Exercise 1 Polarization observable and measurement principle

Consider the “measurement apparatus” (in the below figure) constituted of “an analyzer and a detector”. The incoming (initial) state of the photon is linearly polarized:

$$|\theta\rangle = \cos \theta |x\rangle + \sin \theta |y\rangle.$$  

![Diagram of polarization observable and measurement principle](image)

When the photodetector clicks we record $+1$ and when it does not click we record $-1$. Thus the “polarization observable” is represented by the $2 \times 2$ matrix

$$P_\alpha = (+1) |\alpha\rangle \langle \alpha| + (-1) |\alpha\rangle \langle \alpha\rangle$$

where $|\alpha\rangle = \cos \alpha |x\rangle + \sin \alpha |y\rangle$ and $|\alpha\rangle = -\sin \alpha |x\rangle + \cos \alpha |y\rangle$ are the two vectors of the measurement basis. Note that the two orthogonal projectors of the measurement basis are $\Pi_\alpha = |\alpha\rangle \langle \alpha|$ and $\Pi_{\alpha\perp} = |\alpha\rangle \langle \alpha\rangle$.

1) Show that $\Pi_\alpha^2 = \Pi_\alpha$, $\Pi_{\alpha\perp}^2 = \Pi_{\alpha\perp}$ and $\Pi_\alpha \Pi_{\alpha\perp} = \Pi_{\alpha\perp} \Pi_\alpha = 0$.

2) Check the following formulas:

$$|\langle \theta|\alpha\rangle|^2 = \langle \theta| \Pi_\alpha |\theta\rangle,$$

$$|\langle \theta|\alpha\rangle|^2 = \langle \theta| \Pi_{\alpha\perp} |\theta\rangle.$$  

3) Let $p = \pm 1$ the random variable corresponding to the event click / no-click of the detector. Express $\text{Prob}(p = \pm 1)$ with simple trigonometric functions and check that the two probabilities sum to one.

4) Deduce from 3) $E(p)$ and $\text{Var}(p)$ and check that you find the same expressions by directly computing $\langle \theta| P_\alpha |\theta\rangle$ and $\langle \theta| P_\alpha^2 |\theta\rangle - \langle \theta| P_\alpha |\theta\rangle^2$ in Dirac notation.
Exercise 2 Interferometer with an atom on the upper path

Consider the following set-up where an atom may absorb the photon on the upper arm of the interferometer.

\[
\begin{align*}
\left| H \right\rangle & \rightarrow S \rightarrow R \rightarrow D_1 \\
\left| V \right\rangle & \rightarrow S \rightarrow R \rightarrow D_2 \\
\left| \text{abs} \right\rangle & \rightarrow S \rightarrow R \rightarrow D_1
\end{align*}
\]

The Hilbert space of the photon is here \( \mathbb{C}^3 \) with basis states

\[
\left| H \right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \left| V \right\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \left| \text{abs} \right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

the semi-transparent and reflecting mirrors are modeled by the unitary matrices

\[
S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

and the “absorption-reemission” process\(^1\) is modeled by the unitary matrix

\[
A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

Note that in Dirac notation

\[
A = \left| H \right\rangle \langle \text{abs} \right| + \left| V \right\rangle \langle V \right| + \left| \text{abs} \right\rangle \langle H \right|
\]

This models three possible transitions: \( A \left| H \right\rangle = \left| \text{abs} \right\rangle \) (absorption); \( A \left| \text{abs} \right\rangle = \left| H \right\rangle \) (emission); and \( A \left| V \right\rangle = \left| V \right\rangle \) (nothing happens).

1) Write down all matrices in Dirac notation and then compute the unitary operator \( U = SARS \) representing the total evolution process of this interferometer.

2) Given that the initial state is \( \left| H \right\rangle \), what is the state after the second semi-transparent mirror? What are the probabilities of the following three events: click in \( D_1 \); or click in \( D_2 \); or no clicks in \( D_1 \) nor \( D_2 \)? Verify the probabilities sum to to 1.

\(^1\)On the picture \( E_0 \) and \( E_1 \) are two energy levels of the atom corresponding to ground state and excited state; but you can ignore this aspect in this problem.
3) Suppose the photon-atom interaction is not absorption-reemission but some other process modeled by a matrix. Which of the two following matrices would be legitimate in QM?

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\text{ or }
\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix}
\]

and why?

**Exercise 3 Entanglement for two quantum bits**

Consider two quantum bits in the state:

\[
|\Psi\rangle = \frac{1}{\sqrt{3}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + i |1\rangle \otimes |0\rangle)
\]

where \(|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\) are the canonical orthonormal basis of \(\mathbb{C}^2\).

1) Write this state in 4-component-form as a column vector (use the conventions of class for the tensor product).

2) Prove that this state is “entangled” (in french “intriquer”) in the sense that it is impossible to express it in “product form”

\[
(\alpha |0\rangle + \beta |1\rangle) \otimes (\gamma |0\rangle + \delta |1\rangle)
\]

for any \(\alpha, \beta, \gamma, \delta \in \mathbb{C}\).