}$$

4) As

$$\begin{aligned}
|\alpha\rangle &= \cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle\,,\\
|\alpha_\perp\rangle &= -\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle\,,
\end{aligned}$$

we find $|x\rangle\,,|y\rangle$ by linear combination of the above two equations:

$$\begin{aligned}
\cos\alpha\,|\alpha\rangle - \sin\alpha\,|\alpha_\perp\rangle &= \left(\cos^2\alpha + \sin^2\alpha\right)|x\rangle = |x\rangle\,,\\
\sin\alpha\,|\alpha\rangle + \cos\alpha\,|\alpha_\perp\rangle &= \left(\sin^2\alpha + \cos^2\alpha\right)|y\rangle = |y\rangle\,.
\end{aligned}$$

Then we have

$$\begin{aligned}
|x\rangle\,\langle x| &= \big(\cos\alpha\,|\alpha\rangle - \sin\alpha\,|\alpha_\perp\rangle\big)\big(\cos\alpha\,\langle\alpha| - \sin\alpha\,\langle\alpha_\perp|\big)\\
&= \cos^2\alpha\,|\alpha\rangle\,\langle\alpha| - \sin\alpha\cos\alpha\,|\alpha\rangle\,\langle\alpha_\perp| - \sin\alpha\cos\alpha\,|\alpha_\perp\rangle\,\langle\alpha| + \sin^2\alpha\,|\alpha_\perp\rangle\,\langle\alpha_\perp|\,,\\
|x\rangle\,\langle y| &= \big(\cos\alpha\,|\alpha\rangle - \sin\alpha\,|\alpha_\perp\rangle\big)\big(\sin\alpha\,\langle\alpha| + \cos\alpha\,\langle\alpha_\perp|\big)\\
&= \sin\alpha\cos\alpha\,|\alpha\rangle\,\langle\alpha| + \cos^2\alpha\,|\alpha\rangle\,\langle\alpha_\perp| - \sin^2\alpha\,|\alpha_\perp\rangle\,\langle\alpha| - \sin\alpha\cos\alpha\,|\alpha_\perp\rangle\,\langle\alpha_\perp|\,,\\
|y\rangle\,\langle x| &= \big(\sin\alpha\,|\alpha\rangle + \cos\alpha\,|\alpha_\perp\rangle\big)\big(\cos\alpha\,\langle\alpha| - \sin\alpha\,\langle\alpha_\perp|\big)\\
&= \sin\alpha\cos\alpha\,|\alpha\rangle\,\langle\alpha| - \sin^2\alpha\,|\alpha\rangle\,\langle\alpha_\perp| + \cos^2\alpha\,|\alpha_\perp\rangle\,\langle\alpha| - \sin\alpha\cos\alpha\,|\alpha_\perp\rangle\,\langle\alpha_\perp|\,,\\
|y\rangle\,\langle y| &= \big(\sin\alpha\,|\alpha\rangle + \cos\alpha\,|\alpha_\perp\rangle\big)\big(\sin\alpha\,\langle\alpha| + \cos\alpha\,\langle\alpha_\perp|\big)\\
&= \sin^2\alpha\,|\alpha\rangle\,\langle\alpha| + \sin\alpha\cos\alpha\,|\alpha\rangle\,\langle\alpha_\perp| + \sin\alpha\cos\alpha\,|\alpha_\perp\rangle\,\langle\alpha| + \cos^2\alpha\,|\alpha_\perp\rangle\,\langle\alpha_\perp|\,.
\end{aligned}$$

