Midterm

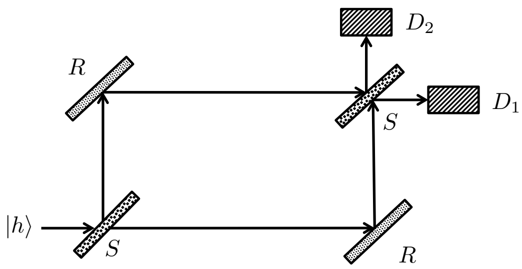
Nom: .... **Prénom:** .... **Section:** ....

- Vous pouvez répondre aux questions en Francais ou en Anglais.
- Ecrivez votre nom sur chaque feuille double, rendez la donnée et tous les brouillons SVP.
- Durée: 8h15-10h00.
- Les formules suivantes de trigonométrie peuvent être utiles:

$$\cos^2\alpha + \sin^2\alpha = 1, \quad \cos^2\alpha - \sin^2\alpha = \cos(2\alpha), \quad 2\cos\alpha\sin\alpha = \sin(2\alpha).$$

- Vous avez le droit de faire les calculs en notation de Dirac ou en composantes si vous préférez (on prend tout ce qui est juste).

Exercise 1 Interferometer with imperfect semi-transparent mirrors



We consider the interferometer constituted of two semi-transparent and two reflecting mirrors. We assume the Hilbert space of the photon is $\mathcal{H} = \mathbb{C}^2 = \left\{ |\Psi\rangle = \alpha|h\rangle + \beta|v\rangle, |\alpha|^2 + |\beta|^2 = 1 \right\}$ where $|h\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|v\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The incoming photon on the left is in state $|h\rangle$. The semi-transparent mirrors have "imperfections" and modeled by the matrix

$$S = \cos\gamma\,|h\rangle\langle h| + \sin\gamma\,|h\rangle\langle v| + \sin\gamma\,|v\rangle\langle h| - \cos\gamma\,|v\rangle\langle v| = \begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix}$$

The perfectly reflecting mirrors are modeled by

$$R = |h\rangle\langle v| + |v\rangle\langle h| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(a) Compute the state after the second semi-transparent mirrors (just before the detectors).

Answer: In Dirac notation: The state after the second semi-transparent mirrors is

$$\begin{aligned} SRS\,|h\rangle &= SR(\cos\gamma\,|h\rangle + \sin\gamma\,|v\rangle) \\ &= (\cos\gamma)SR\,|h\rangle + (\sin\gamma)SR\,|v\rangle \\ &= (\cos\gamma)S\,|v\rangle + (\sin\gamma)S\,|h\rangle \\ &= \cos\gamma(\sin\gamma\,|h\rangle - \cos\gamma\,|v\rangle) + \sin\gamma(\cos\gamma\,|h\rangle + \sin\gamma\,|v\rangle) \\ &= \sin(2\gamma)\,|h\rangle - \cos(2\gamma)\,|v\rangle \end{aligned}$$

Answer: In matrix component notation: The state after the second semi-transparent mirrors is

$$\begin{aligned} SRS\,|h\rangle &= \begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix} \\ &= \begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix} \begin{pmatrix} \sin\gamma \\ \cos\gamma \end{pmatrix} \\ &= \begin{pmatrix} \sin 2\gamma \\ -\cos 2\gamma \end{pmatrix} \end{aligned}$$

(b) Compute the probabilities of detecting the photon in the detectors $P(D_1)$ and $P(D_2)$.

Answer: Using the measurement postulate (in the simple form of the Born rule), we have

$$\begin{aligned} P(D_1) &= |\,\langle h|\,SRS\,|h\rangle\,|^2 = (\sin 2\gamma)^2 \\ P(D_2) &= |\,\langle v|\,SRS\,|h\rangle\,|^2 = (\cos 2\gamma)^2 \end{aligned}$$

(c) What are the probabilities for $\gamma = \frac{\pi}{4}$? For $\gamma = \frac{\pi}{8}$? (express the probabilities as simple rational fractions using $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$).

Answer: When $\gamma = \pi/4$, we have $P(D_1) = 1$ and $P(D_2) = 0$. When $\gamma = \pi/8$, we have $P(D_1) = P(D_2) = 1/2$.

(d) Prove that S and R are unitary.

