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Handout 20a Principles of Digital Communications
Midterm solutions Apr. 23, 2018

PROBLEM 1.

(a) $E[N^2] = \sum_j \alpha_j^2 \sigma_j^2$, so we can choose $v = (\alpha_1\sigma_1, \dots, \alpha_n\sigma_n)$ to make $E[N^2] = \|v\|^2$. As $S = \pm\sum_j \alpha_j h_j$, choosing $u = (h_1/\sigma_1, \dots, h_n/\sigma_n)$ will ensure $S = \pm\langle u, v\rangle$.

(b) Cauchy–Schwarz inequality says $\langle u, v\rangle^2 \le \|u\|^2\|v\|^2$. As $\|u\|^2 = \sum_j h_j^2/\sigma_j^2$, the conclusion follows.

(c) Equality in Cauchy-Schwarz holds if u and v are scalar multiples of each other, in particular if $u = v$, or equivalently, choosing $h_j/\sigma_j = \alpha_j\sigma_j$, again equivalently, choosing $\alpha = (h_1/\sigma_1^2, \dots, h_n/\sigma_n^2)$.

(d) Note that

$$f_{Y|H}(y_1, \dots, y_n|0) = \prod_j (2\pi\sigma_j^2)^{-1/2} \exp\bigl[-(y_j - h_j)^2/(2\sigma_j^2)\bigr]$$
$$f_{Y|H}(y_1, \dots, y_n|1) = \prod_j (2\pi\sigma_j^2)^{-1/2} \exp\bigl[-(y_j + h_j)^2/(2\sigma_j^2)\bigr],$$

so the log-likelihood-ratio equals $-2\sum_j y_j h_j/\sigma^2$ which is $-2t(y)$. Since we know that the log-likelihood ratio is a sufficient statistic, we conclude that T is too.

PROBLEM 2.

(a) Note that

$$2(2\pi\sigma^2) f_{Y|H}(y|0) = \exp\bigl(-\|y - c_0\|^2/(2\sigma^2)\bigr) + \exp\bigl(-\|y + c_0\|^2/(2\sigma^2)\bigr)$$
$$2(2\pi\sigma^2) f_{Y|H}(y|1) = \exp\bigl(-\|y - c_1\|^2/(2\sigma^2)\bigr) + \exp\bigl(-\|y + c_1\|^2/(2\sigma^2)\bigr).$$

So (as c_0 and c_1 have the same norm) the decision rule is to decide 0 or 1 according to

$$\exp(\langle y, c_0\rangle/\sigma^2) + \exp(-\langle y, c_0\rangle/\sigma^2) \gtrless \exp(\langle y, c_1\rangle/\sigma^2) + \exp(-\langle y, c_1\rangle/\sigma^2).$$

As $\langle y, c_0\rangle = \sqrt{\mathcal{E}/2}(y_1 + y_2)$ and $\langle y, c_1\rangle = \sqrt{\mathcal{E}/2}(y_1 - y_2)$ the decision rule is, with $\tilde{y}_i = \sqrt{\mathcal{E}/2}\, y_i/\sigma^2$,

$$e^{\tilde{y}_1 + \tilde{y}_2} + e^{-\tilde{y}_1 - \tilde{y}_2} - e^{\tilde{y}_1 - \tilde{y}_2} - e^{\tilde{y}_2 - \tilde{y}_1} \gtrless 0.$$

The left hand side above equals $\bigl(e^{\tilde{y}_1} - e^{-\tilde{y}_1}\bigr)\bigl(e^{\tilde{y}_2} - e^{-\tilde{y}_2}\bigr)$, so we decide 0 if y_1 and y_2 have the same sign, and decide 1 otherwise.

(b) By the symmetry in the problem the error probality is the same for the two hypotheses, and is the same regardless of the value of A. So we can assume c_0 is sent and $A = 1$. We will make an error if either $Y_1 > 0$ and $Y_2 < 0$, or $Y_1 < 0$ and $Y_2 > 0$. Again by symmetry, these two events have the same probability, and thus the error probability is

$$2\Pr\bigl(Z_1 > -\sqrt{\mathcal{E}/2}\bigr)\Pr\bigl(Z_2 < -\sqrt{\mathcal{E}/2}\bigr) = 2Q\bigl(\sqrt{\mathcal{E}/(2\sigma^2)}\bigr)\bigl[1 - Q\bigl(\sqrt{\mathcal{E}/(2\sigma^2)}\bigr)\bigr].$$

