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Handout 5 Principles of Digital Communications
Solutions to Problem Set 2 Mar. 2, 2018

Solution 1.

(a) Let $l(y)$ be the number of 0's in the sequence y.

$$P_{Y|H}(y|0) = \frac{1}{2^{2k}}$$
$$P_{Y|H}(y|1) = \begin{cases} \frac{1}{\binom{2k}{k}}, & \text{if } l = k \\ 0, & \text{otherwise} \end{cases}$$

(b) The ML decision rule is:

$$P_{Y|H}(y|1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} P_{Y|H}(y|0)$$

Because $\frac{1}{\binom{2k}{k}} > \frac{1}{2^{2k}}$ for any value of k, the ML decision rule becomes

$$\hat{H} = \begin{cases} 0, & \text{if } l(y) \neq k \\ 1, & \text{if } l(y) = k. \end{cases}$$

The single number needed is $l(y)$, the number of 0's in the sequence y.

(c) The decision rule that minimizes the error probability is the MAP rule:

$$P_{Y|H}(y|1)P_H(1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} P_{Y|H}(y|0)P_H(0).$$

The MAP decision rule gives $\hat{H} = 0$ whenever $l(y) \neq k$. When $l(y) = k$:

$$\hat{H} = \begin{cases} 0, & \text{if } \frac{\binom{2k}{k}}{2^{2k}} \geq \frac{P_H(1)}{P_H(0)} \\ 1, & \text{otherwise.} \end{cases}$$

(d) Trivial solution: If $P_H(1) = 1$ then $\hat{H} = 1$ for all y (In this case, $l(y) = k$ is guaranteed). Similarly, if $P_H(0) = 1$ then $\hat{H} = 0$ for all y.

Now assume $P_H(1) \neq 1$. Then there is a nonzero probability that $l(y) \neq k$, in which case $\hat{H} = 0$. The MAP decision rule always chooses $\hat{H} = 0$ if

$$\frac{\binom{2k}{k}}{2^{2k}} \geq \frac{P_H(1)}{P_H(0)} \iff P_H(0) \geq \frac{\frac{1}{\binom{2k}{k}}}{\frac{1}{\binom{2k}{k}} + \frac{1}{2^{2k}}}.$$

SOLUTION 2.

(a) A and B must be chosen such that the suggested functions become valid probability density functions, i.e. $\int_0^1 f_{Y|H}(y|i)dy = 1$ for $i = 0, 1$. This yields $A = 4/3$ and $B = 6/7$. (A quicker way is to draw the functions and find the area by looking at the drawings.)

(b) Let us first find the marginal of Y, i.e.

$$f_Y(y) = f_{Y|H}(y|0)P_H(0) + f_{Y|H}(y|1)P_H(1) = C - Dy,$$

where we find $C = 23/21$ and $D = 4/21$. Then, applying Bayes' rule gives

$$P_{H|Y}(0|y) = \frac{f_{Y|H}(y|0)P_H(0)}{f_Y(y)} = \frac{1}{2}\frac{A - \frac{A}{2}y}{C - Dy} = \frac{1}{2}\frac{4/3 - 2/3y}{23/21 - 4/21y},$$

and similarly

$$P_{H|Y}(1|y) = \frac{f_{Y|H}(y|1)P_H(1)}{f_Y(y)} = \frac{1}{2}\frac{B + \frac{B}{3}y}{C - Dy} = \frac{1}{2}\frac{6/7 + 2/7y}{23/21 - 4/21y}.$$

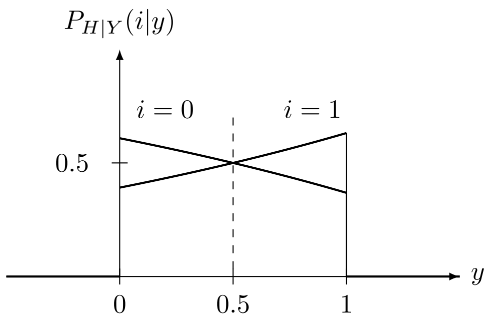
(c) The threshold is where the two a posteriori probabilities are equal,

$$\frac{1}{2}\frac{4/3 - 2/3y}{23/21 - 4/21y} = \frac{1}{2}\frac{6/7 + 2/7y}{23/21 - 4/21y},$$

or equivalently,

$$4/3 - 2/3y = 6/7 + 2/7y.$$

The y that satisfies this equation is our threshold θ, thus $\theta = 0.5$.



