1) The “Ket” and the associated Dirac or usual vector notations are:

- $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\langle H| = \begin{pmatrix} 1 & 0 \end{pmatrix}$
- $|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\langle H| = \begin{pmatrix} 0 & 1 \end{pmatrix}$
- $\alpha |H\rangle + \beta |V\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $\alpha^* \langle H| + \beta^* \langle V| = (\alpha^* \beta^*)$

2) In Dirac notation:

$$(\gamma^* \langle H| + \delta^* \langle V|) (\alpha |H\rangle + \beta |V\rangle)$$

$= \gamma^* \alpha \langle H|H\rangle + \gamma^* \beta \langle H|V\rangle + \delta^* \alpha \langle V|H\rangle + \delta^* \beta \langle V|V\rangle$

$= \gamma^* \alpha + \delta^* \beta$

because $\langle H|V\rangle = \langle V|H\rangle = 0$ and $\langle H|H\rangle = \langle V|V\rangle = 1$.

The equivalent vector notation is

$$(\gamma^* \delta^*) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \gamma^* \alpha + \delta^* \beta.$$

3) We have $R^\dagger = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $R^\dagger = R^{\dagger,*} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$. Thus

$RR^\dagger = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$R^\dagger R = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

Matrices satisfying $MM^\dagger = M^{\dagger}M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are called unitary matrices.

Let us compute $R (\alpha |H\rangle + \beta |V\rangle)$ in Dirac notation. By linearity of matrix operations,

$$R (\alpha |H\rangle + \beta |V\rangle) = \alpha R |H\rangle + \beta R |V\rangle$$

$$= \alpha i |V\rangle + \beta i |H\rangle$$

$$= i (\alpha |V\rangle + \beta |H\rangle).$$
4) We have

\[ S |H\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \]

\[ S |V\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}, \]

\[ S (\alpha |H\rangle + \beta |V\rangle) = \alpha S |H\rangle + \beta S |V\rangle \]

\[ = \frac{\alpha}{\sqrt{2}} (|H\rangle + i |V\rangle) + \frac{\beta}{\sqrt{2}} (i |H\rangle + |V\rangle) \]

\[ = \frac{\alpha + i \beta}{\sqrt{2}} |H\rangle + \frac{i \alpha + \beta}{\sqrt{2}} |V\rangle. \]

Refer to question 5 for the picture of the operations.

5) The semi-transparent mirror prepares photons in state \( S (\alpha |H\rangle + \beta |V\rangle) \). It is then measured by the detector \( D \) which detects photons in state \( |V\rangle \). Therefore, the probability of finding a photon in \( D \) is the probability of finding a photon in state \( |V\rangle \) given that photons in state \( S (\alpha |H\rangle + \beta |V\rangle) \) are produced. By measurement postulate (which will be formally introduced in Chapter 3 of the lecture note) we have

\[ \text{Prob}(D) = |\langle V | S (\alpha |H\rangle + \beta |V\rangle \rangle|^2. \]

From the previous question we have

\[ S (\alpha |H\rangle + \beta |V\rangle) = \frac{\alpha + i \beta}{\sqrt{2}} |H\rangle + \frac{i \alpha + \beta}{\sqrt{2}} |V\rangle, \]

\[ \langle V | S (\alpha |H\rangle + \beta |V\rangle \rangle = \frac{\alpha + i \beta}{\sqrt{2}} \langle V | |H\rangle + \frac{i \alpha + \beta}{\sqrt{2}} \langle V | |V\rangle \]

\[ = \frac{i \alpha + \beta}{\sqrt{2}}. \]

So we find

\[ \text{Prob}(D) = \left| \frac{i \alpha + \beta}{\sqrt{2}} \right|^2 = \frac{1}{2} (i \alpha + \beta)^2 = \frac{1}{2} (\alpha^2 + \beta^2) = \frac{1}{2}. \]
6) The state after $S$ is

$$S |H\rangle = \frac{1}{\sqrt{2}} (|H\rangle + i |V\rangle)$$

The state after $R$ is

$$RS |H\rangle = \frac{1}{\sqrt{2}} (R |H\rangle + iR |V\rangle)$$

$$= \frac{i}{\sqrt{2}} (|V\rangle + i |H\rangle).$$

The state after the second $S$ is

$$SRS |H\rangle = \frac{i}{\sqrt{2}} (S |V\rangle + iS |H\rangle)$$

$$= \frac{i}{\sqrt{2}} \left( \frac{i |H\rangle + |V\rangle}{\sqrt{2}} + i \cdot \frac{|H\rangle + i |V\rangle}{\sqrt{2}} \right)$$

$$= - |H\rangle.$$

Thus

$$\text{Prob}(D_1) = |\langle V | |H\rangle|^2 = 0$$

$$\text{Prob}(D_2) = |\langle H | |H\rangle|^2 = 1.$$

All photons so in detection $D_2$! For “classical balls” we would expect a split between $D_1$ and $D_2$. For example, if $S$ act as half–half splitters we would expect $\text{Prob}(D_1) = \text{Prob}(D_2) = 1/2$. The quantum behavior is completely different!