SOLUTION 1.

(a) (i) The plots are shown below:

(ii) The joint density function is invariant under rotation for $\alpha = 2$ only. For this value of $\alpha$, we have $X, Y \sim \mathcal{N}(0, \frac{1}{2})$.

(b) (i) We know that we can write $(x, y)$ in polar coordinates $(r, \theta)$. Hence in general the joint distribution of $X$ and $Y$ is a function of $r$ and $\theta$. Because of circular symmetry the joint distribution should not depend on $\theta$, which means that $f_{X,Y}(x, y)$ can be written as a function of $r$. Hence if we denote this function by $\psi$ and use the independence of $X$ and $Y$, we have $f_X(x)f_Y(y) = \psi(r)$.

(ii) Taking the partial derivative with respect to $x$ and using the chain rule for differentiation, we have $\frac{f'_X(x)f_Y(y)}{x_f(x)} = \frac{\psi'(r)}{r\psi(r)} = \frac{\psi'(r)}{x_f}. \quad \text{If we divide both sides by } x_f(x)f_Y(y) \text{ we have } \frac{f'_X(x)}{x_f(x)} = \frac{\psi'(r)}{r\psi(r)}. \quad \text{Proceeding similarly for } y, \text{ we obtain}$

$$\frac{f'_X(x)}{x_f(x)} = \frac{\psi'(r)}{r\psi(r)} = \frac{f'_Y(y)}{y_f(y)}.$$
(iii) \( f_x(x) \) is a function of \( x \) while \( f_Y(y) \) is a function of \( y \). Hence the only way for the equality to hold is that both of them equal a constant. If we denote this constant by \(-\frac{1}{\sigma^2}\), we reach the final result.

(iv) We have \( f_x(x) = \frac{x}{\sigma^2} \). Integrating both sides we have \( \log(f_x(x)) = -\frac{x^2}{2\sigma^2} \). Hence \( f_x(x) = C \exp(-\frac{x^2}{2\sigma^2}) \). Hence \( f_X(x) = \sqrt{\frac{2}{\pi \sigma^2}} \exp(-\frac{x^2}{2\sigma^2}) \), which shows that \( X \) and \( Y \) are Gaussian random variables.

**Solution 2.**

(a) Let \( x_E(t) = x_R(t) + jx_I(t) \). Then

\[
x(t) = \sqrt{2} \Re \{ x_E(t) e^{j2\pi f_c t} \}
\]

\[
= \sqrt{2} \Re \{ x_R(t) + jx_I(t) \} e^{j2\pi f_c t} \}
\]

\[
= \sqrt{2} [ x_R(t) \cos(2\pi f_c t) - x_I(t) \sin(2\pi f_c t) ] .
\]

Hence, we have

\( x_{EI}(t) = \sqrt{2} \Re \{ x_E(t) \} \)

and

\( x_{EQ}(t) = \sqrt{2} \Im \{ x_E(t) \} \).

(b) Let \( x_E(t) = \alpha(t) e^{j\beta(t)} \). Then

\[
x(t) = \sqrt{2} \Re \{ x_E(t) e^{j2\pi f_c t} \}
\]

\[
= \sqrt{2} \Re \{ \alpha(t) e^{j\beta(t)} e^{j2\pi f_c t} \}
\]

\[
= \sqrt{2} \Re \{ \alpha(t) e^{j(2\pi f_c t + \beta(t))} \}
\]

\[
= \sqrt{2} \alpha(t) \cos[2\pi f_c t + \beta(t)] .
\]

We thus have

\( x_{E}(t) = \alpha(t) e^{j\beta(t)} = \frac{a(t)}{\sqrt{2}} e^{j\theta(t)} \).

(c) From (b) we see that

\( x_{E}(t) = \frac{A(t)}{\sqrt{2}} e^{j\varphi} \).

