ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 30 Principles of Digital Communications

Problem Set 12 May 23, 2018

PROBLEM 1.

(a) Suppose X and Y are real-valued i.i.d. random variables with probability density function $f_X(s) = f_Y(s) = c\exp(-|s|^\alpha)$, where α is a parameter and $c = c(\alpha)$ is the normalizing factor.

  (i) Draw the contour of the joint density function for $\alpha = \frac{1}{2}$, $\alpha = 1$, $\alpha = 2$, and $\alpha = 3$.

  Hint: For simplicity, draw the set of points (x, y) for which $f_{X,Y}(x, y) = c^2(\alpha)e^{-1}$.

  (ii) For which value of α is the joint density function invariant under rotation? What is the corresponding distribution?

(b) In general we can show that if X and Y are i.i.d. random variables and $f_{X,Y}(x, y)$ is circularly symmetric, then X and Y are Gaussian. Use the following steps to prove this.

  (i) Show that if X and Y are i.i.d. and $f_{X,Y}(x, y)$ is circularly symmetric, then $f_X(x)f_Y(y) = \psi(r)$ where ψ is a univariate function and $r = \sqrt{x^2 + y^2}$.

  (ii) Take the partial derivative with respect to x and y to show that

$$\frac{f'_X(x)}{xf_X(x)} = \frac{\psi'(r)}{r\psi(r)} = \frac{f'_Y(y)}{yf_Y(y)}$$

  (iii) Argue that the only way for the above equalities to hold is that they be equal to a constant value, i.e. $\frac{f'_X(x)}{xf_X(x)} = \frac{\psi'(r)}{r\psi(r)} = \frac{f'_Y(y)}{yf_Y(y)} = -\frac{1}{\sigma^2}$.

  (iv) Integrate the above equations and show that X and Y should be Gaussian random variables.

PROBLEM 2. A real-valued passband signal $x(t)$ can be written as $x(t) = \sqrt{2}\Re\{x_E(t)e^{j2\pi f_c t}\}$, where $x_E(t)$ is the baseband-equivalent signal (complex-valued in general) with respect to the carrier frequency f_c. Also, a general complex-valued signal $x_E(t)$ can be written in terms of two real-valued signals, either as $x_E(t) = u(t) + jv(t)$ or as $x_E(t) = \alpha(t)\exp(j\beta(t))$.

(a) Show that a real-valued passband signal $x(t)$ can always be written as

$$x_{EI}(t)\cos(2\pi f_c t) - x_{EQ}(t)\sin(2\pi f_c t)$$

and relate $x_{EI}(t)$ and $x_{EQ}(t)$ to $x_E(t)$.

Comment: This formula can be used at the sender to produce $x(t)$ without doing complex-valued operations. The signals $x_{EI}(t)$ and $x_{EQ}(t)$ are called the in-phase and the quadrature components respectively.

(b) Show that a real-valued passband signal $x(t)$ can always be written as

$$a(t)\cos(2\pi f_c t + \theta(t))$$

and relate $a(t)$ and $\theta(t)$ to $x_E(t)$.

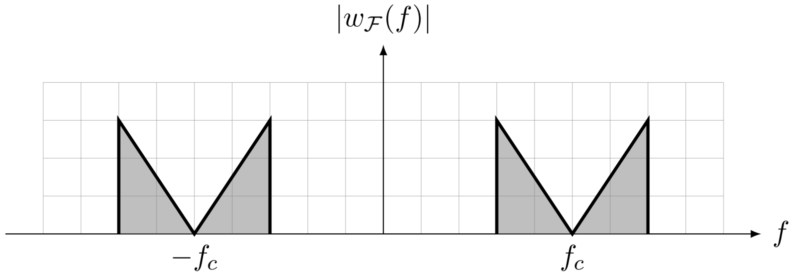
Comment: This explains why sometimes people make the claim that a passband signal is modulated in amplitude and in phase.

(c) Use part (b) to find the baseband-equivalent of the signal

$$x(t) = A(t)\cos(2\pi f_c t + \varphi),$$

where $A(t)$ is a real-valued lowpass signal. Verify your answer with Example 7.9 where we assumed $\varphi = 0$.

Problem 3. Let f_c be a positive carrier frequency and consider an arbitrary real-valued function $w(t)$ whose Fourier transform is shown below:



(a) Argue that there are two different functions, $a_1(t)$ and $a_2(t)$, such that, for $i = \{1, 2\}$,

$$w(t) = \sqrt{2}\Re\{a_i(t)\exp(\mathrm{j}2\pi f_c t)\}$$

This shows that, without some constraint on the input signal, the operation performed by the circuit of Figure 7.4b is not reversible, even in the absence of noise. This was already pointed out in the discussion preceding Lemma 7.8.

