## ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 25

Principles of Digital Communications

Solutions to Problem Set 9

May 4, 2018

Solution 1.

(a)

$$R_{\xi}(\tau) = \int_{-\infty}^{\infty} \xi(t+\tau)\xi^{*}(t) dt = \langle \xi(t+\tau), \xi(t) \rangle$$

$$\stackrel{(1)}{\leq} \|\xi(t+\tau)\| \cdot \|\xi(t)\| = \|\xi\| \cdot \|\xi\| = \|\xi\|^{2} \stackrel{(2)}{=} R_{\xi}(0),$$

where (1) follows from the Cauchy-Schwarz inequality and (2) from the fact that  $R_{\xi}(0) = \int_{-\infty}^{\infty} \xi(t)\xi^{*}(t) dt = ||\xi||^{2}.$ 

(b)

$$R_{\xi}(-\tau) = \int_{-\infty}^{\infty} \xi(t-\tau)\xi^{*}(t) dt$$
$$= \left(\int_{-\infty}^{\infty} \xi(t)\xi^{*}(t-\tau)dt\right)^{*}$$
$$\stackrel{t \to t+\tau}{=} R_{\xi}^{*}(\tau).$$

(c)

$$R_{\xi}(\tau) = \int_{-\infty}^{\infty} \xi(t+\tau)\xi^{*}(t) dt$$

$$\stackrel{t \to t - \tau}{=} \int_{-\infty}^{\infty} \xi(t)\xi^{*}(t-\tau) dt$$

$$= \xi(\tau) \star \xi^{*}(-\tau).$$

(d) By Parseval's identity, we have

$$R_{\xi}(\tau) = \langle \xi(t+\tau), \xi(t) \rangle$$

$$= \langle \xi_{\mathcal{F}}(f)e^{j2\pi f\tau}, \xi_{\mathcal{F}}(f) \rangle$$

$$= \int_{-\infty}^{\infty} \xi_{\mathcal{F}}(f)\xi_{\mathcal{F}}^{*}(f)e^{j2\pi f\tau} df$$

$$= \int_{-\infty}^{\infty} |\xi_{\mathcal{F}}(f)|^{2}e^{j2\pi f\tau} df,$$

which is the inverse Fourier transform of  $|\xi_{\mathcal{F}}(f)|^2$ .

SOLUTION 2.

(a) We have

$$y(t) = \int_{-\infty}^{\infty} w(\tau)\psi(\tau - t)d\tau.$$

The samples of this waveform at multiples of T are

$$y(mT) = \int_{-\infty}^{\infty} w(\tau)\psi(\tau - mT)d\tau$$

$$= \int_{-\infty}^{\infty} \left[\sum_{k=1}^{K} d_k \ \psi(\tau - kT)\right] \psi(\tau - mT)d\tau$$

$$= \sum_{k=1}^{K} d_k \int_{-\infty}^{\infty} \psi(\tau - kT)\psi(\tau - mT)d\tau$$

$$= \sum_{k=1}^{K} d_k \mathbb{1}\{k = m\}$$

$$= d_m.$$

(b) Let  $\tilde{w}(t)$  be the channel output. Then,  $\tilde{y}(t)$  is  $\tilde{w}(t)$  filtered by  $\psi(-t)$ . We have

$$\tilde{w}(t) = w(t) + \rho w(t - T)$$

and

$$\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{w}(\tau)\psi(\tau - t)d\tau.$$

The samples of this waveform at multiples of T are

$$\tilde{y}(mT) = \int_{-\infty}^{\infty} \tilde{w}(\tau)\psi(\tau - mT)d\tau$$

$$= \int_{-\infty}^{\infty} [w(\tau) + \rho w(\tau - T)]\psi(\tau - mT)d\tau$$

$$= \int_{-\infty}^{\infty} \left[\sum_{k=1}^{K} d_k \ \psi(\tau - kT)\right] \psi(\tau - mT)d\tau +$$

$$\rho \int_{-\infty}^{\infty} \left[\sum_{k=1}^{K} d_k \ \psi(\tau - T - kT)\right] \psi(\tau - mT)d\tau$$

$$= \sum_{k=1}^{K} d_k \mathbb{1}\{k = m\} + \rho \sum_{k=1}^{K} d_k \mathbb{1}\{k = m - 1\}$$

