ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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Handout 20b Principles of Digital Communications

Midterm solutions Apr. 23, 2018

PROBLEM 1.

(a) $E[N^2] = \sum_j \alpha_j^2 \sigma_j^2$, so we can choose $v = (\alpha_1\sigma_1, \dots, \alpha_n\sigma_n)$ to make $E[N^2] = \|v\|^2$. As $S = \pm \sum_j \alpha_j h_j$, choosing $u = (h_1/\sigma_1, \dots, h_n/\sigma_n)$ will ensure $S = \pm\langle u, v\rangle$.

(b) Cauchy–Schwarz inequality says $\langle u, v\rangle^2 \le \|u\|^2\|v\|^2$. As $\|u\|^2 = \sum_j h_j^2/\sigma_j^2$, the conclusion follows.

(c) Equality in Cauchy-Schwarz holds if u and v are scalar multiples of each other, in particular if $u = v$, or equivalently, choosing $h_j/\sigma_j = \alpha_j\sigma_j$, again equivalently, choosing $\alpha = (h_1/\sigma_1^2, \dots, h_n/\sigma_n^2)$.

(d) Note that

$$f_{Y|H}(y_1, \dots, y_n|0) = \prod_j (2\pi\sigma_j^2)^{-1/2} \exp\bigl[-(y_j - h_j)^2/(2\sigma_j^2)\bigr]$$
$$f_{Y|H}(y_1, \dots, y_n|1) = \prod_j (2\pi\sigma_j^2)^{-1/2} \exp\bigl[-(y_j + h_j)^2/(2\sigma_j^2)\bigr],$$

so the log-likelihood-ratio equals $-2\sum_j y_j h_j/\sigma^2$ which is $-2t(y)$. Since we know that the log-likelihood ratio is a sufficient statistic, we conclude that T is too.

PROBLEM 2.

(a) Note that

$$2(2\pi\sigma^2) f_{Y|H}(y|0) = \exp\bigl(-\|y - c_0\|^2/(2\sigma^2)\bigr) + \exp\bigl(-\|y + c_0\|^2/(2\sigma^2)\bigr)$$
$$2(2\pi\sigma^2) f_{Y|H}(y|1) = \exp\bigl(-\|y - c_1\|^2/(2\sigma^2)\bigr) + \exp\bigl(-\|y + c_1\|^2/(2\sigma^2)\bigr).$$

So (as c_0 and c_1 have the same norm) the decision rule is to decide 0 or 1 according to

$$\exp(\langle y, c_0\rangle/\sigma^2) + \exp(-\langle y, c_0\rangle/\sigma^2) \gtrless \exp(\langle y, c_1\rangle/\sigma^2) + \exp(-\langle y, c_1\rangle/\sigma^2).$$

As $\langle y, c_0\rangle = \sqrt{\mathcal{E}/2}(y_1 + y_2)$ and $\langle y, c_1\rangle = \sqrt{\mathcal{E}/2}(y_1 - y_2)$ the decision rule is, with $\tilde{y}_i = \sqrt{\mathcal{E}/2}\,y_i/\sigma^2$,

$$e^{\tilde{y}_1+\tilde{y}_2} + e^{-\tilde{y}_1-\tilde{y}_2} - e^{\tilde{y}_1-\tilde{y}_2} - e^{\tilde{y}_2-\tilde{y}_1} \gtrless 0.$$

The left hand side above equals $\bigl(e^{\tilde{y}_1} - e^{-\tilde{y}_1}\bigr)\bigl(e^{\tilde{y}_2} - e^{-\tilde{y}_2}\bigr)$, so we decide 0 if y_1 and y_2 have the same sign, and decide 1 otherwise.

(b) By the symmetry in the problem the error probality is the same for the two hypotheses, and is the same regardless of the value of A. So we can assume c_0 is sent and $A = 1$. We will make an error if either $Y_1 > 0$ and $Y_2 < 0$, or $Y_1 < 0$ and $Y_2 > 0$. Again by symmetry, these two events have the same probability, and thus the error probability is

$$2\Pr\bigl(Z_1 > -\sqrt{\mathcal{E}/2}\bigr)\Pr\bigl(Z_2 < -\sqrt{\mathcal{E}/2}\bigr) = 2Q\bigl(\sqrt{\mathcal{E}/(2\sigma^2)}\bigr)\bigl[1 - Q\bigl(\sqrt{\mathcal{E}/(2\sigma^2)}\bigr)\bigr].$$