(d) The probability that we decide $\hat{H}_\gamma(y) = 1$ when in reality $H = 0$ is just the probability that y is *larger* than the threshold given that $H = 0$, which is

$$\begin{aligned}
\Pr\{Y > \gamma | H = 0\} &= \int_\gamma^1 f_{Y|H}(y|0)dy = \int_\gamma^1 \left(A - \frac{A}{2}y\right) dy \\
&= A(1-\gamma) - \frac{A}{2}\frac{1-\gamma^2}{2} \\
&= \frac{4(1-\gamma)}{3} - \frac{1-\gamma^2}{3}.
\end{aligned}$$

(e) By analogy to the previous question,

$$\begin{aligned}
\Pr\{Y < \gamma | H = 1\} &= \int_0^\gamma f_{Y|H}(y|1) dy = \int_0^\gamma \left(B + \frac{B}{3} y\right) dy \\
&= B\gamma + \frac{B}{3}\frac{\gamma^2}{2} \\
&= \frac{6\gamma}{7} + \frac{\gamma^2}{7}.
\end{aligned}$$

$$\begin{aligned}
P_e(\gamma) &= \Pr\{Y > \gamma | H = 0\} P_H(0) + \Pr\{Y < \gamma | H = 1\} P_H(1) \\
&= \frac{1}{2}\left(\frac{4(1-\gamma)}{3} - \frac{1-\gamma^2}{3} + \frac{6\gamma}{7} + \frac{\gamma^2}{7}\right).
\end{aligned}$$

For $\gamma = \theta = 0.5$, we find $P_e(\theta) = 0.44$.

(f) To minimize P_e over γ, we take the derivative of P_e with respect to γ, i.e.,

$$\frac{d}{d\gamma} P_e(\gamma) = \frac{1}{2}\left(-\frac{4}{3} + \frac{2\gamma}{3} + \frac{6}{7} + \frac{2\gamma}{7}\right).$$

Setting this equal to zero, we find $\gamma = 0.5$. We observe that the value of γ which minimizes $P_e(\gamma)$ is equal to θ. This was expected, because the MAP decision rule minimizes the error probability.

Solution 3.

(a) Let $H \in \{T, L\}$.

$$\begin{aligned}
H = T \text{ (telling truth):} \qquad & f_{Y|H}(y|T) = \alpha e^{-\alpha y},\ y \geq 0 \\
H = L \text{ (telling lie):} \qquad & f_{Y|H}(y|L) = \beta e^{-\beta y},\ y \geq 0.
\end{aligned}$$

The MAP decision rule is

$$p\beta e^{-\beta y} \underset{\hat{H}=T}{\overset{\hat{H}=L}{\gtrless}} (1-p)\alpha e^{-\alpha y}.$$

After taking the logarithm, we obtain

$$-\beta y + \ln(p\beta) \underset{\hat{H}=T}{\overset{\hat{H}=L}{\gtrless}} -\alpha y + \ln((1-p)\alpha).$$

Or, equivalently

$$y \underset{\hat{H}=L}{\overset{\hat{H}=T}{\gtrless}} \frac{1}{\alpha - \beta} \ln\left[\frac{\alpha}{\beta}\frac{(1-p)}{p}\right] = \theta$$

(b)

$$P_{L|T} = \int_0^\theta \alpha e^{-\alpha y} dy = 1 - e^{-\alpha\theta}.$$

(c)

$$P_{T|L} = \int_\theta^\infty \beta e^{-\beta y} dy = e^{-\beta\theta}.$$

(d)

$$
\begin{aligned}
H = T: \qquad & f_{Y|H}(y|T) = \alpha^n e^{-\alpha(y_1+\cdots+y_n)} = \alpha^n e^{-\alpha z} \\
H = L: \qquad & f_{Y|H}(y|L) = \beta^n e^{-\beta(y_1+\cdots+y_n)} = \beta^n e^{-\beta z},
\end{aligned}
$$

where Y is the random vector $(Y_1, \ldots, Y_n)$ and where $z = \sum_{i=1}^n y_i$. With this new definition, the test becomes $z \underset{\hat{H}=L}{\overset{\hat{H}=T}{\gtrless}} \theta$, with the new threshold $\theta = \frac{1}{\alpha-\beta} \ln\left[\left(\frac{\alpha}{\beta}\right)^n \frac{(1-p)}{p}\right]$.

$$
P_{L|T} = \int_0^\theta f_{Z|H}(z|T)dz,
$$

where $Z = \sum_{i=1}^n Y_i$ and

$$
f_{Z|H}(z|T) = \frac{\alpha^n}{(n-1)!} z^{(n-1)} e^{-\alpha z}.
$$

This is the density of the Erlang distribution. Putting things together, we get

$$
P_{L|T} = \int_0^\theta \frac{\alpha^n}{(n-1)!} z^{(n-1)} e^{-\alpha z} dz.
$$

SOLUTION 4.

