Exercise 1  Réalisation physique de la porte SWAP

a) To find the matrix representation, it is sufficient to find how SWAP port operates on the basis vectors.

\[
\begin{align*}
\text{SWAP } |↑↑⟩ &= |↑↑⟩ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, & \text{SWAP } |↓↑⟩ &= |↓↑⟩ = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \\
\text{SWAP } |↑↓⟩ &= |↑↓⟩ = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & \text{SWAP } |↓↓⟩ &= |↓↓⟩ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]

Putting the resulting columns together we obtain the matrix representation

\[
\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

Now it is easy to check that \((\text{SWAP})(\text{SWAP}^†) = I\) which shows that SWAP is a unitary matrix.

b) The Heisenberg Hamiltonian is obtained in the lecture notes and has the following matrix representation

\[
H = \hbar J \vec{σ}_1 \cdot \vec{σ}_2 = \hbar J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

To compute the evolution operator \(e^{-\frac{iH}{\hbar}}\), we notice that the matrix for \(H\) has the matrix representation

\[
\begin{pmatrix} A & 0 \\ B & C \end{pmatrix},
\]

1
where $A = (1)$ and $C = (1)$ are $1 \times 1$ matrices and $B = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$ is a $2 \times 2$ matrix. It is easy to show that for any complex number $\alpha$

$$e^{\alpha H} = \begin{pmatrix} e^{\alpha A} & 0 \\ 0 & e^{\alpha C} \end{pmatrix}. $$

Thus it is sufficient to find these three matrix exponentials. $A$ and $C$ are numbers equal to 1, thus $e^{\alpha A} = e^{\alpha C} = e^\alpha$.

Now it remains to find $e^{\alpha B}$. Notice that we can write $B = -I + 2X$ where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Notice that $I$ and $X$ commute with each other, i.e., $IX = XI$. It is not difficult to show that the matrices that commute with each other can be treated like number while taking exponentials, namely, for any commuting matrix $M, N$, $e^{M+N} = e^M e^N$. (Notice that this formula is not in general correct). Therefore, we have

$$e^{i\beta B} = e^{-i\beta I} e^{2i\beta X} = e^{-i\beta (I \cos(2\beta) + iX \sin(2\beta))}, $$

where we used the Euler’s formula for $X$. It can be checked that at time $t = \frac{\pi}{4}$, $\alpha = -i\frac{\pi}{4}$, thus $\beta = -\frac{\pi}{4}$. Hence, $e^{\alpha A} = e^{\alpha B} = e^{-i\frac{\pi}{4}}$, and

$$e^{i\beta B} = e^{i\frac{\pi}{4}} \left( \cos \left( \frac{\pi}{2} \right) I - i \sin \left( \frac{\pi}{2} \right) X \right) = -ie^{i\frac{\pi}{4}} X = e^{-i\frac{\pi}{4}} X. $$

Putting all together, the evolution operator at time $t = \frac{\pi}{4}$ is

$$e^{-i\frac{\pi}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which neglecting the constant phase $-\frac{\pi}{4}$ is equal to the matrix for SWAP.

c) We can implement SWAP using three CNOT gates as depicted in Figure 1. To show this, one can simply check that starting from a general state $|x,y\rangle$ with $x, y \in \{0,1\}$, after the first CNOT the resulting state is $|x, y \oplus x\rangle$, after the second CNOT the state is

$$|x \oplus (x \oplus y), x \oplus y\rangle = |x \oplus x \oplus y, x \oplus y\rangle = |y, x\rangle$$

where we used the identity $x \oplus x = 0$ for $x \in \{0,1\}$. Finally after the third CNOT the state is $|y, (x \oplus y) \oplus y\rangle = |y, x\rangle$. Therefore the combination of the three gates just swaps $x$ and $y$. Note that this gives another proof that SWAP is a unitary matrix because it can be implemented as a combination of quantum gates and we know that all quantum gates are unitary.
Figure 1: Implementation of SWAP gate using three CNOT