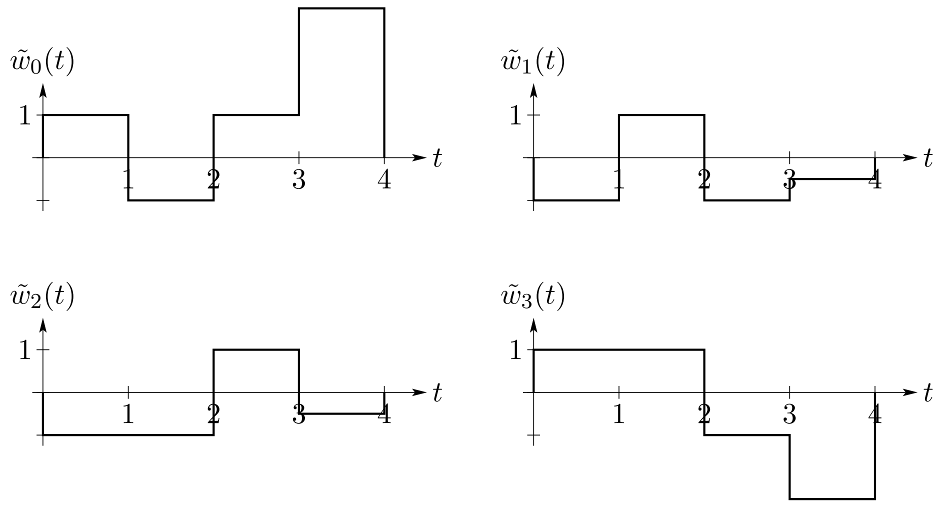
(c) If the receiver knows that $A = 1$, it knows that its observation is a noisy version $c_0 = \sqrt{\mathcal{E}/2}(1,1)$ or $c_1 = \sqrt{\mathcal{E}/2}(1,-1)$. The MAP rule will decide 0 if $Y_2 > 0$ and 1 if $Y_2 < 0$. Similarly, if the receiver knows $A = -1$, it will decide 0 if $Y_2 < 0$ and 1 if $Y_2 > 0$. In either case the error probablity is $Q\big(\sqrt{\mathcal{E}/(2\sigma^2)}\big)$.

(d) If the receiver is told the correct value of A, then the error proabability is the value $q = Q\big(\sqrt{\mathcal{E}/(2\sigma^2)}\big)$ we found (c). If it is told the incorrect value of A the receiver's decision regions are flipped, and thus the error probabililty is $1-q$. Combining these, we find the error probability as $(1-p)q + p(1-q)$.

Problem 3.

(a) An orthonormal basis for the four waveforms is given by $p(t)$, $p(t-1)$, $p(t-2)$, $p(t-3)$ where $p(t)$ is the rectangular pulse $\mathbb{1}\{t \in [0,1)\}$. The map decoder would work with Y_1, Y_2, Y_2, Y_3, the inner product of $R(t)$ with these basis functions. But these can be computed from the filter output as $Y_1 = R_1 + R_2$, $Y_2 = R_3 + R_4$, $Y_3 = R_5 + R_6$, $Y_4 = R_7 + R_8$ with R_i denoting the filter output at time $t_i = i/2$.

(b) The translate that gives the minimum energy should make the resulting constellation have average equal to 0. In the original constellation the average signal $[w_0(t) + w_1(t) + w_2(t) + w_3(t)]/4$ is a piecewise constant signal taking the values 1, 2, -1, -2 on the intervals $[0,1)$, $[1,2)$, $[2,3)$, $[3,4)$. The translated constellation is obtained by subtracting this from each signal, and we obtain the constellation:



(c) Note that the signal set is two dimentional, spanned by the ortonormal basis $\psi_1 = \tilde{w}_0/2$ and $\psi_2 = \tilde{w}_2/2$. In this basis the codewords are $c_0 = (2,0)$, $c_1 = (-2,0)$, $c_2 = (0,2)$, $c_3 = (0,-2)$. This is a 4-QAM constellation consisting of four corners of a square of side length $2\sqrt{2}$. The error probability is thus $q(2-q)$ with $q = Q(\sqrt{2}/\sqrt{N_0/2}) = Q(2/\sqrt{N_0})$.

(d) Since isometric transforms do not change the probability of error the implementation in (a) has the same error probability as we found in (c).

Problem 4.

(a) For the case $m = 2$, the U–B bound is $\iint \sqrt{f_{Y|H}(y_1,y_2|1) f_{Y|H}(y_1,y_2|2)}\, dy_1 dy_2$. This evaluates as

$$\iint \sqrt{q(y_1)p(y_2)p(y_1)q(y_2)}\, dy_1 dy_2 = \int \sqrt{p(y_1)q(y_1)}\, dy_1 \int \sqrt{p(y_2)q(y_2)}\, dy_2 = B^2.$$

(b) For general m, we need to first evaluate

$$\int \cdots \int \sqrt{f_{Y|H}(y_1, \ldots, y_m|i) f_{Y|H}(y_1, \ldots, y_m|j)}\, dy_1 \cdots dy_m.$$

The integrand above equals $\sqrt{p(y_i)q(y_i)}\sqrt{p(y_j)q(y_j)} \prod_{k \neq i,j} p(y_k)$, so the integral splits as a product of integrals. The integrals for y_i and y_j both give B, the other integrals give 1. Thus the value of the above integral is B^2. The U–B bound thus evaluates to $(m-1)B^2$.

(c) For this p and q, $B = m^{-1/2} \int_0^\infty \exp(-(1+m^{-1})y/2)\, dy = 2\sqrt{m}/(m+1)$. The U–B bound in (b) is thus $4m^2/(m+1)^2$, which approaches 4 as m gets large — a rather useless bound on error probability.

(d) The decision will be wrong only if, either $Y_1 \leq t$, or, for some $j = 2, \ldots, m$, $Y_j > t$. The union bound on these m events gives

$$\Pr(Y_1 \leq t) + \sum_{j=2}^{m} \Pr(Y_j > t)$$

as an upper bound to the probability of error. But since since Y_2, …, Y_m have the same distribution p, $\Pr(Y_j > t) = P(Y_2 > t)$. Which yields the upper bound $\Pr(Y_1 \leq t) + (m-1)\Pr(Y_2 > t)$. As $\Pr(Y_2 > t) = \exp(-t)$ and $\Pr(Y_1 > t) = \exp(-t/m)$ we obtain the desired conclusion.

Note that by choosing, for example, $t = \sqrt{m}$, this upper bound on error probability approches zero as m gets large, and shows that the U–B bound may be very pessimistic.