REMARK. Independent and identically distributed (i.i.d.) *means that all* $Y_1, \ldots, Y_k$ *have the same probability mass function and are independent of each other.* First-order Markov *means that* $Y_1, \ldots, Y_k$ *depend on each other in a particular way: the probability mass function* Y_i *depends on the value of* Y_{i-1}*, but given the value of* Y_{i-1}*, it is independent of* $Y_1, \ldots, Y_{i-2}$*. Thus, in this problem, we observe a binary sequence, and we want to know whether it has been generated by an* i.i.d. *source or by a first-order Markov source.*

(a) Since the two hypotheses are equally likely, we find

$$
\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{P_H(0)}{P_H(1)} = 1.
$$

Plugging in, we obtain

$$
\frac{1/2 \cdot (1/4)^l \cdot (3/4)^{k-l-1}}{(1/2)^k} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 1,
$$

where l is the number of times the observed sequence changes either from zero to one or from one to zero, i.e. the number of transitions in the observed sequence.

(b) The sufficient statistic here is simply the number of transitions l; this entirely specifies the likelihood ratio.

(c) In this case, the number of non-transitions is $(k - l) = s$, and the log-likelihood ratio becomes

$$
\begin{aligned}
\log \frac{1/2 \cdot (1/4)^{k-s} \cdot (3/4)^{s-1}}{(1/2)^k} &= \log \frac{(1/4)^{k-s} \cdot (3/4)^{s-1}}{(1/2)^{k-1}} \\
&= (k-s)\log(1/4) + (s-1)\log(3/4) - (k-1)\log(1/2) \\
&= s \log \frac{3/4}{1/4} + k \log \frac{1/4}{1/2} + \log \frac{1/2}{3/4} \\
&= s \log 3 + k \log 1/2 + \log 2/3.
\end{aligned}
$$

Thus, in terms of this log-likelihood ratio, the decision rule becomes

$$s \log 3 + k \log 1/2 + \log 2/3 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0.$$

That is, we have to find the smallest possible s such that this expression becomes larger or equal to zero. Therefore,

$$s \;\geq\; \left\lceil \frac{k \log 1/2 + \log 2/3}{\log 1/3} \right\rceil.$$

Solution 5. Since noise samples are i.i.d., the conditional probability distribution functions under H_0 and H_1 will respectively be

$$f_{Y|H}(y|0) = \prod_{k=1}^{n} f_Z(y_k)$$
$$f_{Y|H}(y|1) = \prod_{k=1}^{n} f_Z(y_k - 2A)$$

where $f_Z(z)$ is the p.d.f. of Z_k, $k = 1, \ldots, n$. Furthermore, since the two hypotheses are equi-probable, the MAP decision reduces to the ML decision rule.

(a) Plugging the density of Z the MAP decision rule becomes

$$\frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k-2A)^2} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{n} y_k^2}.$$

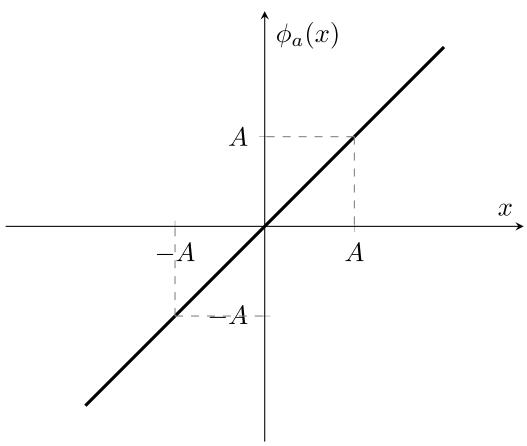
Simplifying the common factor $\frac{1}{(2\pi\sigma^2)^{n/2}}$ and taking the logarithm we have

$$-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k - 2A)^2 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} -\frac{1}{2\sigma^2}\sum_{k=1}^{n} y_k^2.$$

Further simplifications reduce the MAP decision rule to

$$\sum_{k=1}^{n} y_k \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} nA \quad \Longleftrightarrow \quad \sum_{k=1}^{n}(y_k - A) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0.$$

Hence $\phi_a(x) = x$.



