ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 7 Principles of Digital Communications

Problem Set 3 Mar. 7, 2018

PROBLEM 1. Consider the ternary hypothesis testing problem

$$H_0 : Y = c_0 + Z, \qquad H_1 : Y = c_1 + Z, \qquad H_2 : Y = c_2 + Z,$$

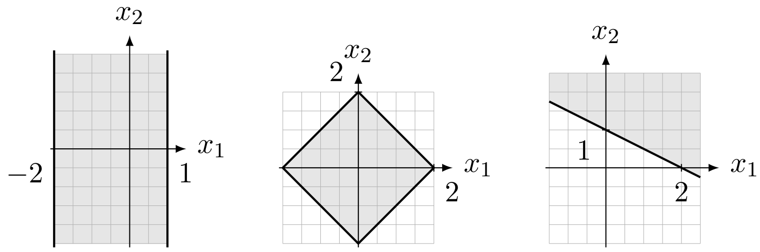
where $Y = [Y_1, Y_2]^\mathsf{T}$ is the two-dimensional observation vector, $c_0 = \sqrt{\mathcal{E}}[1, 0]^\mathsf{T}$, $c_1 = \frac{1}{2}\sqrt{\mathcal{E}}[-1, \sqrt{3}]^\mathsf{T}$, $c_2 = \frac{1}{2}\sqrt{\mathcal{E}}[-1, -\sqrt{3}]^\mathsf{T}$, and $Z = [Z_1, Z_2]^\mathsf{T} \sim \mathcal{N}(0, \sigma^2 I_2)$.

(a) Assuming the three hypotheses are equally likely, draw the optimal decision regions in the (Y_1, Y_2) plane.

(b) Assume now that the apriori probabilities for the hypotheses are $\Pr\{H = 0\} = \frac{1}{2}$, $\Pr\{H = 1\} = \Pr\{H = 2\} = \frac{1}{4}$. Draw the decision regions in the (L_1, L_2) plane where

$$L_i := \frac{f_{Y|H}(Y|i)}{f_{Y|H}(Y|0)}, \quad i = 1, 2.$$

PROBLEM 2. Let $X \sim \mathcal{N}(0, \sigma^2 I_2)$. For each of the three diagrams shown below, express the probability that X lies in the shaded region. You may use the Q function when appropriate.



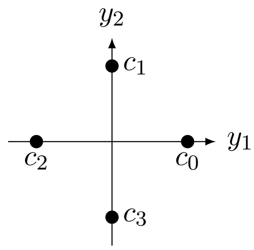
(a) (b) (c)

PROBLEM 3. Let $H \in \{0, 1, 2, 3\}$ and assume that when $H = i$ you transmit the codeword c_i shown in the following diagram. Under $H = i$, the receiver observes $Y = c_i + Z$.

(a) Draw the decoding regions assuming that $Z \sim \mathcal{N}(0, \sigma^2 I_2)$ and that $P_H(i) = 1/4$, $i \in \{0, 1, 2, 3\}$.

(b) Draw the decoding regions (qualitatively) assuming $Z \sim \mathcal{N}(0, \sigma^2 I_2)$ and $P_H(0) = P_H(2) > P_H(1) = P_H(3)$. Justify your answer.

(c) Assume again that $P_H(i) = 1/4$, $i \in \{0, 1, 2, 3\}$ and that $Z \sim \mathcal{N}(0, K)$, where $K = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 4\sigma^2 \end{pmatrix}$. How do you decode now?



PROBLEM 4. The following problem relates to the design of multi-antenna systems. Consider the binary equiprobable hypothesis testing problem:

$$H = 0 : Y_1 = A + Z_1, \quad Y_2 = A + Z_2$$

$$H = 1 : Y_1 = -A + Z_1, \quad Y_2 = -A + Z_2$$

where Z_1, Z_2 are independent Gaussian random variables with different variances $\sigma_1^2 \neq \sigma_2^2$, that is, $Z_1 \sim \mathcal{N}(0, \sigma_1^2)$ and $Z_2 \sim \mathcal{N}(0, \sigma_2^2)$. $A > 0$ is a constant.

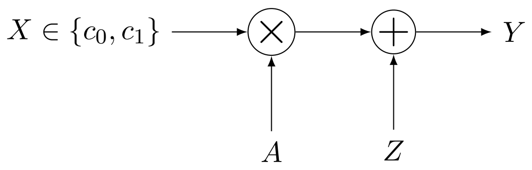
(a) Show that the decision rule that minimizes the probability of error (based on the observable Y_1 and Y_2) can be stated as

$$\sigma_2^2 y_1 + \sigma_1^2 y_2 \underset{1}{\overset{0}{\gtrless}} 0$$

(b) Draw the decision regions in the (Y_1, Y_2) plane for the special case where $\sigma_1 = 2\sigma_2$.

(c) Evaluate the probability of the error for the optimal detector as a function of σ_1^2, σ_2^2 and A.

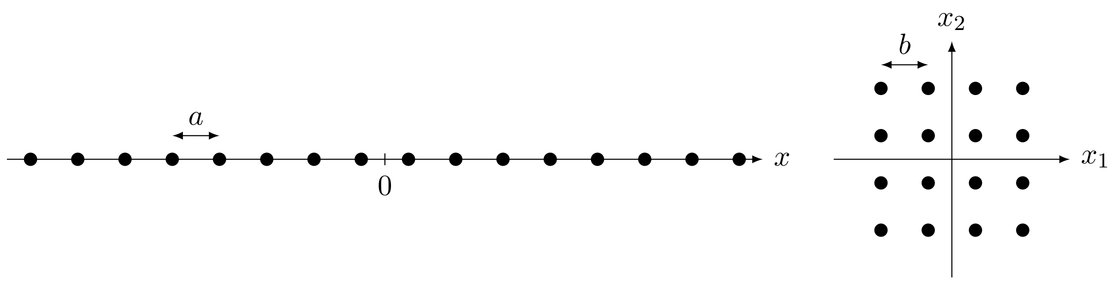
PROBLEM 5. Consider the communication system depicted below. There are two equiprobable hypotheses. When $H = 0$, we transmit $c_0 = -b$, where b is an arbitrary but fixed positive number. When $H = 1$, we transmit $c_1 = b$. The channel is as shown in the diagram, where $Z \sim \mathcal{N}(0, \sigma^2)$ represents noise, $A \in \{0, 1\}$ represents a random attenuation (fading) with $P_A(0) = \frac{1}{2}$, and Y is the channel output. The random variables H, A, and Z are independent.



(a) Find the decision rule that the receiver should implement to minimize the probability of error. Sketch the decision regions.

(b) Calculate the probability of error P_e, based on the above decision rule.

PROBLEM 6. The following two signal constellations are used to communicate across an additive white Gaussian noise channel. Let the noise variance be σ^2. Each point represents a codeword c_i for some i. Assume each codeword is used with the same probability.



16-PAM

16-QAM

(a) For each signal constellation, compute the average probability of error P_e as a function of the parameters a and b, respectively.

(b) For each signal constellation, compute the average energy per symbol $\mathcal{E}$ as a function of parameters a and b, respectively:

$$\mathcal{E} = \sum_{i=1}^{16} P_H(i) \|c_i\|^2$$

(c) Plot P_e versus $\frac{\mathcal{E}}{\sigma^2}$ for both signal constellations and comment.