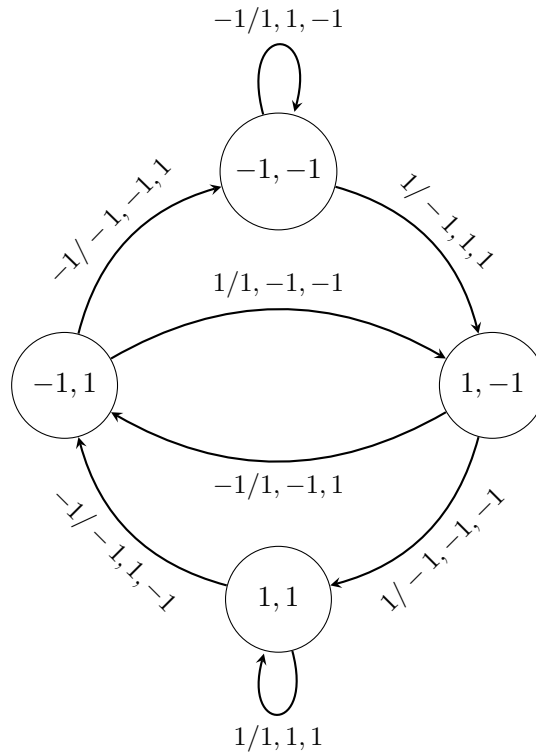
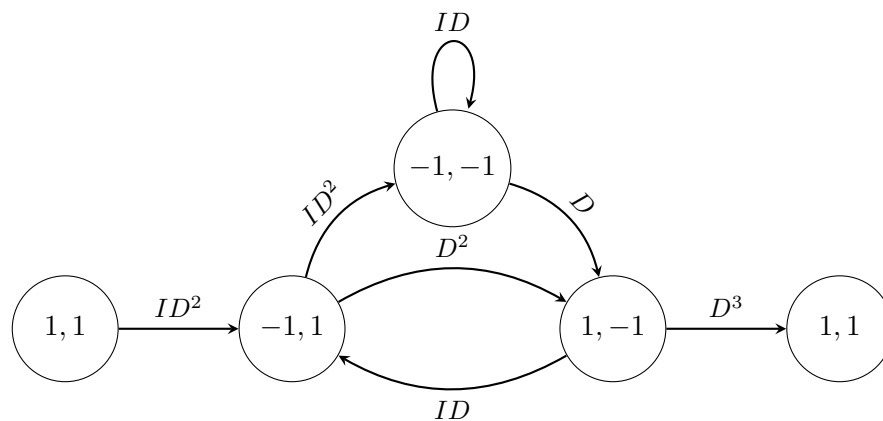


SOLUTION 1.

(a) The state diagram and detour flow graph are respectively shown below:



State diagram



Detour flow graph

(b) Let  $a, b, c, d, e$  respectively represent the states  $(1, 1), (-1, 1), (-1, -1), (1, -1)$  and  $(1, 1)$ . We have

$$T_b = T_d ID + T_a ID^2$$

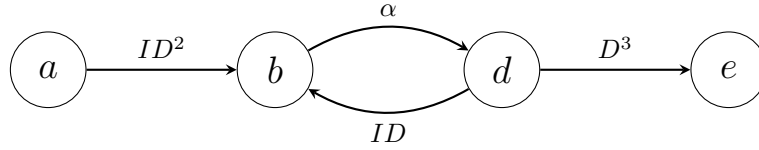
$$T_c = T_c ID + T_b ID^2$$

$$T_d = T_b D^2 + T_c D.$$

Substituting  $T_c = T_b \frac{ID^2}{1-ID}$  in the third equation above,

$$\begin{aligned} T_d &= T_b D^2 + T_b \frac{ID^3}{1-ID} \\ &= T_b \left( D^2 + \frac{ID^3}{1-ID} \right) \\ &= T_b \frac{D^2}{1-ID} \\ &= T_b \alpha, \end{aligned}$$

with  $\alpha = \frac{D^2}{1-ID}$ . The detour flow graph can thus be simplified to:



In  $T_b = T_d ID + T_a ID^2$ , we substitute for  $T_d$  to get

$$T_b = T_a \frac{ID^2(1-ID)}{1-ID-ID^3}.$$

It follows that

$$T_d = T_b \frac{D^2}{1-ID} = T_a \frac{ID^4}{1-ID-ID^3},$$

and that

$$T(I, D) = T_e = T_a \frac{ID^7}{1-ID-ID^3}.$$

Taking the derivative yields

$$\frac{\partial T(I, D)}{\partial I} = \frac{D^7(1-ID-ID^3) - ID^7(-D-D^3)}{(1-ID-ID^3)^2} = \frac{D^7}{(1-ID-ID^3)^2}.$$

