Homework 11 : 30 Novembre 2017
Traitement Quantique de l'Information

Exercise 1 *Rotations on the Bloch sphere*

A general vector can be written in the form $\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|\downarrow\rangle$ in the Bloch sphere.

a) The eigenvectors for σ_z basis are $|\uparrow\rangle$ and $|\downarrow\rangle$, corresponding to $(\theta = 0, \phi = 0)$ and $(\theta = \pi, \phi = 0)$, respectively.

The eigenvectors for σ_y basis are $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{i}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{i}{\sqrt{2}}|\downarrow\rangle$, corresponding to $(\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2})$ and $(\theta = \frac{\pi}{2}, \phi = -\frac{\pi}{2})$, respectively.

The eigenvectors for σ_x basis are $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$, corresponding to $(\theta = \frac{\pi}{2}, \phi = 0)$ and $(\theta = \frac{\pi}{2}, \phi = \pi)$, respectively.

The corresponding representation over the Bloch sphere is shown in Figure 1.

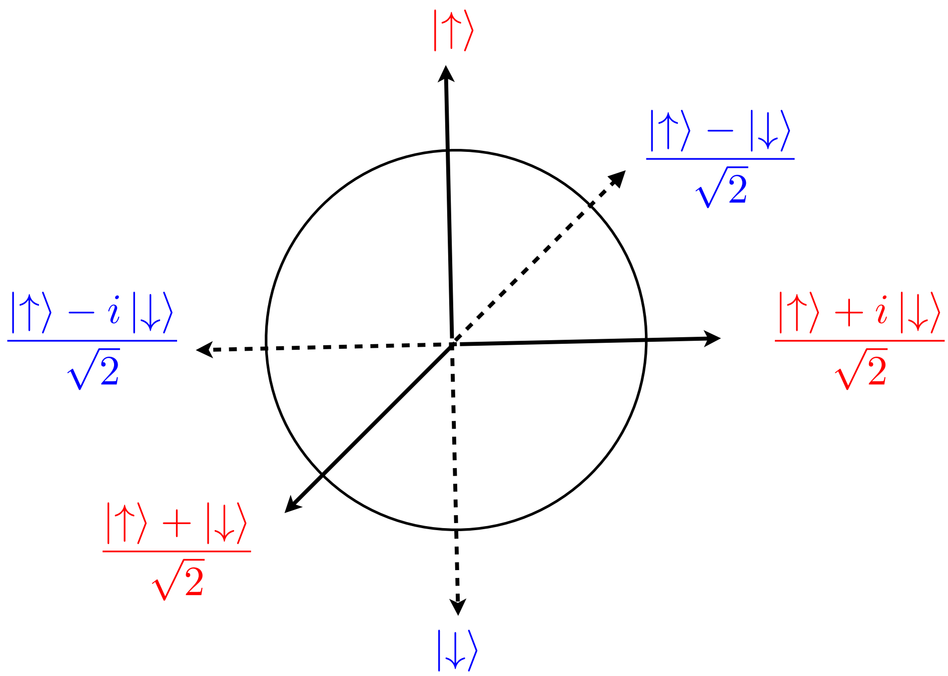

FIGURE 1 – Representation of basis vectors on Bloch Sphere

b) Using the general formula

$$\exp\left(i\frac{\theta}{2}(\vec{\sigma}\cdot\vec{n})\right) = \cos\left(\frac{\theta}{2}\right)I + i\vec{\sigma}\cdot\vec{n}(\sin\left(\frac{\theta}{2}\right)),$$

we obtain

$$\exp\left(-i\frac{\alpha}{2}\sigma_x\right) = \cos\left(\frac{\alpha}{2}\right)I - i\sigma_x(\sin\left(\frac{\alpha}{2}\right))$$
$$= \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) & -i\sin\left(\frac{\alpha}{2}\right) \\ -i\sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{pmatrix},$$