Answer:

$$S^\dagger S = SS^\dagger = \begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix}^2 = \begin{pmatrix} \cos^2\gamma + \sin^2\gamma & \cos\gamma\sin\gamma - \sin\gamma\cos\gamma \\ \cos\gamma\sin\gamma - \sin\gamma\cos\gamma & \cos^2\gamma + \sin^2\gamma \end{pmatrix} = I$$

$$R^\dagger R = RR^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = I$$

Exercise 2 Quantum key distribution when Alice has an imperfect encoder

Consider the BB84 protocol with the following modification.

- Alice generates two random sequences $x_1 \dots x_N$ with $P(x_i = 0) = P(x_i = 1) = 1/2$ and $e_1 \dots e_N$ with $P(e_i = 0) = P(e_i = 1) = 1/2$.
- For $e_i = 0$ she encodes the i-th qubit in the state $|x_i\rangle$. For $e_i = 1$ she encodes the i-th qubit in the state $U|x_i\rangle$, where

$$U = \cos\frac{\pi}{8}|0\rangle\langle 0| + \sin\frac{\pi}{8}|0\rangle\langle 1| + \sin\frac{\pi}{8}|1\rangle\langle 0| - \cos\frac{\pi}{8}|1\rangle\langle 1|.$$

- Bob generates a random sequence $d_1 \dots d_N$ with $P(d_i = 0) = P(d_i = 1) = 1/2$.
  – If $d_i = 0$, he measures the received qubit in the basis $\{|0\rangle, |1\rangle\}$ and registers $y_i = 0$ (respectively $y_i = 1$) if the outcome is $|0\rangle$ (respectively $|1\rangle$).
  – If $d_i = 1$, he measures the received qubit in the basis $\{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$ and registers $y_i = 0$ (respectively $y_i = 1$) if the outcome is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ (respectively $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$).

(a) Compute the probabilities $P(x_i = y_i | e_i = d_i = 0)$ and $P(x_i = y_i | e_i = d_i = 1)$.
Hint: For any event A you have:

$$\begin{aligned} P(x_i = y_i|A) =& P(x_i = y_i = 0|A, x_i = 0)P(x_i = 0) \\ &+ P(x_i = y_i = 1|A, x_i = 1)P(x_i = 1) \end{aligned}$$

Answer: Let A_0 be the event that $e_i = d_i = 0$ and A_1 be the event that $e_i = d_i = 1$. When $e_i = d_i = 0$, the probability is

$$\begin{aligned} P(x_i = y_i|e_i = d_i = 0) &= P(x_i = 0) \cdot P(x_i = y_i = 0|A_0, x_i = 0) \\ &\quad + P(x_i = 1) \cdot P(x_i = y_i = 1|A_0, x_i = 1) \\ &= \frac{1}{2} \cdot |\langle 0|0\rangle|^2 + \frac{1}{2} \cdot |\langle 1|1\rangle|^2 \\ &= 1. \end{aligned}$$

When $e_i = d_i = 1$, we compute

$$\begin{aligned} P(x_i = 0) \cdot P(y_i = 0|A_1, x_i = 0) &= \frac{1}{2} \cdot \left|\frac{(\langle 0| + \langle 1|)}{\sqrt{2}} U|0\rangle\right|^2 \\ &= \frac{1}{4}\left|(\langle 0| + \langle 1|)(\cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle)\right|^2 \\ &= \frac{1}{4}\left|\cos\frac{\pi}{8} + \sin\frac{\pi}{8}\right|^2 \\ &= \frac{1}{4}\left(1 + \sin\frac{\pi}{4}\right) \\ &= \frac{2 + \sqrt{2}}{8} \end{aligned}$$

and similarly

$$\begin{aligned}
P(x_i = 1) \cdot P(y_i = 1 | A_1, x_i = 1) &= \frac{1}{2} \cdot \Big| \frac{(\langle 0| - \langle 1|)}{\sqrt{2}} U |1\rangle \Big|^2 \\
&= \frac{1}{4} \Big| (\langle 0| - \langle 1|)(\sin\frac{\pi}{8} |0\rangle - \cos\frac{\pi}{8} |1\rangle) \Big|^2 \\
&= \frac{1}{4} \Big| \cos\frac{\pi}{8} + \sin\frac{\pi}{8} \Big|^2 \\
&= \frac{2+\sqrt{2}}{8}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
P(x_i = y_i | e_i = d_i = 1) &= P(x_i = 0) \cdot P(x_i = y_i = 0 | A_1, x_i = 0) \\
&\quad + P(x_i = 1) \cdot P(x_i = y_i = 1 | A_1, x_i = 1) \\
&= \frac{2+\sqrt{2}}{8} + \frac{2+\sqrt{2}}{8} \\
&= \frac{2+\sqrt{2}}{4}.
\end{aligned}$$

(b) Deduce the probability $P(x_i = y_i | e_i = d_i)$. How does it compare to the corresponding probability in the perfect BB84 protocol?

