ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 17 | Principles of Digital Communications

Problem Set 7 | Apr. 13, 2018

PROBLEM 1. In this problem, we develop further intuition about matched filters. You may assume that all waveforms are real-valued. Let $R(t) = \pm w(t) + N(t)$ be the channel output, where $N(t)$ is additive white Gaussian noise of power spectral density $\frac{N_0}{2}$ and $w(t)$ is an arbitrary but fixed pulse. Let $\phi(t)$ be a unit-norm but otherwise arbitrary pulse, and consider the receiver operation

$$Y = \langle R, \phi \rangle = \langle w, \phi \rangle + \langle N, \phi \rangle$$

The signal-to-noise ratio (SNR) is defined as

$$\text{SNR} \triangleq \frac{|\langle w, \phi \rangle|^2}{\mathbb{E}[|\langle N, \phi \rangle|^2]}$$

Notice that the SNR remains the same if we scale $\phi(t)$ by a constant factor. Notice also that

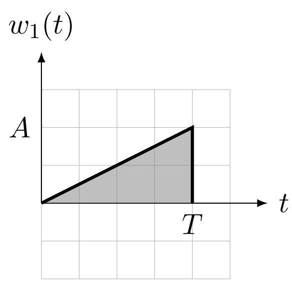
$$\mathbb{E}[|\langle N, \phi \rangle|^2] = \frac{N_0}{2}$$

(a) Use the Cauchy–Schwarz inequality to give an upper bound on the SNR. What is the condition for equality in the Cauchy–Schwarz inequality? Find the $\phi(t)$ that maximizes the SNR. What is the relationship between the maximizing $\phi(t)$ and the signal $w(t)$?

(b) Let us verify that we would get the same result using a pedestrian approach. Instead of waveforms we consider tuples. So let $c = (c_1, c_2)^\mathsf{T} \in \mathbb{R}^2$ and use calculus (instead of the Cauchy–Schwarz inequality) to find the $\phi = (\phi_1, \phi_2)^\mathsf{T} \in \mathbb{R}^2$ that maximizes $\langle c, \phi \rangle$ subject to the constraint that ϕ has unit norm.

(c) Verify with a picture (convolution) that the output at time T of a filter with input $w(t)$ and impulse response $h(t) = w(T - t)$ is indeed $\langle w, w \rangle = \int_{-\infty}^{\infty} w^2(t) dt$.

PROBLEM 2. Let $w_1(t)$ be as shown below and let $w_2(t) = w_1(t - T_d)$, where $T_d \geq T$ is a fixed number known to the receiver. One of the two pulses is selected at random and transmitted across the AWGN channel of noise power spectral density $\frac{N_0}{2}$.



(a) Describe an ML receiver that decides which pulse was transmitted. The n-tuple former must contain a single causal matched filter. Finally, draw the matched filter impulse response.

(b) Express the error probability of the receiver in (a) in terms of A, T, T_d, N_0. Consider both cases $T_d \geq T$ and $T_d < T$.

PROBLEM 3. In this problem, we consider the implementation of matched filter receivers. In particular, we consider frequency-shift keying (FSK) with the following signals:

$$w_j(t) = \begin{cases} \sqrt{\frac{2}{T}} \cos 2\pi \frac{n_j}{T} t & 0 \leq t \leq T \\ 0 & \text{otherwise,} \end{cases}$$

where $n_j \in \mathbb{Z}$ and $0 \leq j \leq m-1$. Thus, the communication scheme consists of m signals $w_j(t)$ of different frequencies $\frac{n_j}{T}$.

(a) Determine the impulse response $h_j(t)$ of a causal matched filter for the signal $w_j(t)$. Plot $h_j(t)$ and specify the sampling time.

(b) Sketch the matched filter receiver. How many matched filters are needed?

(c) Sketch the output of the matched filter with impulse response $h_j(t)$ when the input is $w_j(t)$.

PROBLEM 4. Let $\mathcal{W} = \{w_0(t), w_1(t)\}$ be the signal constellation used to communicate an equiprobable bit across an additive Gaussian noise channel. In this exercise, we verify that the projection of the channel output onto the inner product space $\mathcal{V}$ spanned by $\mathcal{W}$ is not necessarily a sufficient statistic, unless the noise is white.
Let $\psi_1(t), \psi_2(t)$ be an orthonormal basis for $\mathcal{V}$. We choose the additive noise to be $N(t) = Z_1\psi_1(t) + Z_2\psi_2(t) + Z_3\psi_3(t)$ for some normalized $\psi_3(t)$ that is orthogonal to $\psi_1(t)$ and $\psi_2(t)$, and choose Z_1, Z_2, Z_3 to be zero-mean jointly Gaussian random variables of identical variance σ^2. Let $c_i = (c_{i,1}, c_{i,2}, 0)^\mathsf{T}$ be the codeword associated to $w_i(t)$ with respect to the extended orthonormal basis $\psi_1(t), \psi_2(t), \psi_3(t)$. There is a one-to-one correspondence between the channel output $R(t)$ and $Y = (Y_1, Y_2, Y_3)^\mathsf{T}$, where $Y_i = \langle R, \psi_i \rangle$. In terms of Y, the hypothesis testing problem is

