Midterm

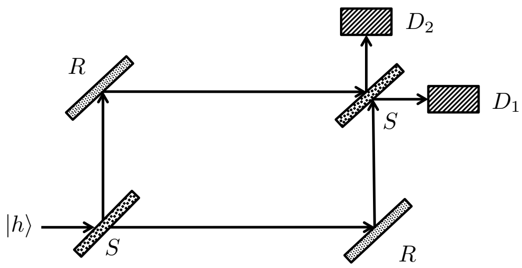
Nom: .... **Prénom:** .... **Section:** ....

- Vous pouvez répondre aux questions en Francais ou en Anglais.
- Ecrivez votre nom sur chaque feuille double, rendez la donnée et tous les brouillons SVP.
- Durée: 8h15-10h00.
- Les formules suivantes de trigonométrie peuvent être utiles:

$$\cos^2\alpha + \sin^2\alpha = 1, \quad \cos^2\alpha - \sin^2\alpha = \cos(2\alpha), \quad 2\cos\alpha\sin\alpha = \sin(2\alpha).$$

- Vous avez le droit de faire les calculs en notation de Dirac ou en composantes si vous préférez (on prend tout ce qui est juste).

Exercise 1 Interferometer with imperfect semi-transparent mirrors



We consider the interferometer constituted of two semi-transparent and two reflecting mirrors. We assume the Hilbert space of the photon is $\mathcal{H} = \mathbb{C}^2 = \{ |\Psi\rangle = \alpha|h\rangle + \beta|v\rangle, |\alpha|^2 + |\beta|^2 = 1 \}$ where $|h\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|v\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. The incoming photon on the left is in state $|h\rangle$. The semi-transparent mirrors have "imperfections" and modeled by the matrix

$$S = \cos\gamma\,|h\rangle\langle h| + \sin\gamma\,|h\rangle\langle v| + \sin\gamma\,|v\rangle\langle h| - \cos\gamma\,|v\rangle\langle v| = \begin{pmatrix}\cos\gamma & \sin\gamma\\ \sin\gamma & -\cos\gamma\end{pmatrix}$$

The perfectly reflecting mirrors are modeled by

$$R = |h\rangle\langle v| + |v\rangle\langle h| = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

(a) Compute the state after the second semi-transparent mirrors (just before the detectors).

(b) Compute the probabilities of detecting the photon in the detectors $P(D_1)$ and $P(D_2)$.

(c) What are the probabilities for $\gamma = \frac{\pi}{4}$? For $\gamma = \frac{\pi}{8}$? (express the probabilities as simple rational fractions using $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$).

(d) Prove that S and R are unitary.

Exercise 2 Quantum key distribution when Alice has an imperfect encoder

Consider the BB84 protocol with the following modification.

- Alice generates two random sequences $x_1 \dots x_N$ with $P(x_i = 0) = P(x_i = 1) = 1/2$ and $e_1 \dots e_N$ with $P(e_i = 0) = P(e_i = 1) = 1/2$.
- For $e_i = 0$ she encodes the i-th qubit in the state $|x_i\rangle$. For $e_i = 1$ she encodes the i-th qubit in the state $U|x_i\rangle$, where
$$U = \cos\frac{\pi}{8}|0\rangle\langle 0| + \sin\frac{\pi}{8}|0\rangle\langle 1| + \sin\frac{\pi}{8}|1\rangle\langle 0| - \cos\frac{\pi}{8}|1\rangle\langle 1|.$$
- Bob generates a random sequence $d_1 \dots d_N$ with $P(d_i = 0) = P(d_i = 1) = 1/2$.
  – If $d_i = 0$, he measures the received qubit in the basis $\{|0\rangle, |1\rangle\}$ and registers $y_i = 0$ (respectively $y_i = 1$) if the outcome is $|0\rangle$ (respectively $|1\rangle$).
  – If $d_i = 1$, he measures the received qubit in the basis $\{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$ and registers $y_i = 0$ (respectively $y_i = 1$) if the outcome is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ (respectively $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$).

(a) Compute the probabilities $P(x_i = y_i | e_i = d_i = 0)$ and $P(x_i = y_i | e_i = d_i = 1)$.
Hint: For any event A you have:
$$\begin{aligned} P(x_i = y_i | A) =& P(x_i = y_i = 0 | A, x_i = 0) P(x_i = 0) \\ &+ P(x_i = y_i = 1 | A, x_i = 1) P(x_i = 1) \end{aligned}$$

(b) Deduce the probability $P(x_i = y_i | e_i = d_i)$. How does it compare to the corresponding probability in the perfect BB84 protocol?

Exercise 3 Entanglement for two quantum bits

The state of two quantum bits has a general form
$$|\Psi\rangle = a_{00}|0\rangle \otimes |0\rangle + a_{01}|0\rangle \otimes |1\rangle + a_{10}|1\rangle \otimes |0\rangle + a_{11}|1\rangle \otimes |1\rangle,$$
where $\{|0\rangle, |1\rangle\}$ is the canonical orthonormal basis of $\mathbb{C}^2$.

(a) Show that $|\Psi\rangle$ is a product state in the sense that *it admits a "product form"* $(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$ if and only if $\det A = 0$, where A is the matrix
$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}.$$
Hint: Recall that $\det A = a_{00}a_{11} - a_{10}a_{01}$.

(b) Using part (a), show that
$$|\Psi_1\rangle = \frac{1}{\sqrt{3}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + i|1\rangle \otimes |0\rangle)$$
is entangled (in french "intriquer").