ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 19a Principles of Digital Communications
Midterm exam Apr. 20, 2018

4 problems, each with 4 parts, each part worth 4 points
165 minutes
1 sheet (2 pages) of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

PROBLEM 1. In a binary hypothesis testing problem the observation $Y = (Y_1, \dots, Y_n)$, when $H = i$, is given by

$$Y_j = (-1)^i h_j + Z_j, \qquad j = 1, \dots, n, \qquad i = 0, 1.$$

Here $h = (h_1, \dots, h_n)$ is a fixed vector in $\mathbb{R}^n$, and $Z_1, \dots, Z_n$ are independent zero mean Gaussians, with $E[Z_j^2] = \sigma_j^2$.

For a given vector $\beta = (\beta_1, \dots, \beta_n)$ of coefficients, consider the statistic

$$T = \sum_j \beta_j Y_j = (-1)^i \Big(\sum_j \beta_j h_j \Big) + \sum_j \beta_j Z_j.$$

which consists of a 'signal' $S = (-1)^i \sum_j \beta_j h_j$ and 'noise' $N = \sum_j \beta_j Z_j$.

(a) Find n dimensional vectors u and v such that $E[N^2] = \|v\|^2$ and $S^2 = \langle u, v \rangle^2$.

(b) Show that

$$S^2 \le \Big(\sum_j h_j^2 / \sigma_j^2 \Big) E[N^2],$$

in other words, the 'signal-power to noise-power ratio' is upper bounded by $\sum_j h_j^2/\sigma_j^2$. [Hint: (a) and Cauchy–Schwarz.]

(c) Find an $\beta = (\beta_1, \dots, \beta_n)$ such that the signal to noise ratio equals the upper bound found in (b). [Hint: equality holds in Cauchy–Schwarz if $u = v$.]

(d) Show that for the choice of β as in (c), the statistic T is a sufficient statistic. [Hint: compute the log-likelihood-ratio of the observation Y.]

Problem 2. Suppose we have a communication system where the sent codeword $c_i = (c_{i1}, c_{i2})$ is first multiplied by a scalar gain A to form $\tilde{c}_i = (\tilde{c}_{i1}, \tilde{c}_{i2}) = (Ac_{i1}, Ac_{i2})$, $\tilde{c}_i$ is further corrupted by zero mean additive Gaussian noise $Z = (Z_1, Z_2)$ with covariance $\left[\begin{smallmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{smallmatrix}\right]$ to form the receiver's observation $Y = \tilde{c}_i + Z$.

The multiplicative factor A takes on the values $+1$ or -1 with equal probability, and is independent of the additive noise Z.

(a) For the constellation $c_0 = \sqrt{\frac{\mathcal{E}}{2}}(1, 1)$, $c_1 = \sqrt{\frac{\mathcal{E}}{2}}(1, -1)$ sketch the decision regions of the optimal rule, assuming that 0 and 1 are equally likely.

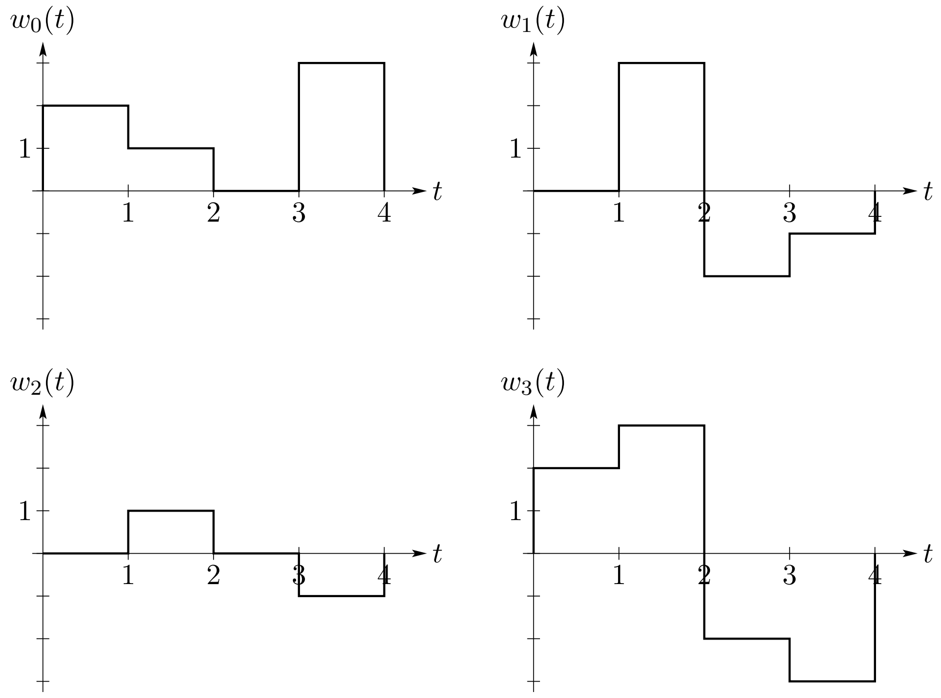
(b) Find the probability of error.