This is consistent with Example 7.9 (DSB-SC) given in the text. We can also verify:

\[
x(t) = \sqrt{2} \Re \{ x_{E}(t) e^{j2\pi f_c t} \}
\]

\[
= \sqrt{2} \Re \left\{ \frac{A(t)}{\sqrt{2}} e^{j\varphi} e^{j2\pi f_c t} \right\}
\]

\[
= \Re \{ A(t) e^{j(2\pi f_c t + \varphi)} \}
\]

\[
= A(t) \cos(2\pi f_c t + \varphi) .
\]
Solution 3.

(a) The key observation is that while \( e^{j2\pi f_1 t} \) and \( e^{-j2\pi f_1 t} \) are two different signals if \( f_1 \neq 0 \), \( \mathbb{R}\{e^{j2\pi f_1 t}\} \) and \( \mathbb{R}\{e^{-j2\pi f_1 t}\} \) are identical.

Therefore, if we fix \( f_1 \neq 0 \) and choose \( a_1(t) \) and \( a_2(t) \) so that \( a_1(t)e^{j2\pi f_1 t} = e^{j2\pi f_1 t} \) and \( a_2(t)e^{j2\pi f_1 t} = e^{-j2\pi f_1 t} \), we get \( a_1(t) \neq a_2(t) \) and \( \mathbb{R}\{a_1(t)e^{j2\pi f_1 t}\} = \mathbb{R}\{a_2(t)e^{j2\pi f_1 t}\} \).

Let \( a_1(t) = e^{-j2\pi(f_c-f_1)t} \) and \( a_2(t) = e^{-j2\pi(f_1+f_c)t} \). Then \( a_1(t) \neq a_2(t) \) and

\[
\sqrt{2}\mathbb{R}\{a_1(t)e^{j2\pi f_1 t}\} = \sqrt{2}\mathbb{R}\{a_2(t)e^{j2\pi f_1 t}\}.
\]

(b) Let \( b(t) = a(t)e^{j2\pi f_c t} \), which represents a translation of \( a(t) \) in the frequency domain. If \( a_F(f) = 0 \) for \( f < -f_c \), then \( b_F(f) = 0 \) for \( f < 0 \). Because \( \mathbb{R}\{b(t)\} = \frac{1}{2}(a(t)e^{j2\pi f_c t} + a^*(t)e^{-j2\pi f_c t}) \), taking the real part has a scaling effect and adds a negative-frequency component. The negative spectrum is canceled by the \( h_f \) filter, and the scaling is compensated by the \( \sqrt{2} \) factors from the up-converter and down-converter. Multiplying by \( e^{-j2\pi f_c t} \) translates the spectrum back to the initial position. In conclusion, we obtain \( a(t) \).

(c) Take any baseband signal \( u(t) \) with frequency domain support \([-f_c-\Delta, f_c+\Delta] \), \( \Delta > 0 \). The signal can be real-valued or complex-valued (for example \( u_F(f) = 1_{[-f_c-\Delta,f_c+\Delta]}(f) \), which is a sinc in time domain). After we up-convert, the support of \( u_F(f) \) will not extend beyond \( 2f_c + \Delta \). When we chop the negative frequencies we obtain a support contained in \([0, 2f_c+\Delta]\) and when we shift back to the left the support will be contained in \([-f_c, f_c+\Delta]\), which is too small to be the support of \( u_F(f) \).

(d) In time domain:

\[
w(t) = \sqrt{2}\mathbb{R}\{a(t)e^{j2\pi f_c t}\} \quad \text{a.e.} \quad \sqrt{2}a(t)\cos(2\pi f_c t).
\]

Therefore,

\[
a(t) = \frac{w(t)}{\sqrt{2}\cos(2\pi f_c t)}.
\]

In frequency domain: If \( a_F(f) = 0 \) for \( f < -f_c \), we obtain \( a(t) \) as described in (b). In the following, we consider the case \( a_F(f) \neq 0 \) for \( f < -f_c \).

