SOLUTION 1. If $H = 0$, we have $Y_2 = Z_1Z_2 = Y_1Z_2$, and if $H = 1$, we have $Y_2 = -Z_1Z_2 = Y_1Z_2$. Therefore, $Y_2 = Y_1Z_2$ in all cases. Now since $Z_2$ is independent of $H$, we clearly have $H \rightarrow Y_1 \rightarrow (Y_1, Y_1Z_2)$. Hence, $Y_1$ is a sufficient statistic.

SOLUTION 2.

(a) The MAP decoder $\hat{H}(y)$ is given by

$$
\hat{H}(y) = \arg\max_i P_Y|H(y|i) = \begin{cases} 
0 & \text{if } y = 0 \text{ or } y = 1 \\
1 & \text{if } y = 2 \text{ or } y = 3.
\end{cases}
$$

$T(Y)$ takes two values with the conditional probabilities

$$
P_{T|H}(t|0) = \begin{cases} 
0.7 & \text{if } t = 0 \\
0.3 & \text{if } t = 1
\end{cases} \quad \quad P_{T|H}(t|1) = \begin{cases} 
0.3 & \text{if } t = 0 \\
0.7 & \text{if } t = 1.
\end{cases}
$$

Therefore, the MAP decoder $\hat{H}(T(y))$ is

$$
\hat{H}(T(y)) = \arg\max_i P_{T|Y|H}(t|i) = \begin{cases} 
0 & \text{if } t = 0 \quad (y = 0 \text{ or } y = 1) \\
1 & \text{if } t = 1 \quad (y = 2 \text{ or } y = 3).
\end{cases}
$$

Hence, the two decoders are equivalent.

(b) We have

$$
\Pr\{Y = 0|T(Y) = 0, H = 0\} = \frac{\Pr\{Y = 0, T(Y) = 0|H = 0\}}{\Pr\{T(Y) = 0|H = 0\}} = \frac{0.4}{0.7} = \frac{4}{7}
$$

and

$$
\Pr\{Y = 0|T(Y) = 0, H = 1\} = \frac{\Pr\{Y = 0, T(Y) = 0|H = 1\}}{\Pr\{T(Y) = 0|H = 1\}} = \frac{0.1}{0.3} = \frac{1}{3}
$$

Thus $\Pr\{Y = 0|T(Y) = 0, H = 0\} \neq \Pr\{Y = 0|T(Y) = 0, H = 1\}$, hence $H \rightarrow T(Y) \rightarrow Y$ is not true, although the MAP decoders are equivalent.

SOLUTION 3.

(a) The MAP decision rule can always be written as

$$
\hat{H}(y) = \arg\max_i f_{Y|H}(y|i)P_H(i) \\
= \arg\max_i g_i(T(y))h(y)P_H(i) \\
= \arg\max_i g_i(T(y))P_H(i).
$$

The last step is valid because $h(y)$ is a non-negative constant which is independent of $i$ and thus does not give any further information for our decision.
(b) Let us define the event $B = \{ y : T(y) = t \}$. Then,

$$f_{Y|H,T(Y)}(y|i,t) = \frac{f_{Y,Y|H,T(Y)}(y|t)P_H(i)}{f_{T|H,T(Y)}(t|i)P_H(i)}$$

$$= \frac{\Pr\{ Y = y, T(Y) = t | H = i \}}{\Pr\{ T(Y) = t | H = i \}} = \frac{\Pr\{ Y = y, Y \in B | H = i \}}{\Pr\{ Y \in B | H = i \}}$$

$$= \frac{f_{Y|H}(y|i)\mathbf{1}_B(y)}{\int_B f_{Y|H}(y|i)dy}.$$

If $f_{Y|H}(y|i) = g_i(T(y))h(y)$, then

$$f_{Y|H,T(Y)}(y|i,t) = \frac{g_i(T(y))h(y)\mathbf{1}_B(y)}{\int_B g_i(T(y))h(y)dy}$$

$$= \frac{g_i(t)h(y)\mathbf{1}_B(y)}{\int_B h(y)dy}.$$

Hence, we see that $f_{Y|H,T(Y)}(y|i,t)$ does not depend on $i$, so $H \to T(Y) \to Y$.

(c) Note that $P_{Y_i|H}(1|i) = p_i, P_{Y_i|H}(0|i) = 1 - p_i$ and

$$P_{Y_1,\ldots,Y_n|H}(y_1,\ldots,y_n|i) = P_{Y_1|H}(y_1|i)\cdots P_{Y_n|H}(y_n|i).$$

Thus, we have

$$P_{Y_1,\ldots,Y_n|H}(y_1,\ldots,y_n|i) = p_i^{(n-t)}(1 - p_i)^t,$$

where $t = \sum_k y_k$.

Choosing $g_i(t) = p_i^{(n-t)}(1 - p_i)^t$ and $h(y) = 1$, we see that $P_{Y_1,\ldots,Y_n|H}(y_1,\ldots,y_n|i)$ fulfills the condition in the question.