(The following parts can be solved without solving (a) and (b).)

(c) Suppose now that the value of A is revealed to the receiver, in addition to Y. Describe the optimal decision rule (i.e., sketch the decision regions for $A = +1$ and $A = -1$) and find the probability of error.

(d) Suppose we use the decision rule in (c), but the value of A revealed to the receiver may be incorrect with probability p. Find the probability of error.

Problem 3. We have designed a tranmitter to communicate one of four equally likely messages using the waveforms w_0, w_1, w_2, w_3 sketched below. The transmission takes place over an additive white Gaussian noise channel with noise spectral density $N_0/2$.



Our employer, PDC Inc., does not want to design a receiver completely from scratch, and asks us to process the received signal $R(t)$ only using the filter $h(t) = \mathbb{1}\{t \in [0, 1/2)\}$ (which was designed for a previous product). We are thus restricted to base our decision on $R_1, \ldots, R_n$ where $R_i = (R * h)(t_i)$, $i = 1, \ldots, n$, are the samples of the filter output at time instants $t_1, \ldots, t_n$. We are free to choose these time instants.

(a) Show that in spite of the restriction imposed by our employer, we can implement the MAP rule by appropriately choosing n, and $t_1, \ldots, t_n$.

(b) Find a translation of the signal set to minimize the average energy. Sketch the new signal set.

(c) Suppose the signal set we found in (b) is used for communication. Find the error probability.

(d) What is the error probability of the implementation in part (a).

Problem 4. Suppose $p(y)$ and $q(y)$ are two probability density functions on the real line. Let $B = \int \sqrt{p(y)q(y)}\,dy$.

Consider an m-ary hypothesis testing problem with equally likely hypotheses $\{1,\dots,m\}$ where the observation is $(Y_1,\dots,Y_m)$ is an m-dimensional real vector. When $H=i$, the observations $Y_1,\dots,Y_m$ are independent, Y_i has probability density q, and for $j \neq i$, Y_j had probability density p. For example, with $m=2$, we have $f_{Y|H}(y_1,y_2|1) = q(y_1)p(y_2)$, $f_{Y|H}(y_1,y_2|2) = p(y_1)q(y_2)$.

(a) For the case $m=2$, express the union–Bhattacharyya bound on the probablity of error in terms of B. [Hint: the answer is not B or $B/2$ or $2B$.]

(b) For general m, express the union–Bhattacharyya bound on the probability of error in terms of m and B.

(c) Suppose $p(y) = \exp(-y)\mathbb{1}\{y \geq 0\}$ and $q(y) = m\exp(-my)\mathbb{1}\{y \geq 0\}$. What value does the union–Bhattacharyya bound you found in (b) take as m goes to ∞?

(d) With the same supposition as in (c), consider the following decision rule: fix a threshold $t > 0$, and decide in favor of hypothesis i if (1) $Y_i > t$, and (2) for every $j \neq i$, $Y_j \leq t$.

When $H=1$, show that the error probability of this rule is upper bounded by

$$\Pr(Y_1 \leq t) + (m-1)\Pr(Y_2 > t) = 1 - \exp(-tm) + (m-1)\exp(-t).$$