$$\exp\left(-i\frac{\beta}{2}\sigma_y\right) = \cos\left(\frac{\beta}{2}\right)I - i\sigma_y(\sin\left(\frac{\beta}{2}\right))$$
$$= \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) & -\sin\left(\frac{\beta}{2}\right) \\ \sin\left(\frac{\beta}{2}\right) & \cos\left(\frac{\beta}{2}\right) \end{pmatrix},$$

$$\exp\left(-i\frac{\gamma}{2}\sigma_z\right) = \cos\left(\frac{\gamma}{2}\right)I - i\sigma_z(\sin\left(\frac{\gamma}{2}\right))$$
$$= \begin{pmatrix} \cos\left(\frac{\gamma}{2}\right) - i\sin\left(\frac{\gamma}{2}\right) & 0 \\ 0 & \cos\left(\frac{\gamma}{2}\right) + i\sin\left(\frac{\gamma}{2}\right) \end{pmatrix}$$
$$= \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}.$$

c) The matrix $\exp\left(-i\frac{\alpha}{2}\sigma_x\right)$ is a rotation matrix of angle α around the X-axis, thus the state vector $\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle$ is transformed to the vector $\cos\left(\frac{\theta-\alpha}{2}\right)|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\theta-\alpha}{2}\right)|\downarrow\rangle$. One can see the transformation geometrically on the Bloch sphere, however one can also show by direct calculation :

$$\exp\left(-i\frac{\alpha}{2}\sigma_x\right)\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\theta}{2}\right)\right)|\downarrow\rangle = \cos\left(\frac{\theta-\alpha}{2}\right)|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\theta-\alpha}{2}\right)|\downarrow\rangle.$$

Similarly, one can see that $\exp\left(i\frac{\gamma}{2}\sigma_z\right)$ is a rotation of angle γ around the Z-axis. Therefore,

$$\exp\left(-i\frac{\gamma}{2}\sigma_z\right)\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\theta}{2}\right)\right)|\downarrow\rangle = e^{-i\frac{\gamma}{2}}\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i(\frac{\pi}{2}+\gamma)}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle\right).$$

Exercise 2 *Création d'intrication par une interaction magnétique*

L'état final est (en utilisant que $|\uparrow\rangle, |\downarrow\rangle$ sont des vecteurs propres de σ_z avec valeurs propres $+1$ et -1).

$$\begin{aligned} e^{-\frac{it}{\hbar}\mathcal{H}}|\psi_0\rangle &= e^{-itJ\sigma_1^z\otimes\sigma_2^z}\cdot\frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) \\ &= \frac{1}{2}\left(e^{-itJ}|\uparrow\uparrow\rangle - e^{itJ}|\uparrow\downarrow\rangle + e^{itJ}|\downarrow\uparrow\rangle - e^{-itJ}|\downarrow\downarrow\rangle\right) \\ &= \frac{e^{-itJ}}{2}\left(|\uparrow\uparrow\rangle - e^{2itJ}|\uparrow\downarrow\rangle + e^{2itJ}|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right). \end{aligned}$$

a) Pour $t = \frac{\pi}{4J}$ on a $e^{2itJ} = e^{\frac{i\pi}{2}} = i$
$\Rightarrow |\psi_t\rangle = \frac{e^{-\frac{i\pi}{4}}}{2}\left(|\uparrow\uparrow\rangle - i\,|\uparrow\downarrow\rangle + i\,|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$.

b) Supposons que l'état puisse s'écrire
$(\alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle)\otimes(\gamma\,|\uparrow\rangle + \delta\,|\downarrow\rangle) = \alpha\gamma\,|\uparrow\uparrow\rangle + \alpha\delta\,|\uparrow\downarrow\rangle + \beta\gamma\,|\downarrow\uparrow\rangle + \beta\delta\,|\downarrow\downarrow\rangle$,
alors $\alpha\gamma = 1,\ \alpha\delta = -i,\ \beta\gamma = i,\ \beta\delta = -1$.
On peut toujours poser $\alpha = 1$ (phase globale). Donc $\gamma = 1,\ \delta = -i,\ \beta = i$ et $\delta = i \Rightarrow$ contradiction sur δ. Vous pouvez aussi prendre n'importe quelle valeur fixée pour α pour montrer qu'une contradiction apparait.