We have \( w_F(f) = \frac{1}{\sqrt{2}} [a_F(f - f_c) + a_F(f + f_c)] = a^+_F(f) + a^-_F(f) \), with \( a^+_F(f) = \frac{1}{\sqrt{2}} a_F(f - f_c) \) and \( a^-_F(f) = \frac{1}{\sqrt{2}} a_F(f + f_c) \), respectively. These two components have overlapping support in some interval centered at 0. However, there is no overlap for sufficiently large frequencies. This means that for sufficiently large frequencies \( f \) we have \( w_F(f) = \frac{1}{\sqrt{2}} a^+_F(f) \), which implies that from \( w_F(f) \) we can observe the right tail of \( a^+_F(f) \) and use that information to remove the right tail of \( a^-_F(f) \) from \( w_F(f) \) (the right tails of \( a^+_F(f) \) and \( a^-_F(f) \) are the same because \( a(t) \) is real). Hence, from \( w_F(f) \) we can read more of the right tail of \( a^+_F(f) \). The procedure can be repeated until we get to see \( a^+_F(f) \) for all frequencies above \( f_c \). At this point, using \( a_F(f) = a^+_F(f + f_c)\sqrt{2} \) and the fact that \( a(t) \) is real-valued, we have \( a_F(f) \) for the positive frequencies, hence for all frequencies.
\( x(t) \sqrt{2} \cos(2\pi f_s t) = x(t) \left[ \frac{e^{j 2\pi f_s t} + e^{-j 2\pi f_s t}}{\sqrt{2}} \right] \)

\[ = \sqrt{2} \Re \{ x_E(t) e^{j 2\pi f_s t} \} \left[ \frac{e^{j 2\pi f_s t} + e^{-j 2\pi f_s t}}{\sqrt{2}} \right] \]

\[ = \left[ x_E(t) e^{j 2\pi f_s t} + x_E^*(t) e^{-j 2\pi f_s t} \right] \left[ \frac{e^{j 2\pi f_s t} + e^{-j 2\pi f_s t}}{2} \right] \]

At the lowpass filter output we have

\[ \frac{x_E(t) + x_E^*(t)}{2} = \Re \{ x_E(t) \} . \]

The calculation for the other path is similar.

**Solution 5.**

(a) Notice that the sinusoids of \( w(t) \) have a period of \( T_s = 4 \text{ ms} \) units of time, which implies that \( f_c = \frac{1}{T_c} = \frac{1}{4 \text{ ms}} = 250 \text{ Hz} \).

(b) Notice that the phase of the sinusoidal signal changes every \( T_s = 4 \text{ ms} \). (Here we have \( T_s = T_c \), but in general it is not the case. In practice we usually have \( T_s \gg T_c \). See the note at the end.)

The expression of \( w(t) \) as a function of \( t \) is:

\[
\begin{align*}
\{ 4 \cos(2\pi f_s t - \pi) \} & \quad t \in ]0, T_s[ \quad \rightarrow \quad \Re \{ 4 e^{j(2\pi f_s t - \pi)} \} \quad t \in ]0, T_s[ \\
\{ 4 \cos(2\pi f_s t) \} & \quad t \in ]T_s, 2T_s[ \quad \rightarrow \quad \Re \{ 4 e^{j(2\pi f_s t)} \} \quad t \in ]T_s, 2T_s[ \\
\{ 4 \cos(2\pi f_s t + \pi) \} & \quad t \in ]2T_s, 3T_s[ \quad \rightarrow \quad \Re \{ 4 e^{j(2\pi f_s t + \pi)} \} \quad t \in ]2T_s, 3T_s[ \\
\{ 4 \cos(2\pi f_s t + \pi) \} & \quad t \in ]3T_s, 4T_s[ \quad \rightarrow \quad \Re \{ 4 e^{j(2\pi f_s t)} \} \quad t \in ]3T_s, 4T_s[ \\
\{ -4 j e^{j2\pi f_s t} \} & \quad t \in ]0, T_s[ \\
\{ 4 e^{j2\pi f_s t} \} & \quad t \in ]T_s, 2T_s[ \quad \rightarrow \quad \sqrt{2} \Re \{ w_E(t) e^{j2\pi f_s t} \} , \\
\{ -4 e^{j2\pi f_s t} \} & \quad t \in ]2T_s, 3T_s[ \\
\{ 4 j e^{j2\pi f_s t} \} & \quad t \in ]3T_s, 4T_s[ 
\end{align*}
\]

