Homework 13: 13 December 2017
Traitement Quantique de l'Information

Exercise 1 *Réalisation physique de la porte SWAP*

a) To find the matrix representaion, it is sufficient to find how SWAP port operates on the basis vectors.

$$\text{SWAP}\,|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \text{SWAP}\,|\downarrow\uparrow\rangle = |\downarrow\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},$$

$$\text{SWAP}\,|\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad \text{SWAP}\,|\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Putting the resulting columns together we obtain the matrix representation

$$\text{SWAP} = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

Now it is easy to check that $(\text{SWAP})(\text{SWAP}^\dagger) = I$ which shows that SWAP is a unitary matrix.

b) The Heisenberg Hamiltonian is obtained in the lecture notes and has the following matrix representation

$$H = \hbar J \vec{\sigma}_1 \cdot \vec{\sigma}_2 = \hbar J \begin{pmatrix}1&0&0&0\\0&-1&2&0\\0&2&-1&0\\0&0&0&1\end{pmatrix}$$

To compute the evolution operator $e^{-\frac{itH}{\hbar}}$, we notice that the matrix for H has the matrix representation

$$\begin{pmatrix}A&&0\\&B&\\0&&C\end{pmatrix},$$

where $A = (1)$ and $C = (1)$ are 1×1 matrices and $B = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$ is a 2×2 matrix. It is easy to show that for any complex number α

$$e^{\alpha H} = \begin{pmatrix} e^{\alpha A} & & 0 \\ & e^{\alpha B} & \\ 0 & & e^{\alpha C} \end{pmatrix}.$$

Thus it is sufficient to find these three matrix exponentials. A and C are numbers equal to 1, thus $e^{\alpha A} = e^{\alpha C} = e^{\alpha}$.

Now it remains to find $e^{\alpha B}$. Notice that we can write $B = -I + 2X$ where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Notice that I and X commute with each other, i.e., $IX = XI$. It is not difficult to show that the matrices that commute with each other can be treated like number while taking exponentials, namely, for any commuting matrix M, N, $e^{M+N} = e^M e^N$. (**Notice that this formula is not in general correct**). Therefore, we have

$$e^{i\beta B} = e^{-i\beta I} e^{2i\beta X} = e^{-i\beta}\big(I\cos(2\beta) + iX\sin(2\beta)\big),$$

where we used the Euler's formula for X. It can be checked that at time $t = \frac{\pi}{4J}$, $\alpha = -i\frac{\pi}{4}$, thus $\beta = -\frac{\pi}{4}$. Hence, $e^{\alpha A} = e^{\alpha B} = e^{-i\frac{\pi}{4}}$, and

$$e^{i\beta B} = e^{i\frac{\pi}{4}}\left(\cos\left(\frac{\pi}{2}\right) I - i\sin\left(\frac{\pi}{2}\right) X\right) = -ie^{i\frac{\pi}{4}} X = e^{-i\frac{\pi}{4}} X.$$

Putting all together, the evolution operator at time $t = \frac{\pi}{4J}$ is

$$e^{-i\frac{\pi}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which neglecting the constant phase $-\frac{\pi}{4}$ is equal to the matrix for SWAP.

c) We can implement SWAP using three CNOT gates as depicted in Figure 1. To show this, one can simply check that starting from a general state $|x, y\rangle$ with $x, y \in \{0, 1\}$, after the first CNOT the resulting state is $|x, y \oplus x\rangle$, after the second CNOT the state is

$$|x \oplus (x \oplus y), x \oplus y\rangle = |x \oplus x \oplus y, x \oplus y\rangle = |y, x \oplus y\rangle$$

where we used the identity $x \oplus x = 0$ for $x \in \{0, 1\}$. Finally after the third CNOT the state is $|y, (x \oplus y) \oplus y\rangle = |y, x\rangle$. Therefore the combination of the three gates just swaps x and y. Note that this gives another proof that SWAP is a unitary matrix because it can be implemented as a combination of quantum gates and we know that all quantum gates are unitary.

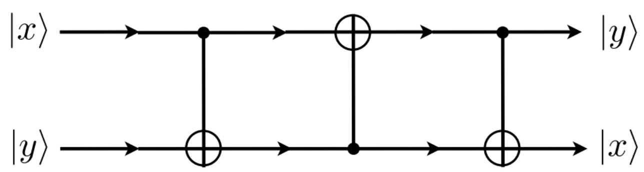

Figure 1: Implementation of SWAP gate using three CNOT