Homework 2: 28 September 2017
Traitement Quantique de l'Information

1) The "Ket" and the associated Dirac or usual vector notations are:

- $|H\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $\langle H| = \begin{pmatrix}1 & 0\end{pmatrix}$
- $|V\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ and $\langle H| = \begin{pmatrix}0 & 1\end{pmatrix}$
- $\alpha\,|H\rangle + \beta\,|V\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ and $\alpha^*\,\langle H| + \beta^*\,\langle V| = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}$

2) In Dirac notation:

$$\begin{aligned}
&(\gamma^*\,\langle H| + \delta^*\,\langle V|)\,(\alpha\,|H\rangle + \beta\,|V\rangle)\\
&= \gamma^*\alpha\,\langle H|H\rangle + \gamma^*\beta\,\langle H|V\rangle + \delta^*\alpha\,\langle V|H\rangle + \delta^*\beta\,\langle V|V\rangle\\
&= \gamma^*\alpha + \delta^*\beta
\end{aligned}$$

because $\langle H|V\rangle = \langle V|H\rangle = 0$ and $\langle H|H\rangle = \langle V|V\rangle = 1$.
The equivalent vector notation is

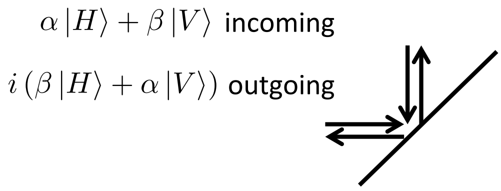
$$\begin{pmatrix}\gamma^* & \delta^*\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \gamma^*\alpha + \delta^*\beta.$$

3) We have $R^\top = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$ and $R^\dagger = R^{\top,*} = \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}$. Thus

$$\begin{aligned}
RR^\dagger &= \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
R^\dagger R &= \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.
\end{aligned}$$

Matrices satisfying $MM^\dagger = M^\dagger M = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ are called unitary matrices.
Let us compute $R\,(\alpha\,|H\rangle + \beta\,|V\rangle)$ in Dirac notation. By linearity of matrix operations,

$$\begin{aligned}
R\,(\alpha\,|H\rangle + \beta\,|V\rangle) &= \alpha R\,|H\rangle + \beta R\,|V\rangle\\
&= \alpha i\,|V\rangle + \beta i\,|H\rangle\\
&= i\,(\alpha\,|V\rangle + \beta\,|H\rangle)\,.
\end{aligned}$$



4) We have

$$S|H\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}} (|H\rangle + i|V\rangle),$$

$$S|V\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}} (i|H\rangle + |V\rangle),$$

$$\begin{aligned} S(\alpha|H\rangle + \beta|V\rangle) &= \alpha S|H\rangle + \beta S|V\rangle \\ &= \frac{\alpha}{\sqrt{2}} (|H\rangle + i|V\rangle) + \frac{\beta}{\sqrt{2}} (i|H\rangle + |V\rangle) \\ &= \frac{\alpha + i\beta}{\sqrt{2}} |H\rangle + \frac{i\alpha + \beta}{\sqrt{2}} |V\rangle . \end{aligned}$$

Refer to question 5 for the picture of the operations.

5) The semi-transparent mirror prepares photons in state $S\big(\alpha|H\rangle + \beta|V\rangle\big)$. It is then measured by the detector D which detects photons in state $|V\rangle$. Therefore, the probability of finding a photon in D is the probability of finding a photon in state $|V\rangle$ given that photons in state $S\big(\alpha|H\rangle + \beta|V\rangle\big)$ are produced. By measurement postulate (which will be formally introduced in Chapter 3 of the lecture note) we have

$$\text{Prob}(D) = \big|\langle V| S\big(\alpha|H\rangle + \beta|V\rangle\big)\big|^2 .$$

From the previous question we have

$$S(\alpha|H\rangle + \beta|V\rangle) = \frac{\alpha + i\beta}{\sqrt{2}} |H\rangle + \frac{i\alpha + \beta}{\sqrt{2}} |V\rangle ,$$

$$\begin{aligned} \langle V| S(\alpha|H\rangle + \beta|V\rangle) &= \frac{\alpha + i\beta}{\sqrt{2}} \langle V| |H\rangle + \frac{i\alpha + \beta}{\sqrt{2}} \langle V| |V\rangle \\ &= \frac{i\alpha + \beta}{\sqrt{2}}. \end{aligned}$$

So we find

$$\text{Prob}(D) = \left|\frac{i\alpha + \beta}{\sqrt{2}}\right|^2 = \frac{1}{2}|i\alpha + \beta|^2 = \frac{1}{2}\big(\alpha^2 + \beta^2\big) = \frac{1}{2}.$$

6) The state after S is

$$S|H\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)$$

The state after R is

$$\begin{aligned} RS|H\rangle &= \frac{1}{\sqrt{2}}(R|H\rangle + iR|V\rangle) \\ &= \frac{i}{\sqrt{2}}(|V\rangle + i|H\rangle). \end{aligned}$$

The state after the second S is

$$\begin{aligned} SRS|H\rangle &= \frac{i}{\sqrt{2}}(S|V\rangle + iS|H\rangle) \\ &= \frac{i}{\sqrt{2}}\left(\frac{i|H\rangle + |V\rangle}{\sqrt{2}} + i \cdot \frac{|H\rangle + i|V\rangle}{\sqrt{2}}\right) \\ &= -|H\rangle. \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{Prob}(D_1) &= |\langle V|\,|H\rangle|^2 = 0 \\ \mathrm{Prob}(D_2) &= |\langle H|\,|H\rangle|^2 = 1. \end{aligned}$$

All photons so in detection D_2! For "classical balls" we would expect a split between D_1 and D_2. For example, if S act as half–half splitters we would expect $\mathrm{Prob}(D_1) = \mathrm{Prob}(D_2) = 1/2$. **The quantum behavior is completely different**!