where

\[
\begin{align*}
w_E(t) &= -\frac{4 j}{\sqrt{2}} \mathbb{1} \{ t \in ]0, T_s[ \} + \frac{4}{\sqrt{2}} \mathbb{1} \{ t \in ]T_s, 2T_s[ \} \\
& \quad - \frac{4}{\sqrt{2}} \mathbb{1} \{ t \in ]2T_s, 3T_s[ \} + \frac{4 j}{\sqrt{2}} \mathbb{1} \{ t \in ]3T_s, 4T_s[ \} \\
& = - j \sqrt{8 T_s} \frac{1}{\sqrt{T_s}} \mathbb{1} \{ t \in ]0, T_s[ \} + \sqrt{8 T_s} \frac{1}{\sqrt{T_s}} \mathbb{1} \{ t \in ]T_s, 2T_s[ \} \\
& \quad - \sqrt{8 T_s} \frac{1}{\sqrt{T_s}} \mathbb{1} \{ t \in ]2T_s, 3T_s[ \} + j \sqrt{8 T_s} \frac{1}{\sqrt{T_s}} \mathbb{1} \{ t \in ]3T_s, 4T_s[ \} .
\end{align*}
\]
If we define \( \psi(t) = \frac{1}{\sqrt{T_s}} 1 \{ t \in [0, T_s] \} \), \( c_0 = -j \sqrt{8T_s} \), \( c_1 = \sqrt{8T_s} \), \( c_2 = -\sqrt{8T_s} \) and \( c_3 = j \sqrt{8T_s} \), we get

\[
  w_E(t) = \sum_{i=0}^{3} c_i \psi(t - iT_s).
\]

(1)

Therefore, the pulse used in the waveform former is \( \psi(t) = \frac{1}{\sqrt{T_s}} 1 \{ t \in [0, T_s] \} \), and the waveform former output signal is given by (1). The orthonormal basis that is used is \( \{ \psi(t - iT_s) \}^3_{i=0} \).

(c) The symbol sequence is \( \{ c_0, c_1, c_2, c_3 \} = \{ -j \sqrt{E_s}, \sqrt{E_s}, -\sqrt{E_s}, j \sqrt{E_s} \} \), where \( E_s = 8T_s \). We can see that the symbol alphabet is \( \{ \sqrt{E_s}, j \sqrt{E_s}, -\sqrt{E_s}, -j \sqrt{E_s} \} \).

(d) We have:

- The output sequence of the encoder is the symbol sequence, which is
  \[ \{ c_0, c_1, c_2, c_3 \} = \{ -j \sqrt{E_s}, \sqrt{E_s}, -\sqrt{E_s}, j \sqrt{E_s} \} \].
- The symbol alphabet contains 4 symbols. This means that each symbol represents two bits. Since the symbol rate is \( f_s = \frac{1}{T_s} = 250 \) symbols/s, the bit rate is \( 2 \times 250 = 500 \) bits/s.
- The input/output mapping can be obtained by assigning two bits for each symbol in the symbol alphabet. Keeping in mind that it is better to minimize the number of bit-differences between close symbols, we obtain the following input/output mapping (which is not unique, i.e., we can obtain other mappings that satisfy the mentioned criterion): \( \sqrt{E_s} \leftrightarrow 00 \), \( j \sqrt{E_s} \leftrightarrow 01 \), \( -\sqrt{E_s} \leftrightarrow 11 \) and \( -j \sqrt{E_s} \leftrightarrow 10 \).
- Assuming that the above input/output mapping was used, we can obtain the input sequence of the encoder: 10001101.

Note that in this example, we have \( T_s = T_c \), so \( f_c = f_s \). This is very unusual. In practice we almost always have \( f_c \gg f_s \), especially if we are using electromagnetic waves.