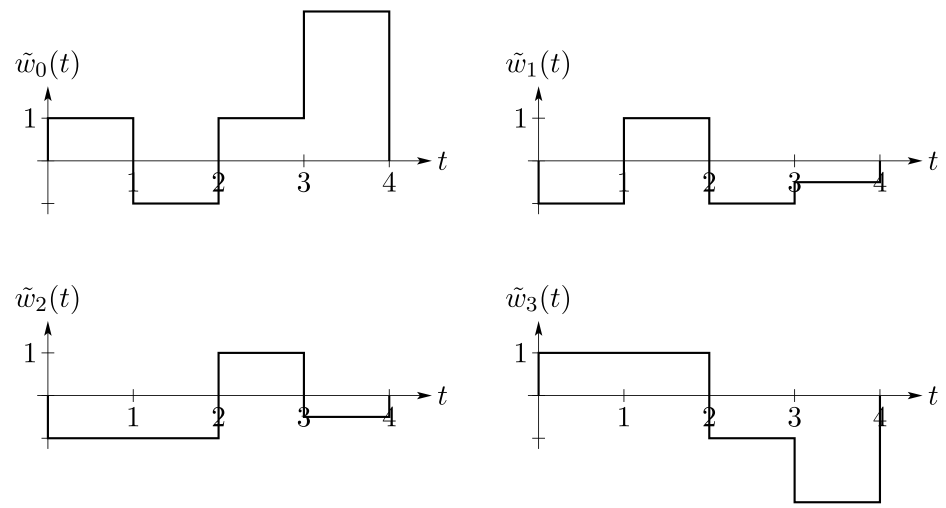
(c) If the receiver knows that $A = 1$, it knows that its observation is a noisy version $c_0 = \sqrt{\mathcal{E}/2}(1, 1)$ or $c_1 = \sqrt{\mathcal{E}/2}(1, -1)$. The MAP rule will decide 0 if $Y_2 > 0$ and 1 if $Y_2 < 0$. Similarly, if the receiver knows $A = -1$, it will decide 0 if $Y_2 < 0$ and 1 if $Y_2 > 0$. In either case the error probablity is $Q\big(\sqrt{\mathcal{E}/(2\sigma^2)}\big)$.

(d) If the receiver is told the correct value of A, then the error proabability is the value $q = Q\big(\sqrt{\mathcal{E}/(2\sigma^2)}\big)$ we found (c). If it is told the incorrect value of A the receiver's decision regions are flipped, and thus the error probabililty is $1-q$. Combining these, we find the error probability as $(1 - p)q + p(1 - q)$.

PROBLEM 3.

(a) An orthonormal basis for the four waveforms is given by $p(t), p(t-1), p(t-2), p(t-3)$ where $p(t)$ is the rectangular pulse $\mathbb{1}\{t \in [0, 1)\}$. The map decoder would work with Y_1, Y_2, Y_2, Y_3, the inner product of $R(t)$ with these basis functions. But these can be computed from the filter output as $Y_1 = R_1 + R_2$, $Y_2 = R_3 + R_4$, $Y_3 = R_5 + R_6$, $Y_4 = R_7 + R_8$ with R_i denoting the filter output at time $t_i = i/2$.

(b) The translate that gives the minimum energy should make the resulting constellation have average equal to 0. In the original constellation the average signal $[w_0(t)+w_1(t)+w_2(t) + w_3(t)]/4$ is a piecewise constant signal taking the values 1, 2, -1, $-1/2$ on the intervals $[0, 1)$, $[1, 2)$, $[2, 3)$, $[3, 4)$. The translated waveforms are obtained by subtracting this from each signal, and we obtain:



(c) The original waveworms were meant to be such that w_0 should have height -3 instead of $+3$. This would have made the translated waveforms form a two dimensional QAM constellation whose error probability is simple to compute. As asked, however, the translated signals have no such structure, and form a constellation in three dimensions. The error probability would be possible to compute numerially, but a closed from expression, if there is one, will be extremely ugly.

(d) Since isometric transforms do not change the probability of error the implementation in (a) has the same (but very ugly) error probability as in (c).

PROBLEM 4.

(a) For the case $m = 2$, the U–B bound is $\iint \sqrt{f_{Y|H}(y_1, y_2|1) f_{Y|H}(y_1, y_2|2)}\, dy_1 dy_2$. This evaluates as

$$\iint \sqrt{q(y_1)p(y_2)p(y_1)q(y_2)}\, dy_1 dy_2 = \int \sqrt{p(y_1)q(y_1)}\, dy_1 \int \sqrt{p(y_2)q(y_2)}\, dy_2 = B^2.$$

(b) For general m, we need to first evaluate

$$\int \cdots \int \sqrt{f_{Y|H}(y_1, \ldots, y_m|i) f_{Y|H}(y_1, \ldots, y_m|j)}\, dy_1 \cdots dy_m.$$

The integrand above equals $\sqrt{p(y_i)q(y_i)}\sqrt{p(y_j)q(y_j)} \prod_{k \neq i,j} p(y_k)$, so the integral splits as a product of integrals. The integrals for y_i and y_j both give B, the other integrals give 1. Thus the value of the above integral is B^2. The U–B bound thus evaluates to $(m-1)B^2$.

(c) For this p and q, $B = \sqrt{m} \int_0^\infty \exp(-(m+1)y/2)\, dy = 2\sqrt{m}/(m+1)$. The U–B bound in (b) is thus $4m^2/(m+1)^2$, which approaches 4 as m gets large — a rather useless bound on error probability.

(d) The decision will be wrong only if, either $Y_1 \leq t$, or, for some $j = 2, \ldots, m$, $Y_j > t$. The union bound on these m events gives

$$\Pr(Y_1 \leq t) + \sum_{j=2}^{m} \Pr(Y_j > t)$$

as an upper bound to the probability of error. But since since $Y_2, \ldots, Y_m$ have the same distribution p, $\Pr(Y_j > t) = P(Y_2 > t)$. Which yields the upper bound $\Pr(Y_1 \leq t) + (m-1)\Pr(Y_2 > t)$. As $\Pr(Y_2 > t) = \exp(-t)$ and $\Pr(Y_1 > t) = \exp(-tm)$ we obtain the desired conclusion.