Exercise 1  
Orthogonal basis and measurement principle

Let \{ |x\rangle, |y\rangle \} an orthogonal basis of \( \mathbb{C}^2 \). This means that \( \langle x | x \rangle = \langle y | y \rangle = 1 \) and \( \langle x | y \rangle = \langle y | x \rangle = 0 \). Let \( |\alpha\rangle = \cos \alpha |x\rangle + \sin \alpha |y\rangle, |\alpha_\perp\rangle = -\sin \alpha |x\rangle + \cos \alpha |y\rangle, |R\rangle = \frac{1}{\sqrt{2}} (|x\rangle + i |y\rangle), |L\rangle = \frac{1}{\sqrt{2}} (|x\rangle - i |y\rangle) \).

1) Check that \{ |\alpha\rangle, |\alpha_\perp\rangle \} and \{ |R\rangle, |L\rangle \} are two orthogonal basis.

2) We measure the polarization with three different measurement apparatus. The first apparatus is modeled by the basis \{ |x\rangle, |y\rangle \}; the second one is modeled by the basis \{ |R\rangle, |L\rangle \}; and the third one by the basis \{ |\alpha\rangle, |\alpha_\perp\rangle \}. Let \( |\psi\rangle = \cos \theta |x\rangle + \sin \theta e^{i\varphi} |y\rangle \) be the polarized state of a photon just before the measurement. For each of the three experiments, give the (two) possible outgoing states just after the measurement and their corresponding probabilities of outcome.

Exercise 2  
Matrices in Dirac’s notation

We have seen that in Dirac notation the “ket” is \( \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha |x\rangle + \beta |y\rangle \) where \( |x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). The “bra” is \( (\gamma^*, \delta^*) = \gamma^* \langle x | + \delta^* \langle y | \). 

1) Verify that the “bracket” \( (\gamma^* \langle x | + \delta^* \langle y |) (\alpha |x\rangle + \beta |y\rangle) \) is equal to \( \gamma^* \alpha + \delta^* \beta \) (this is a complex number, so the bracket is the usual scalar product).

2) Compute the “ketbra” \( (\alpha |x\rangle + \beta |y\rangle) (\gamma^* |x\rangle + \delta^* |y\rangle) \) and write it down as a 2 \( \times \) 2 matrix.

3) Verify that a general matrix \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) is expressed as 
\[ A = a_{11} |x\rangle \langle x| + a_{12} |x\rangle \langle y| + a_{21} |y\rangle \langle x| + a_{22} |y\rangle \langle y| . \]

4) In the basis \{ |\alpha\rangle, |\alpha_\perp\rangle \} we have 
\[ A = \tilde{a}_{11} |\alpha\rangle \langle \alpha| + \tilde{a}_{12} |\alpha\rangle \langle \alpha_\perp| + \tilde{a}_{21} |\alpha_\perp\rangle \langle \alpha| + \tilde{a}_{22} |\alpha_\perp\rangle \langle \alpha_\perp| \].

Find \( |x\rangle \) and \( |y\rangle \) in terms of \( |\alpha\rangle \) and \( |\alpha_\perp\rangle \) and then \( \tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{21}, \tilde{a}_{22} \) in terms of \( a_{11}, a_{12}, a_{21}, a_{22} \).
Exercise 3 Interferometer revisited

Let \( S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \), \( R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \) be the two matrices representing a semi-transparent mirror and a perfectly reflecting mirror.

1) Express \( S \) and \( R \) in Dirac notation in the \(|H⟩ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(|V⟩ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\) basis.

2) Compute the product \( SRS \) in Dirac notation. Verify that you find the same with usual matrix rules.

3) Compute (in Dirac notation) the state \( SRS |H⟩ \) as well as the probabilities \( |⟨H| SRS |H⟩|^2 \) and \( |⟨V| SRS |H⟩|^2 \) (the two probabilities should sum to one).

Recall the picture of the experimental set-up in homework 2 and discuss your computations with your neighbor.

4) We introduce a “dephaser” described by \( D = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \) where \( \varphi_1 \) and \( \varphi_2 \) are two different phases (angles). We consider the operation \( SRDS \).

Make a picture of the experimental situation and discuss it with your neighbor. Compute \( SRDS |H⟩ \), and the probabilities \( |⟨H| SRDS |H⟩|^2 \) and \( |⟨V| SRDS |H⟩|^2 \).

Verify also that the matrix \( SRDS \) is unitary and relate this fact to the other fact that the two probabilities should sum to one.