$$H = i : Y = c_i + Z, \quad i = \{0, 1\},$$

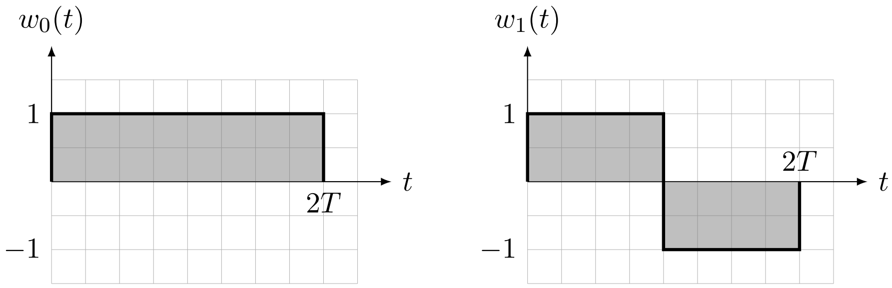
where we have defined $Z = (Z_1, Z_2, Z_3)^\mathsf{T}$.

(a) As a warm-up exercise, let us first assume that Z_1, Z_2, Z_3 are independent. Use the Fisher–Neyman factorization theorem to show that $(Y_1, Y_2)^\mathsf{T}$ is a sufficient statistic.

(b) Now assume that Z_1 and Z_2 are independent, but $Z_3 = Z_2$. Prove that in this case $(Y_1, Y_2)^\mathsf{T}$ is *not* a sufficient statistic.

(c) To check a specific case, consider $c_0 = (1, 0, 0)^\mathsf{T}$ and $c_1 = (0, 1, 0)^\mathsf{T}$. Determine the error probability of an ML receiver that observes $(Y_1, Y_2)^\mathsf{T}$ and that of another ML receiver that observes $(Y_1, Y_2, Y_3)^\mathsf{T}$.

PROBLEM 5. *(Signal translation)*
Consider the signals $w_0(t)$ and $w_1(t)$ shown below, used to communicate 1 bit across the AWGN channel of power spectral density $\frac{N_0}{2}$.



(a) Determine an orthonormal basis $\{\psi_0(t), \psi_1(t)\}$ for the space spanned by $\{w_0(t), w_1(t)\}$ and find the corresponding codewords c_0 and c_1. Work out two solutions, one obtained via Gram–Schmidt and one in which $\psi_1(t)$ is a delayed version of $\psi_0(t)$. Which of the two solutions would you choose if you had to implement the system?

(b) Let X be a uniformly distributed binary random variable that takes values in $\{0, 1\}$. We want to communicate the value of X over an additive white Gaussian noise channel. When $X = 0$, we send $w_0(t)$, and when $X = 1$, we send $w_1(t)$. Draw the block diagram of an ML receiver based on a single matched filter.

(c) Determine the error probability P_e of your receiver as a function of T and N_0.

(d) Find a suitable waveform $v(t)$ such that the signals $\tilde{w}_0(t) = w_0(t) - v(t)$ and $\tilde{w}_1(t) = w_1(t) - v(t)$ have minimum energy. Plot the resulting waveforms.

(e) What is the name of the signaling scheme that uses signals such as $\tilde{w}_0(t)$ and $\tilde{w}_1(t)$? Argue that one obtains this kind of signaling scheme independently of the initial choice of $w_0(t)$ and $w_1(t)$.

PROBLEM 6. *(Orthogonal signal sets)*

Consider a set $\mathcal{W} = \{w_0(t), \ldots, w_{m-1}(t)\}$ of mutually orthogonal signals with squared norm $\mathcal{E}$, each used with equal probability.

(a) Find the minimum-energy signal set $\tilde{\mathcal{W}} = \{\tilde{w}_0(t), \ldots, \tilde{w}_{m-1}(t)\}$ obtained by translating the original set.

(b) Let $\tilde{\mathcal{E}}$ be the average energy of a signal picked at random within $\tilde{\mathcal{W}}$. Determine $\tilde{\mathcal{E}}$ and the energy saving $\mathcal{E} - \tilde{\mathcal{E}}$.

(c) Determine the dimension of the inner product space spanned by $\tilde{\mathcal{W}}$.