(b) Argue that if we limit the input of Figure 7.4b to signals $a(t)$ such that $a_{\mathcal{F}}(f) = 0$ for $f < -f_c$, then the circuit of Figure 7.4a will retrieve $a(t)$ when fed with the output of Figure 7.4b.

(c) Find an example showing that the condition of part (b) is necessary. (Can you find and example with a real-valued $a(t)$?)

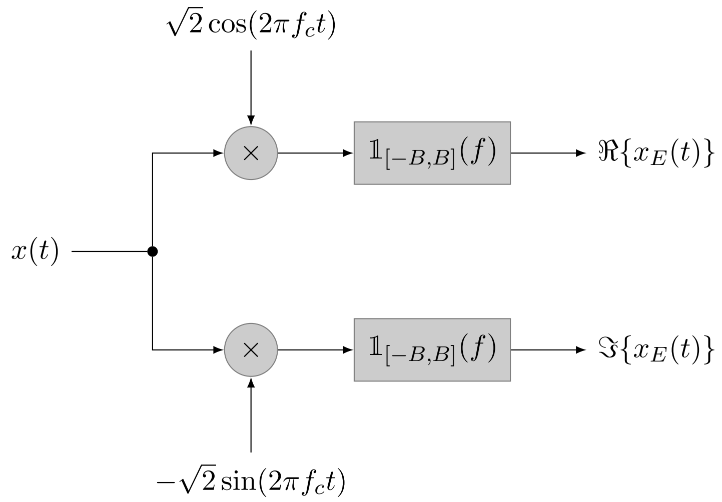
(d) Argue that if we limit the input of Figure 7.4b to signals $a(t)$ that are real-valued, then the input of Figure 7.4b can be retrieved from the output.

Comment: We are not claiming that the circuit of Figure 7.4a will retrieve $a(t)$.

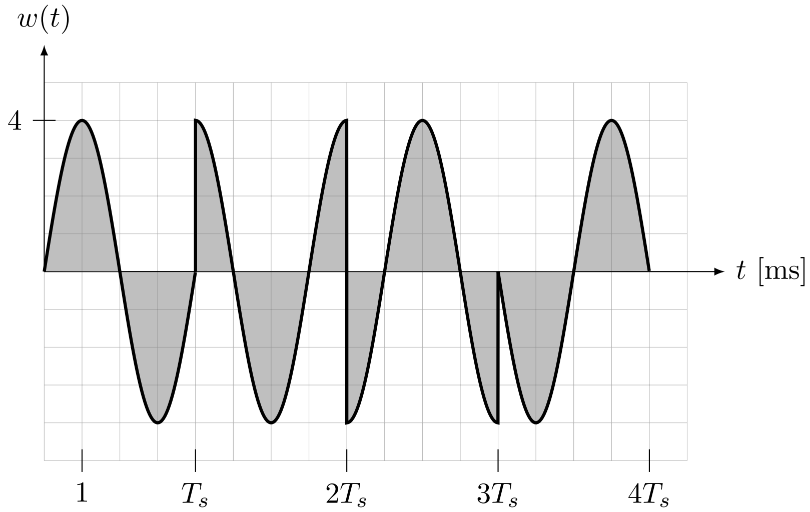
Hint: You may argue in the time domain or in the frequency domain. If you argue in the time domain, you can assume that $a(t)$ is continuous. If you argue in the frequency domain, you can assume that $a(t)$ has finite bandwidth.

Problem 4. Let the signal $x_E(t)$ be band-limited to $[-B, B]$ and let $x(t) = \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\}$, where $0 < B < f_c$. Show that the circuit shown below recovers the real and imaginary part of $x_E(t)$ when fed with $x(t)$. (The two boxes are ideal lowpass filters of cutoff frequency B.)

Comment: The circuit uses only real-valued operations.



PROBLEM 5.



The figure above shows a toy passband signal. (Its carrier frequency is unusually low with respect to its symbol rate.) Specify the three layers of a transmitter that generates the given signal, namely the following:

(a) The carrier frequency f_c used by the up-converter.

(b) The orthonormal basis used by the waveform former to produce the baseband-equivalent signal $w_E(t)$.

(c) The symbol alphabet, seen as a subset of $\mathbb{C}$.

(d) An encoding map, the encoder input sequence that leads to $w(t)$, the bit rate, the encoder output sequence, and the symbol rate.