(d) Because $Y_1,\ldots,Y_n$ are independent,

$$f_{Y_1,\ldots,Y_n|H}(y_1,\ldots,y_n|i) = \prod_{k=1}^n \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_k - m_i)^2}{2}}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{\sum_{k=1}^n (y_k - m_i)^2}{2}}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{\sum_{k=1}^n y_k^2}{2} \frac{1}{\sum_{k=1}^n y_k} - \frac{m_i}{2}}.$$  

Choosing $g_i(t) = e^{nm_i(t - \frac{m_i}{2})}$ and $h(y_1,\ldots,y_n) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\sum_{k=1}^n y_k^2}{2}}$, we see that

$$f_{Y_1,\ldots,Y_n|H}(y_1,\ldots,y_n|i) = g_i(T(y_1,\ldots,y_n))h(y_1,\ldots,y_n).$$

Hence the condition in the question is fulfilled.

**Solution 4.**
(a) With the observation $Y$ being $Y_2$, 

$$f_{Y|X}(y|+1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-1)^2} \quad \text{and} \quad f_{Y|X}(y|-1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y+1)^2}$$

Thus the MAP rule is 

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-1)^2} \begin{array}{c} 1 \\ -1 \end{array} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y+1)^2} (1 - p),$$

which can be further simplified to obtain 

$$y \begin{array}{c} \frac{1}{2} \ln \frac{1 - p}{p}

(b) Observe that 

$$f_{Y_1,Y_2|X}(y_1, y_2|+1) = \frac{1}{2} \mathbb{1} \{y_1 \in [0, 2]\} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y_2-1)^2}$$

$$f_{Y_1,Y_2|X}(y_1, y_2|-1) = \frac{1}{4} \mathbb{1} \{y_1 \in [-3, 1]\} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y_2+1)^2}$$

With

$$g_{+1}(u, y_2) = \frac{1}{2} \mathbb{1} \{u \geq 0\} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y_2-1)^2}$$

$$g_{-1}(u, y_2) = \frac{1}{4} \mathbb{1} \{u \leq 0\} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y_2+1)^2}$$

$$h(y_1, y_2) = \mathbb{1} \{-3 \leq y_1 \leq 2\},$$

we find $f_{Y_1,Y_2|X}(y_1, y_2|x) = g_x(u, y_2)h(y_1, y_2)$ and the Fisher–Neyman theorem lets us conclude that $t = (u, y_2)$ is a sufficient statistic.

(c) The MAP rule minimizes the error probability and is given by the likelihood ratio test 

$$\Lambda(y_1, y_2) = \log \frac{f_{Y_1,Y_2|X}(y_1, y_2|+1)}{f_{Y_1,Y_2|X}(y_1, y_2|-1)} \begin{array}{c} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-1)^2} \\ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y+1)^2} (1 - p), \end{array}$$

Note that 

$$\Lambda(y_1, y_2) = \begin{cases} +\infty & 1 < y_1 \leq 2 \\ 2y_2 + \log 2 & 0 \leq y_1 \leq 1 \\ -\infty & -3 \leq y_1 < 0 \end{cases}$$

So the decision region looks as follows (with $\theta = \frac{1}{2} \log \frac{1-p}{p}$):
(d) When $-1$ is sent an error will happen either when $y_1 > 1$ or when $0 \leq y_1 \leq 1$ and $y_2 \geq \theta$. The first of these cannot happen, and the second happens with probability $\frac{1}{4}Q(1 + \theta)$.

When $+1$ is sent an error will happen either when $y_1 < 0$ or when $0 \leq y_1 \leq 1$ and $y_2 \leq \theta$. The first of these cannot happen, and the second happens with probability $\frac{1}{2}Q(1 - \theta)$.

So the error probability is given by

$$\frac{1 - P}{4}Q(1 + \theta) + \frac{P}{2}Q(1 - \theta)$$

with $\theta = \frac{1}{2}\log \frac{1 - p}{2p}$.

**Solution 5.**

(a) Inequality (a) follows from the Bhattacharyya Bound.

Using the definition of DMC, it is straightforward to see that

$$P_{Y|X}(y|c_0) = \prod_{i=1}^{n} P_{Y|X}(y_i|c_{0,i}) \quad \text{and} \quad P_{Y|X}(y|c_1) = \prod_{i=1}^{n} P_{Y|X}(y_i|c_{1,i}).$$

(b) follows by substituting the above values in (a).

Equality (c) is obtained by observing that $\sum_{y}$ is the same as $\sum_{y_1, \ldots, y_n}$ (the first one being a vector notation for the sum over all possible $y_1, \ldots, y_n$).

In (c), we see that we want the sum of all possible products. This is the same as summing over each $y_i$ and taking the product of the resulting sum for all $y_i$. This results in equality (d). We obtain (e) by writing (d) in a more concise form.