$$= d_m + \rho d_{m-1}.$$

(c) From the symmetry of the problem, we have

$$P_e = P_e(1) = P_e(-1).$$

$$\begin{split} P_e(1) &= \Pr\{\hat{D}_k = -1 | D_k = 1, D_{k-1} = -1\} \Pr\{D_{k-1} = -1\} + \\ &\Pr\{\hat{D}_k = -1 | D_k = 1, D_{k-1} = 1\} \Pr\{D_{k-1} = 1\} \\ &= \frac{1}{2} \left( \Pr\{Y_k < 0 | D_k = 1, D_{k-1} = -1\} + \Pr\{Y_k < 0 | D_k = 1, D_{k-1} = 1\} \right) \\ &= \frac{1}{2} \left( \Pr\{1 - \alpha + Z_k < 0\} + \Pr\{1 + \alpha + Z_k < 0\} \right) \\ &= \frac{1}{2} \left( \Pr\{Z_k < -1 + \alpha\} + \Pr\{Z_k < -1 - \alpha\} \right) \\ &= \frac{1}{2} \left[ Q \left( \frac{1 - \alpha}{\sigma} \right) + Q \left( \frac{1 + \alpha}{\sigma} \right) \right]. \end{split}$$

SOLUTION 3.

(a) We can easily see that

$$\mathbb{E}[X_i|X_{i-1}] = \frac{1}{2}X_{i-1} + \frac{1}{2}(-X_{i-1}) = 0.$$

Consequently (using the law of total expectation)

$$\mathbb{E}[X_i] = \mathbb{E}[\mathbb{E}[X_i|X_{i-1}]] = 0.$$

Therefore,

$$K_X[k] = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_{i-k} - \mathbb{E}[X_{i-k}])^*] = \mathbb{E}[X_i X_{i-k}^*]$$

Moreover, using the fact that  $X_i = X_{i-1} \times (-1)^{D_i}$  repeatedly, we can write

$$X_i = X_{i-k} \times \prod_{j=i-k+1}^{i} (-1)^{D_j}$$

Thus,

$$K_{X}[k] = \mathbb{E}[X_{i}X_{i-k}^{*}]$$

$$= \mathbb{E}\left[X_{i-k}\prod_{j=i-k+1}^{i}(-1)^{D_{j}}X_{i-k}^{*}\right]$$

$$\stackrel{(a)}{=} \mathbb{E}[X_{i-k}X_{i-k}^{*}]\prod_{j=i-k+1}^{i}\mathbb{E}[(-1)^{D_{j}}]$$

$$= \mathcal{E}\prod_{j=i-k+1}^{i}\mathbb{E}[(-1)^{D_{j}}]$$

$$\stackrel{(b)}{=} \begin{cases} \mathcal{E} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

where (a) follows from the independence of data bits  $\{D_i\}$  and (b) since  $\mathbb{E}[(-1)^{D_i}] = 0$ .

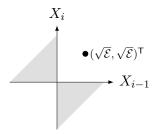
(b) By sampling the signal at the output of the matched filter, Y(t), at multiples of T, we obtain

$$Y(iT) = X_i + Z_i,$$

where  $Z_i$  is normally distributed with zero mean and variance  $N_0/2$ . By looking at the definition of  $X_i$ , we see that it is equal to  $X_{i-1}$  if  $D_i = 0$  and equal to  $-X_{i-1}$  if  $D_i = 1$ . Therefore a simple decoder estimates that  $\hat{D}_i = 0$  if  $Y_i$  and  $Y_{i-1}$  have the same sign, and  $\hat{D}_i = 1$  otherwise. This is equivalent to

$$Y_i Y_{i-1} \overset{\hat{D}_i = 0}{\underset{\hat{D}_i = 1}{\gtrless}} 0.$$

(c) We first compute the error probability when  $D_i = 0$ . If  $X_{i-1} = \sqrt{\mathcal{E}}$ , then  $X_i = \sqrt{\mathcal{E}}$ . When we decode, we will make an error if the signal  $(Y_{i-1}, Y_i)^{\mathsf{T}}$  is in the second or fourth quadrants (shaded regions in the following figure).



Due to the symmetry of the problem, the probability for this to happen is two times the probability for  $(Y_{i-1}, Y_i)^T$  to be in the second quadrant:

$$\Pr\{Z_{i-1} < -\sqrt{\mathcal{E}} \cap Z_i > -\sqrt{\mathcal{E}}\} = Q\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right) Q\left(-\sqrt{\frac{\mathcal{E}}{N_0/2}}\right),$$

so,

$$P_e(D_i = 0 | D_{i-1} = 0) = 2Q\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right) Q\left(-\sqrt{\frac{\mathcal{E}}{N_0/2}}\right).$$

Again, due to the symmetry of the problem,

$$P_e(D_i = 0|D_{i-1} = 1) = P_e(D_i = 0|D_{i-1} = 0) = P_e(D_i = 0),$$

and

$$P_e(D_i = 1) = P_e(D_i = 0);$$

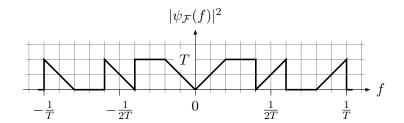
hence

$$P_e = 2Q\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right)Q\left(-\sqrt{\frac{\mathcal{E}}{N_0/2}}\right).$$

SOLUTION 4. Because  $\psi(t)$  is real, its Fourier transform is conjugate symmetric  $(\psi_{\mathcal{F}}(f) = \psi_{\mathcal{F}}^*(-f))$ .

