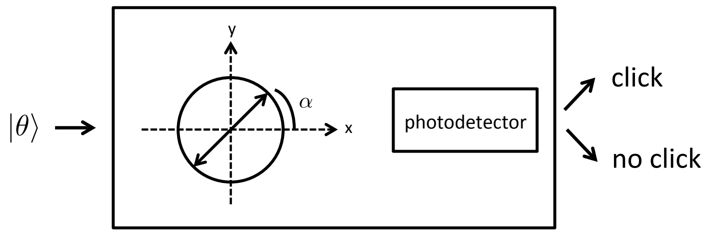
Homework 4: 12 October 2017
Traitement Quantique de l'Information

**Exercise 1** *Polarization observable and measurement principle*

Consider the "measurement apparatus" (in the below figure) constituted of "an analyzer and a detector". The incoming (initial) state of the photon is linearly polarized:

$$|\theta\rangle = \cos\theta\,|x\rangle + \sin\theta\,|y\rangle\,.$$




When the photodetector clicks we record $+1$ and when it does not click we record $-1$. Thus the "polarization observable" is represented by the $2\times 2$ matrix

$$P_\alpha = (+1)\,|\alpha\rangle\,\langle\alpha| + (-1)\,|\alpha_\perp\rangle\,\langle\alpha_\perp|$$

where $|\alpha\rangle = \cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle$ and $|\alpha_\perp\rangle = -\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle$ are the two vectors of the measurement basis. Note that the two orthogonal projectors of the measurement basis are $\Pi_\alpha = |\alpha\rangle\,\langle\alpha|$ and $\Pi_{\alpha_\perp} = |\alpha_\perp\rangle\,\langle\alpha_\perp|$.

1) Show that $\Pi_\alpha^2 = \Pi_\alpha$, $\Pi_{\alpha_\perp}^2 = \Pi_{\alpha_\perp}$ and $\Pi_\alpha\Pi_{\alpha_\perp} = \Pi_{\alpha_\perp}\Pi_\alpha = 0$.

2) Check the following formulas:

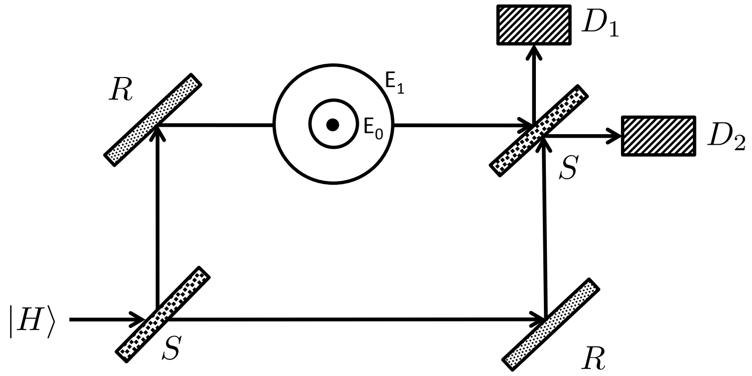
$$|\,\langle\theta|\alpha\rangle\,|^2 = \langle\theta|\,\Pi_\alpha\,|\theta\rangle\,,$$
$$|\,\langle\theta|\alpha_\perp\rangle\,|^2 = \langle\theta|\,\Pi_{\alpha_\perp}\,|\theta\rangle$$

3) Let $p = \pm 1$ the random variable corresponding to the event click / no-click of the detector. Express $\mathrm{Prob}(p = \pm 1)$ with simple trigonometric functions and check that the two probabilities sum to one.

4) Deduce from 3) $\mathbb{E}(p)$ and $\mathrm{Var}(p)$ and check that you find the same expressions by directly computing $\langle\theta|\,P_\alpha\,|\theta\rangle$ and $\langle\theta|\,P_\alpha^2\,|\theta\rangle - \langle\theta|\,P_\alpha\,|\theta\rangle^2$ in Dirac notation.

**Exercise 2** *Interferometer with an atom on the upper path*

Consider the following set-up where an atom may absorb the photon on the upper arm of the interferometer.




The Hilbert space of the photon is here $\mathbb{C}^3$ with basis states

$$|H\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, |V\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, |\text{abs}\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

the semi-transparent and reflecting mirrors are modeled by the unitary matrices

$$S = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad R = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

and the "absorption-reemission" process[1] is modeled by the unitary matrix

$$A = \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$$

Note that in Dirac notation

$$A = |H\rangle\langle\text{abs}| + |V\rangle\langle V| + |\text{abs}\rangle\langle H|$$

This models three possible transitions: $A|H\rangle = |\text{abs}\rangle$ (absorption); $A|\text{abs}\rangle = |H\rangle$ (emission); and $A|V\rangle = |V\rangle$ (nothing happens).

1) Write down all matrices in Dirac notation and then compute the unitary operator $U = SARS$ representing the total evolution process of this interferometer.

2) Given that the initial state is $|H\rangle$, what is the state after the second semi-transparent mirror? What are the probabilities of the following three events: click in $D_1$; or click in $D_2$; or no clicks in $D_1$ nor $D_2$? Verify the probabilities sum to to 1.

[1]On the picture $E_0$ and $E_1$ are two energy levels of the atom corresponding to ground state and excited state; but you can ignore this aspect in this problem.

3) Suppose the photon-atom interaction is not absorption-reemission but some other process modeled by a matrix. Which of the two following matrices would be legitimate in QM?

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and why?

**Exercise 3** *Entanglement for two quantum bits*

Consider two quantum bits in the state:

$$|\Psi\rangle = \frac{1}{\sqrt{3}} \left( |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + i\, |1\rangle \otimes |0\rangle \right)$$

where $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the canonical orthonormal basis of $\mathbb{C}^2$.

1) Write this state in 4-component-form as a column vector (use the conventions of class for the tensor product).

2) Prove that this state is "entangled" (in french "intriquer") in the sense that *it is impossible to express it in "product form"*

$$(\alpha\, |0\rangle + \beta\, |1\rangle) \otimes (\gamma\, |0\rangle + \delta\, |1\rangle)$$

for any $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.