When $c_{0,i} = c_{1,i}$, $\sqrt{P_{Y|X}(y|c_{0,i})P_{Y|X}(y|c_{1,i})} = P_{Y|X}(y|c_{0,i})$. Therefore,

$$\sum_{y} \sqrt{P_{Y|X}(y|c_{0,i})P_{Y|X}(y|c_{1,i})} = \sum_{y} P_{Y|X}(y|c_{0,i}) = 1.$$
This does not affect the product, so we are only interested in the terms where \( c_{0,i} \neq c_{1,i} \). We form the product of all such sums where \( c_{0,i} \neq c_{1,i} \). We then look out for terms where \( c_{0,i} = a \) and \( c_{1,i} = b, a \neq b \), and raise the sum to the appropriate power. (Eg. If we have the product \( prpqrpqrr \), we would write it as \( p^3q^2r^4 \)). Hence equality \((f)\).

(b) For a binary input channel, we have only two source symbols \( X = \{a, b\} \). Thus,

\[
P_e \leq z^{n(a,b)} z^{n(b,a)} = z^{n(a,b)+n(b,a)} = z^{d_H(c_0,c_1)}.
\]

(c) The value of \( z \) is:

(i) For a binary input Gaussian channel,

\[
z = \int_y \sqrt{f_{Y|X}(y|0)f_{Y|X}(y|1)} \, dy = \exp \left( -\frac{E^2 \sigma^2}{2} \right).
\]

(ii) For the Binary Symmetric Channel (BSC),

\[
z = \sqrt{\Pr\{y = 0|x = 0\} \Pr\{y = 0|x = 1\} + \sqrt{\Pr\{y = 1|x = 0\} \Pr\{y = 1|x = 1\}}}
\]

\[
= 2\sqrt{\delta(1 - \delta}).
\]

(iii) For the Binary Erasure Channel (BEC),

\[
z = \sqrt{\Pr\{y = 0|x = 0\} \Pr\{y = 0|x = 1\} + \sqrt{\Pr\{y = E|x = 0\} \Pr\{y = E|x = 1\}}}
\]

\[
+ \sqrt{\Pr\{y = 1|x = 0\} \Pr\{y = 1|x = 1\}}
\]

\[
= 0 + \delta + 0 = \delta.
\]

**Solution 6.**

\[
P_{00} = \Pr\{(N_1 \geq -a) \cap (N_2 \geq -a)\}
\]

\[
= \Pr\{(N_1 \leq a)\} \Pr\{(N_2 \leq a)\}
\]

\[
= \left[1 - Q\left(\frac{a}{\sigma}\right)\right]^2.
\]

By symmetry:

\[
P_{01} = P_{03} = \Pr\{(N_1 \leq -(2b - a)) \cap (N_2 \geq -a)\}
\]

\[
= \Pr\{N_1 \geq 2b - a\} \Pr\{N_2 \leq a\}
\]

\[
= Q\left(\frac{2b - a}{\sigma}\right) \left[1 - Q\left(\frac{a}{\sigma}\right)\right].
\]

\[
P_{02} = \Pr\{(N_1 \leq -(2b - a)) \cap (N_2 \leq -(2b - a))\}
\]

\[
= \Pr\{N_1 \geq 2b - a\} \Pr\{N_2 \geq 2b - a\}
\]

\[
= \left[Q\left(\frac{2b - a}{\sigma}\right)\right]^2.
\]
\[ P_{0δ} = 1 - \Pr\{(Y \in \mathcal{R}_0) \cup (Y \in \mathcal{R}_1) \cup (Y \in \mathcal{R}_2) \cup (Y \in \mathcal{R}_3)|c_0 \text{ was sent}\} \]
\[ = 1 - P_{00} - P_{01} - P_{02} - P_{03} \]
\[ = 1 - \left[ 1 - Q \left( \frac{a}{\sigma} \right) \right]^2 - 2Q \left( \frac{2b-a}{\sigma} \right) \left[ 1 - Q \left( \frac{a}{\sigma} \right) \right] - \left[ Q \left( \frac{2b-a}{\sigma} \right) \right]^2 \]
\[ = 1 - \left[ 1 - Q \left( \frac{a}{\sigma} \right) + Q \left( \frac{2b-a}{\sigma} \right) \right]^2. \]

Equivalently,
\[ P_{0δ} = \Pr\{(N_1 \in [a, 2b-a]) \cup (N_2 \in [a, 2b-a])\} \]
\[ = \Pr\{N_1 \in [a, 2b-a]\} + \Pr\{N_2 \in [a, 2b-a]\} - \Pr\{(N_1 \in [a, 2b-a]) \cap (N_2 \in [a, 2b-a])\} \]
\[ = 2 \left[ Q \left( \frac{a}{\sigma} \right) - Q \left( \frac{2b-a}{\sigma} \right) \right] - \left[ Q \left( \frac{a}{\sigma} \right) - Q \left( \frac{2b-a}{\sigma} \right) \right]^2, \]
which gives the same result as before.