From the condition  $\int \psi(t-kT)\psi(t-lT)dt = \mathbb{1}\{k=l\}$  for every pair k, l, it follows that  $|\psi_{\mathcal{F}}(f)|^2$  satisfies Nyquist's criterion with parameter T,  $\sum_{k\in\mathbb{Z}} |\psi_{\mathcal{F}}(f-k/T)|^2 = T$ . On the other hand, since  $\psi_{\mathcal{F}}(f) = 0$  for  $|f| > \frac{1}{T}$ ,  $|\psi_{\mathcal{F}}(f)|^2$  must have band-edge symmetry.

Putting everything together, we obtain the complete plot of  $|\psi_{\mathcal{F}}(f)|^2$ .



SOLUTION 5. From Theorem 5.6, we know that  $\{\psi(t-jT)\}_{j=-\infty}^{\infty}$  is an orthonormal set if and only if

$$\sum_{k \in \mathbb{Z}} |\psi_{\mathcal{F}}(f - \frac{k}{T})|^2 = T.$$

(a) 
$$\sum_{k\in\mathbb{Z}} T\mathbb{1}_{\left[\frac{k}{T}-\frac{1}{2T},\frac{k}{T}+\frac{1}{2T}\right]}(f) = T \Rightarrow \text{The Nyquist criterion is satisfied}$$

 $\Rightarrow \psi(t)$  is orthonormal to its time-translates by multiples of T.

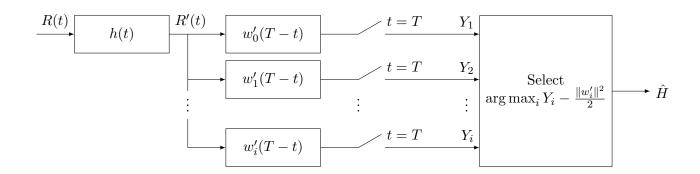
(b) 
$$\sum_{k\in\mathbb{Z}}\frac{T}{2}\mathbb{1}_{\left[\frac{k-1}{T},\frac{k+1}{T}\right]}(f)=T\Rightarrow \text{The Nyquist criterion is satisfied}$$

 $\Rightarrow \psi(t)$  is orthonormal to its time-translates by multiples of T.

- (c) Because  $|\psi_{\mathcal{F}}(f)|^2$  vanishes outside  $\left[-\frac{1}{T}, \frac{1}{T}\right]$ , we verify whether the band-edge symmetry is fulfilled, which is the case. Hence, the Nyquist criterion is satisfied and  $\psi(t)$  is orthonormal to its time-translates by multiples of T. Note: the same reasoning can be applied to (b).
- (d)  $\psi_{\mathcal{F}}(f)$  is a sinc function, therefore  $\psi(t)$  is a box function, equal to  $\frac{1}{T}\mathbbm{1}_{\left[-\frac{T}{2},\frac{T}{2}\right]}(t)$ . This is orthogonal to its time-translates by multiples of T, but does not have unit norm (unless T=1):  $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = \frac{1}{T}$ .

SOLUTION 6.

(a) We pass R(t) through a whitening filter h(t) such that the output R'(t) looks like the output of an AWGN channel. After this step we are facing a familiar situation and can implement a matched filter receiver. The receiver architecture is shown below:



Let  $N'(t) = \int N(\alpha)h(t-\alpha) d\alpha$  be the noise at the output of the whitening filter. We want to select the filter h(t) such that  $\frac{N_0}{2} = G(f)|h_{\mathcal{F}}(f)|^2$ , i.e.,

$$|h_{\mathcal{F}}(f)|^2 = \frac{N_0}{2G(f)}.$$

The output of the filter is

$$R'(t) = \int R(\alpha)h(t-\alpha) \ d\alpha = \int w_i(\alpha)h(t-\alpha) \ d\alpha + \int N(\alpha)h(t-\alpha) \ d\alpha$$
$$= w'_i(t) + N'(t),$$

where N'(t) is white Gaussian noise and  $w_i'(t) = \int w_i(\alpha)h(t-\alpha) d\alpha$ . We need to design the matched filter for the signals  $w_i'(t)$ .

(b) To minimize both the noise and the energy of the signal, we need to select an antipodal signal pair that is frequency-limited to [a, b] and has energy  $\mathcal{E}$ .