(b) Similarly, the MAP decision rule is now

$$\frac{1}{(2\sigma^2)^{n/2}} e^{-\frac{\sqrt{2}}{\sigma}\sum_{k=1}^n |y_k - 2A|} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1}{(2\sigma^2)^{n/2}} e^{-\frac{\sqrt{2}}{\sigma}\sum_{k=1}^n |y_k|}.$$

Simplifying common terms and taking the logarithm gives

$$-\frac{\sqrt{2}}{\sigma}\sum_{k=1}^{n} |y_k - 2A| \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} -\frac{\sqrt{2}}{\sigma}\sum_{k=1}^{n} |y_k|.$$

We can write the above in the desired form by noting that

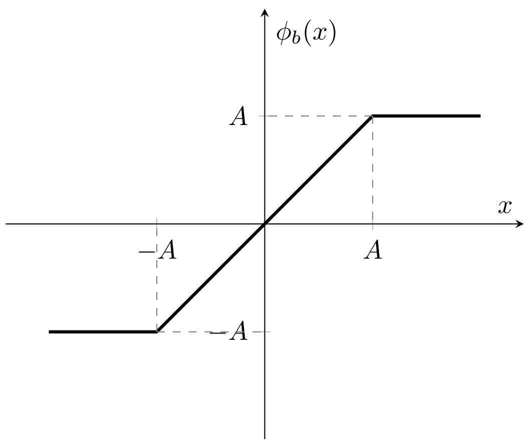
$$|x| - |x - 2A| = 2\phi_b(x - A)$$

where

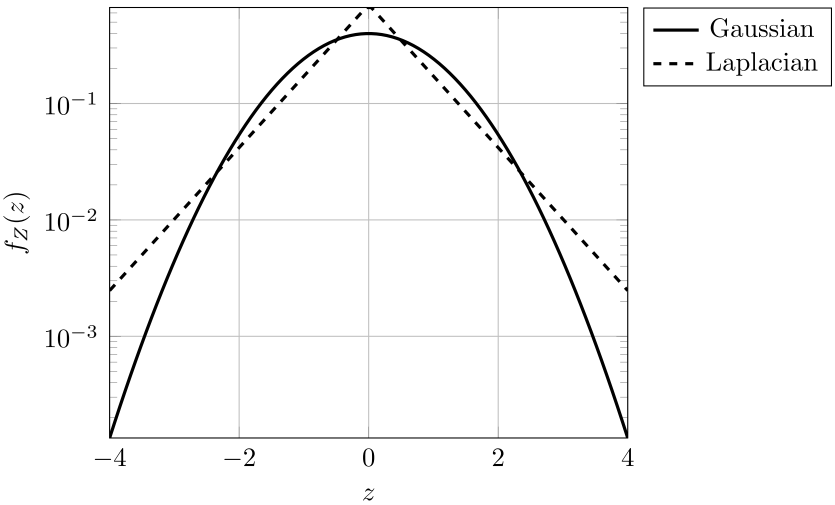
$$\phi_b(x) \triangleq \begin{cases} A & \text{if } x \geq A, \\ x & \text{if } -A \leq x \leq A, \\ -A & \text{if } x \leq -A. \end{cases}$$

Thus the MAP decision rule will be

$$\sum_{k=1}^{n} \phi_b(y_k - A) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0.$$



Here we plot two noise distributions for $\sigma = 1$ (it is convenient to use logarithmic axis):



The Laplacian distribution has larger tails: it puts more mass on zs with very large absolute value. Because of this, for the decision in part (b) the optimal choice is to first "clip" the input data $y_k,\ k = 1, \ldots, n$ so that these high values do not influence the decision.

Solution 6. The MAP decision rule is

$$\frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k - A)^2} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k + A)^2}.$$

Simplifying the common positive factor of $\frac{1}{(2\pi\sigma^2)^{n/2}}$ and taking the logarithm we have

$$-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k - A)^2 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} -\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k + A)^2,$$

which can further be simplified to

$$\sum_{k=1}^{n} y_k \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0.$$

Note that unlike the previous problem, for implementing the decision rule the receiver *does not need to know the value of* A.