Answer: In the perfect BB84 protocol, we have $P(x_i = y_i | e_i = d_i) = 1$. However, when the protocol is replaced by the "imperfect" encoder here, we have

$$\begin{aligned}
P(x_i = y_i | e_i = d_i) &= P(e_i = 0) \cdot P(x_i = y_i | A_0, e_i = 0) \\
&\quad + P(e_i = 1) \cdot P(x_i = y_i | A_1, e_i = 1) \\
&= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{2+\sqrt{2}}{4} \\
&= \frac{6+\sqrt{2}}{8} < 1.
\end{aligned}$$

Exercise 3 Entanglement for two quantum bits

The state of two quantum bits has a general form

$$|\Psi\rangle = a_{00} |0\rangle \otimes |0\rangle + a_{01} |0\rangle \otimes |1\rangle + a_{10} |1\rangle \otimes |0\rangle + a_{11} |1\rangle \otimes |1\rangle,$$

where $\{|0\rangle, |1\rangle\}$ is the canonical orthonormal basis of $\mathbb{C}^2$.

(a) Show that $|\Psi\rangle$ is a product state in the sense that *it admits a "product form"* $(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$ if and only if $\det A = 0$, where A is the matrix

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}.$$

Hint: Recall that $\det A = a_{00}a_{11} - a_{10}a_{01}$.

Answer: If $|\Psi\rangle$ is a product state of the form

$$\begin{aligned}|\Psi\rangle &= (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \\ &= ac|0\rangle \otimes |0\rangle + ad|0\rangle \otimes |1\rangle + bc|1\rangle \otimes |0\rangle + bd|1\rangle \otimes |1\rangle,\end{aligned}$$

then $\det A = \begin{vmatrix} ac & ad \\ bc & bd \end{vmatrix} = abcd - abcd = 0$.

On the other hand, to show that $\det A = 0$ implies that $|\Psi\rangle$ is a product sate, we need to propose a solution of (a, b, c, d) that satisfies

$$\begin{aligned} ac &= a_{00}, \\ ad &= a_{01}, \\ bc &= a_{10}, \\ bd &= a_{11}. \end{aligned}$$

Suppose $a_{00} \neq 0$. Then $\det A = 0$ implies $a_{11} = a_{01}a_{10}/a_{00}$ and this allows us to write

$$\begin{aligned}|\Psi\rangle &= \frac{1}{a_{00}} \left(a_{00}^2 |0\rangle \otimes |0\rangle + a_{00}a_{01}|0\rangle \otimes |1\rangle + a_{00}a_{10}|1\rangle \otimes |0\rangle + a_{01}a_{10}|1\rangle \otimes |1\rangle \right) \\ &= \frac{1}{a_{00}}(a_{00}|0\rangle + a_{10}|1\rangle) \otimes (a_{00}|0\rangle + a_{01}|1\rangle),\end{aligned}$$

which shows that $|\Psi\rangle$ is product state.

Suppose $a_{00} = 0$. Then $\det A = 0$ implies either $a_{01} = 0$ or $a_{10} = 0$. For $a_{00} = a_{01} = 0$,

$$|\Psi\rangle = a_{10}|1\rangle \otimes |0\rangle + a_{11}|1\rangle \otimes |1\rangle = |1\rangle \otimes (a_{10}|0\rangle + a_{11}|1\rangle).$$

For $a_{00} = a_{10} = 0$,

$$|\Psi\rangle = a_{01}|0\rangle \otimes |1\rangle + a_{11}|1\rangle \otimes |1\rangle = (a_{01}|0\rangle + a_{11}|1\rangle) \otimes |1\rangle.$$

(b) Using part (a), show that

$$|\Psi_1\rangle = \frac{1}{\sqrt{3}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + i|1\rangle \otimes |0\rangle)$$

is entangled (in french "intriquer").

Answer: For $|\Psi_1\rangle$, we have $\det A = \begin{vmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{i}{3} & \frac{1}{\sqrt{3}} \end{vmatrix} = \frac{1}{3}$. Hence it is entangled and does not have a product form.