Substituting all these into A, we have

$$\begin{aligned}
A &= a_{11}\,|x\rangle\,\langle x| + a_{12}\,|x\rangle\,\langle y| + a_{21}\,|y\rangle\,\langle x| + a_{22}\,|y\rangle\,\langle y|\\
&= \left(a_{11}\cos^2\alpha + a_{12}\sin\alpha\cos\alpha + a_{21}\sin\alpha\cos\alpha + a_{22}\sin^2\alpha\right)|\alpha\rangle\,\langle\alpha|\\
&\qquad+ \left(-a_{11}\sin\alpha\cos\alpha + a_{12}\cos^2\alpha - a_{21}\sin^2\alpha + a_{22}\sin\alpha\cos\alpha\right)|\alpha\rangle\,\langle\alpha_\perp|\\
&\qquad+ \left(-a_{11}\sin\alpha\cos\alpha - a_{12}\sin^2\alpha + a_{21}\cos^2\alpha + a_{22}\sin\alpha\cos\alpha\right)|\alpha_\perp\rangle\,\langle\alpha|\\
&\qquad+ \left(+a_{11}\sin^2\alpha - a_{12}\sin\alpha\cos\alpha - a_{21}\sin\alpha\cos\alpha + a_{22}\cos^2\alpha\right)|\alpha_\perp\rangle\,\langle\alpha_\perp|\,.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\tilde{a}_{11} &= a_{11}\cos^2\alpha + a_{12}\sin\alpha\cos\alpha + a_{21}\sin\alpha\cos\alpha + a_{22}\sin^2\alpha,\\
\tilde{a}_{12} &= -a_{11}\sin\alpha\cos\alpha + a_{12}\cos^2\alpha - a_{21}\sin^2\alpha + a_{22}\sin\alpha\cos\alpha,\\
\tilde{a}_{21} &= a_{11}\sin\alpha\cos\alpha - a_{12}\sin^2\alpha + a_{21}\cos^2\alpha + a_{22}\sin\alpha\cos\alpha,\\
\tilde{a}_{22} &= a_{11}\sin^2\alpha - a_{12}\sin\alpha\cos\alpha - a_{21}\sin\alpha\cos\alpha + a_{22}\cos^2\alpha.
\end{aligned}$$

An alternative approach to find $\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{21}, \tilde{a}_{22}$:

Suppose we replace the matrix notations by $|\alpha\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\alpha_\perp\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ such that $|x\rangle = \cos\alpha\,|\alpha\rangle - \sin\alpha\,|\alpha_\perp\rangle = \begin{pmatrix}\cos\alpha\\-\sin\alpha\end{pmatrix}$ and $|y\rangle = \sin\alpha\,|\alpha\rangle + \cos\alpha\,|\alpha_\perp\rangle = \begin{pmatrix}\sin\alpha\\\cos\alpha\end{pmatrix}$. Consider the equation

$$\begin{aligned}
&\tilde{a}_{11}\,|\alpha\rangle\langle\alpha| + \tilde{a}_{12}\,|\alpha\rangle\langle\alpha_\perp| + \tilde{a}_{21}\,|\alpha_\perp\rangle\langle\alpha| + \tilde{a}_{22}\,|\alpha_\perp\rangle\langle\alpha_\perp|\\
&\qquad = a_{11}\,|x\rangle\langle x| + a_{12}\,|x\rangle\langle y| + a_{21}\,|y\rangle\langle x| + a_{22}\,|y\rangle\langle y|\,.
\end{aligned}$$

In the $\{|\alpha\rangle, |\alpha_\perp\rangle\}$ basis, the L.H.S. equals $\begin{pmatrix}\tilde{a}_{11} & \tilde{a}_{12}\\ \tilde{a}_{21} & \tilde{a}_{22}\end{pmatrix}$ and the R.H.S. equals

$$\begin{aligned}
&a_{11}\begin{pmatrix}\cos\alpha\\-\sin\alpha\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha\end{pmatrix} + a_{12}\begin{pmatrix}\cos\alpha\\-\sin\alpha\end{pmatrix}\begin{pmatrix}\sin\alpha & \cos\alpha\end{pmatrix}\\
&\qquad + a_{21}\begin{pmatrix}\sin\alpha\\\cos\alpha\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha\end{pmatrix} + a_{22}\begin{pmatrix}\sin\alpha\\\cos\alpha\end{pmatrix}\begin{pmatrix}\sin\alpha & \cos\alpha\end{pmatrix}\\
&= \begin{pmatrix}\cos\alpha\\-\sin\alpha\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix} + \begin{pmatrix}\sin\alpha\\\cos\alpha\end{pmatrix}\begin{pmatrix}a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\\
&= \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\\
&= P\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}P^{-1}
\end{aligned}$$

where $P = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}$. Therefore we find that the usual basis transformation rule

$$\begin{pmatrix}\tilde{a}_{11} & \tilde{a}_{12}\\ \tilde{a}_{21} & \tilde{a}_{22}\end{pmatrix} = P\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}P^{-1}$$

