Homework 3: 5 October 2017
Traitement Quantique de l'Information

Exercise 1 *Orthonormal basis and measurement principle*

Let $\{|x\rangle, |y\rangle\}$ an orthonormal basis of $\mathbb{C}^2$. This means that $\langle x|x\rangle = \langle y|y\rangle = 1$ and $\langle x|y\rangle = \langle y|x\rangle = 0$. Let $|\alpha\rangle = \cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle$, $|\alpha_\perp\rangle = -\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle$, $|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i\,|y\rangle)$, $|L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i\,|y\rangle)$.

1) Check that $\{|\alpha\rangle, |\alpha_\perp\rangle\}$ and $\{|R\rangle, |L\rangle\}$ are two orthonormal basis.

2) We measure the polarization with three different measurement apparatus. The first apparatus is modeled by the basis $\{|x\rangle, |y\rangle\}$; the second one is modeled by the basis $\{|R\rangle, |L\rangle\}$; and the third one by the basis $\{|\alpha\rangle, |\alpha_\perp\rangle\}$. Let

$$|\psi\rangle = \cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle$$

be the polarized state of a photon just before the measurement. For each of the three experiments, give the (two) possible outcoming states just after the measurement and their corresponding probabilities of outcome.

Exercise 2 *Matrices in Dirac's notation*

We have seen that in Dirac notation the "ket" is $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha\,|x\rangle + \beta\,|y\rangle$ where $|x\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}$ and $|y\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}$. The "bra" is $(\gamma^*, \delta^*) = \gamma^*\langle x| + \delta^*\langle y|$.

1) Verify that the "bracket" $(\gamma^*\langle x| + \delta^*\langle y|)(\alpha\,|x\rangle + \beta\,|y\rangle)$ is equal to $\gamma^*\alpha + \delta^*\beta$ (this is a complex number, so the bracket is the usual scalar product).

2) Compute the "ketbra" $(\alpha\,|x\rangle + \beta\,|y\rangle)(\gamma^*\langle x| + \delta^*\langle y|)$ and write it down as a 2×2 matrix.

3) Verify that a general matrix $A = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ is expressed as

$$A = a_{11}\,|x\rangle\langle x| + a_{12}\,|x\rangle\langle y| + a_{21}\,|y\rangle\langle x| + a_{22}\,|y\rangle\langle y|\,.$$

4) In the basis $\{|\alpha\rangle, |\alpha_\perp\rangle\}$ we have

$$A = \tilde{a}_{11}\,|\alpha\rangle\langle\alpha| + \tilde{a}_{12}\,|\alpha\rangle\langle\alpha_\perp| + \tilde{a}_{21}\,|\alpha_\perp\rangle\langle\alpha| + \tilde{a}_{22}\,|\alpha_\perp\rangle\langle\alpha_\perp|\,.$$

Find $|x\rangle$ and $|y\rangle$ in terms of $|\alpha\rangle$ and $|\alpha_\perp\rangle$ and then $\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{21}, \tilde{a}_{22}$ in terms of $a_{11}, a_{12}, a_{21}, a_{22}$.

Exercise 3 *Interferometer revisited*

Let $S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, $R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ be the two matrices representing a semi-transparent mirror and a perfectly reflecting mirror.

1) Express S and R in Dirac notation in the $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ basis.

2) Compute the product SRS in Dirac notation. Verify that you find the same with usual matrix rules.

3) Compute (in Dirac notation) the state $SRS\,|H\rangle$ as well as the probabilities $|\,\langle H|\, SRS\,|H\rangle\,|^2$ and $|\,\langle V|\, SRS\,|H\rangle\,|^2$ (the two probabilities should sum to one).

   Recall the picture of the experimental set-up in homework 2 and discuss your computations with your neighbor.

4) We introduce a "dephaser" described by $D = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix}$ where φ_1 and φ_2 are two different phases (angles). We consider the operation $SRDS$.

   Make a picture of the experimental situation and discuss it with your neighbor. Compute $SRDS\,|H\rangle$, and the probabilities $|\,\langle H|\, SRDS\,|H\rangle\,|^2$ and $|\,\langle V|\, SRDS\,|H\rangle\,|^2$.

   Verify also that the matrix $SRDS$ is unitary and relate this fact to the other fact that the two probabilities should sum to one.