c) At $t = \frac{\pi}{2J}$ with $e^{\pm itJ} = e^{\pm i\frac{\pi}{2}} = \pm i$,

$$\begin{aligned} |\psi_t\rangle &= \frac{1}{2}\left(-i\,|\uparrow\uparrow\rangle - i\,|\uparrow\downarrow\rangle + i\,|\downarrow\uparrow\rangle + i\,|\downarrow\downarrow\rangle\right) \\ &= \frac{-i}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \end{aligned}$$

is a product state. So another $\frac{\pi}{4J}$ time of evolution cancels the entanglement.

d) At $t = \frac{\pi}{J}$ with $e^{\pm itJ} = e^{\pm i\pi} = -1$,

$$\begin{aligned} |\psi_t\rangle &= \frac{1}{2}\left(-\,|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\right) \\ &= \frac{-1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right) \end{aligned}$$

is also a product state.

Complement on the Hamiltonian in matrix and Dirac notation.

1. **Matrix notation.** In the canonical bases, we have $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Using the tensor product rule one obtains that

$$\begin{aligned} \sigma_1^z\otimes\sigma_2^z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\otimes\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

thus the Hamiltonian is

$$\mathcal{H} = \begin{pmatrix} \hbar J & 0 & 0 & 0 \\ 0 & -\hbar J & 0 & 0 \\ 0 & 0 & -\hbar J & 0 \\ 0 & 0 & 0 & \hbar J \end{pmatrix}.$$

2. **Dirac notation.** In the bra-ket formalism one has $\sigma_z = |{\uparrow}\rangle\langle{\uparrow}| - |{\downarrow}\rangle\langle{\downarrow}|$, thus

$$\begin{aligned} \sigma_1^z \otimes \sigma_2^z &= (|{\uparrow}\rangle\langle{\uparrow}| - |{\downarrow}\rangle\langle{\downarrow}|) \otimes (|{\uparrow}\rangle\langle{\uparrow}| - |{\downarrow}\rangle\langle{\downarrow}|) \\ &= |{\uparrow\uparrow}\rangle\langle{\uparrow\uparrow}| - |{\uparrow\downarrow}\rangle\langle{\uparrow\downarrow}| - |{\downarrow\uparrow}\rangle\langle{\downarrow\uparrow}| + |{\downarrow\downarrow}\rangle\langle{\downarrow\downarrow}| . \end{aligned}$$

Therefore, we have

$$\mathcal{H} = \hbar J(|{\uparrow\uparrow}\rangle\langle{\uparrow\uparrow}| - |{\uparrow\downarrow}\rangle\langle{\uparrow\downarrow}| - |{\downarrow\uparrow}\rangle\langle{\downarrow\uparrow}| + |{\downarrow\downarrow}\rangle\langle{\downarrow\downarrow}|).$$

3. **Connection between matrix and Dirac notations.** Notice that to verify this one can use

$$|{\uparrow}\rangle\langle{\uparrow}| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which implies that

$$(|{\uparrow}\rangle\langle{\uparrow}|) \otimes (|{\uparrow}\rangle\langle{\uparrow}|) = |{\uparrow\uparrow}\rangle\langle{\uparrow\uparrow}| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly one can show that

$$|{\uparrow\downarrow}\rangle\langle{\uparrow\downarrow}| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$|{\downarrow\uparrow}\rangle\langle{\downarrow\uparrow}| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$|{\downarrow\downarrow}\rangle\langle{\downarrow\downarrow}| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. **Eigenvalues and eigenvectors.** One can see that the eigen-values are $\hbar J$ corresponding to the eigenvectors $|{\uparrow\uparrow}\rangle, |{\downarrow\downarrow}\rangle$ and $-\hbar J$ corresponding to the eigenvectors $|{\uparrow\downarrow}\rangle, |{\downarrow\uparrow}\rangle$.