Exercise 3 *Interferometer revisited*

1) Using Exercise 2.3, we have

$$\begin{aligned}
S &= |H\rangle\langle H| + i\,|H\rangle\langle V| + i\,|V\rangle\langle H| + |V\rangle\langle V|\\
R &= i\,|H\rangle\langle V| + i\,|V\rangle\langle H|\,.
\end{aligned}$$

2) The computation in Dirac's notation is

$$\begin{aligned}
RS &= \frac{1}{\sqrt{2}}\left(-|H\rangle\langle H| + i\,|H\rangle\langle V| + i\,|V\rangle\langle H| - |V\rangle\langle V|\right),\\
SRS &= -|H\rangle\langle H| - |V\rangle\langle V|\,.
\end{aligned}$$

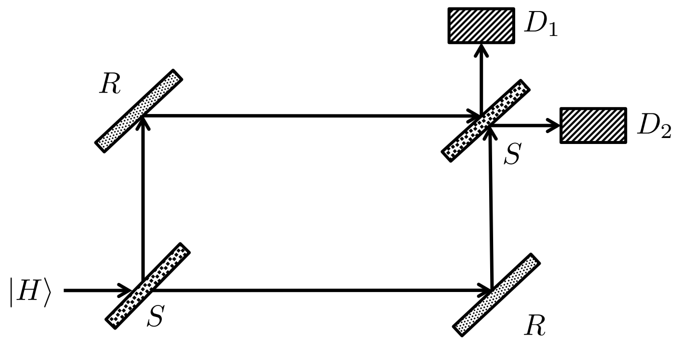
The computation in usual matrix notation is

$$SRS = \frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}-1 & i\\ i & -1\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}.$$

3)

$$\begin{aligned}SRS\,|H\rangle &= \big(-|H\rangle\langle H| - |V\rangle\langle V|\big)\,|H\rangle = -\,|H\rangle\\ |\langle H|\,SRS\,|H\rangle|^2 &= |-\langle H|H\rangle|^2 = 1\\ |\langle V|\,SRS\,|H\rangle|^2 &= |-\langle V|H\rangle|^2 = 0\end{aligned}$$

The experimental set-up:



4) We have

$$SRDS = \frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}e^{i\varphi_1} & 0\\ 0 & e^{i\varphi_2}\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-e^{i\varphi_1} - e^{i\varphi_2} & -ie^{i\varphi_1} + ie^{i\varphi_2}\\ ie^{i\varphi_1} - ie^{i\varphi_2} & -e^{i\varphi_1} - e^{i\varphi_2}\end{pmatrix}$$

and

$$SRDS\,|H\rangle = \frac{1}{2}\begin{pmatrix}-e^{i\varphi_1} - e^{i\varphi_2}\\ ie^{i\varphi_1} - ie^{i\varphi_2}\end{pmatrix},$$

which in Dirac notation is

$$SRDS\,|H\rangle = -\frac{e^{i\varphi_1} + e^{i\varphi_2}}{2}\,|H\rangle + \frac{ie^{i\varphi_1} - ie^{i\varphi_2}}{2}\,|V\rangle\,.$$

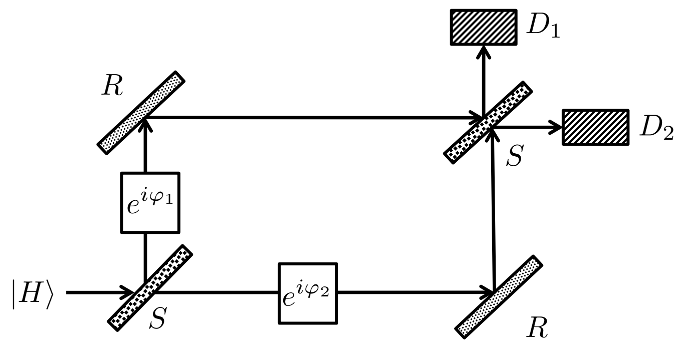
Then we have

$$\begin{aligned}|\langle H|\,SRDS\,|H\rangle|^2 &= \left|\frac{e^{i\varphi_1} + e^{i\varphi_2}}{2}\right|^2\\ &= \frac{1}{4}\,|\cos\varphi_1 + i\sin\varphi_1 + \cos\varphi_1 + i\sin\varphi_2|^2\\ &= \frac{1}{4}\,\big((\cos\varphi_1 + \cos\varphi_2)^2 + (\sin\varphi_1 + \sin\varphi_2)^2\big)\\ &= \frac{1}{4}\,(2 + 2\cos\varphi_1\cos\varphi_2 + 2\sin\varphi_1\sin\varphi_2)\\ &= \frac{1}{2}(1 + \cos(\varphi_1 - \varphi_2))\\ &= \cos^2\left(\frac{\varphi_1 - \varphi_2}{2}\right)\end{aligned}$$

and

$$\begin{aligned}
|\langle V|\, SRDS\, |H\rangle|^2 &= \left|\frac{ie^{i\varphi_1} - ie^{i\varphi_2}}{2}\right|^2 \\
&= \frac{1}{4}\left|-\sin\varphi_1 + i\cos\varphi_1 + \sin\varphi_1 - i\cos\varphi_2\right|^2 \\
&= \frac{1}{4}\left((\sin\varphi_1 - \sin\varphi_2)^2 + (\cos\varphi_1 - \cos\varphi_2)^2\right) \\
&= \frac{1}{4}\left(2 - 2\cos\varphi_1\cos\varphi_2 - 2\sin\varphi_1\sin\varphi_2\right) \\
&= \frac{1}{2}(1 - \cos(\varphi_1 - \varphi_2)) \\
&= \sin^2\left(\frac{\varphi_1 - \varphi_2}{2}\right)
\end{aligned}$$

The experimental set-up:



A proof that $SRDS$ is unitary: Recall the notation $A^\dagger = A^{\top,*}$. We have checked that $SS^\dagger = S^\dagger S = I$ and $RR^\dagger = R^\dagger R = I$ in Homework 2. It is also easy to check $DD^\dagger = D^\dagger D = I$. The product of unitary matrices is unitary, indeed

$$(U_1U_2)(U_1U_2)^\dagger = U_1U_2U_2^\dagger U_1^\dagger = U_1U_1^\dagger = I.$$

Remarks: The matrix elements are $\begin{pmatrix} \langle H|\, SRDS\, |H\rangle & \langle H|\, SRDS\, |V\rangle \\ \langle V|\, SRDS\, |H\rangle & \langle V|\, SRDS\, |V\rangle \end{pmatrix}$.

- We saw that for a unitary matrix the sum of the modulus squares of rows or columns equal 1. For example for the first column we have $|\langle H|\, SRDS\, |H\rangle|^2 + |\langle V|\, SRDS\, |H\rangle|^2 = 1$. This expresses the fact that the two probabilities of finding the photon in state $\langle H|$ or $\langle V|$ after the measurement is 1.

- Similarly if we would do an experiment with a photon coming in state $|V\rangle$ when it enters the interferometer, the probabilities of finding it in state $\langle H|$ or $\langle V|$ at the detectors should sum to 1. This means $|\langle H|\, SRDS\, |V\rangle|^2 + |\langle V|\, SRDS\, |V\rangle|^2 = 1$ which is the sum of the modulus squares of the second column.

- In fact for each of the sum of columns (or rows) that sums to 1, there is an experimental interpretation.