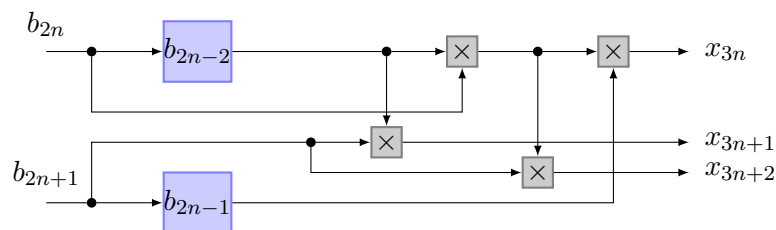
Therefore, we find

$$\begin{aligned} P_b &\leq \left. \frac{\partial T(I, D)}{\partial I} \right|_{I=1, D=z} \\ &= \frac{z^7}{(1-z-z^3)^2}, \end{aligned}$$

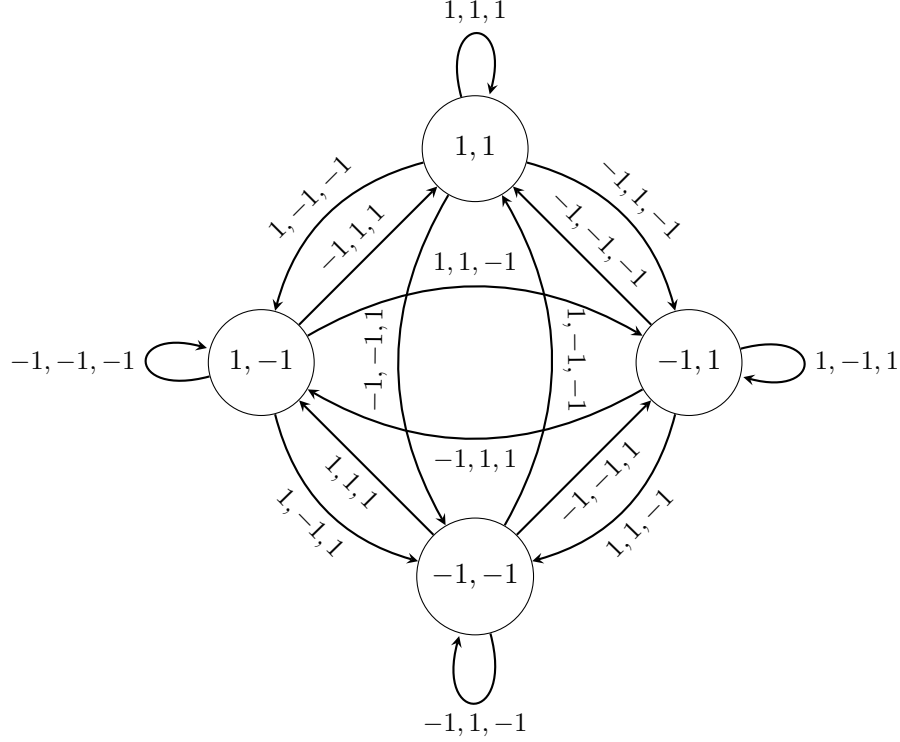
where  $z = e^{-\frac{\epsilon_s}{N_0}}$ .

SOLUTION 2.

(a) An implementation of the encoder will be as follows:



- (b) The state diagram is shown below. We use the following terminology: the state label is  $x, y$ , where  $x$  is the “state of the even sub-sequence”, i.e. contains  $b_{2n-2}$ , and  $y$  is the “state of the odd sub-sequence”, i.e., contains  $b_{2n-1}$ . On the arrows, we only mark the outputs; the input required to make a particular transition is simply the next state, therefore we omitted it. Transitions are labeled with the value of  $x_{3n}, x_{3n+1}, x_{3n+2}$ .



- (c) We use

$$P_b \leq \frac{1}{k_0} \left. \frac{\partial T(I, D)}{\partial I} \right|_{I=1, D=z},$$

where  $z = e^{-\frac{\mathcal{E}_s}{N_0}}$  and  $k_0$  is the number of inputs per section of the trellis. In this problem,  $k_0 = 2$ . Since there are three channel symbols per two source symbols, we find that  $\mathcal{E}_s = 2\mathcal{E}_b/3$ .

From the state diagram we can derive the generating functions of the detour flow graph:

$$\begin{aligned} T(I, D) &= D^3 T_{-1,1} + D^2 T_{-1,-1} + D T_{1,-1} \\ T_{1,-1} &= I D T_{-1,1} + I T_{-1,-1} + I D^3 T_{1,-1} + I D^2 T_{1,1} \\ T_{-1,-1} &= I^2 D T_{-1,1} + I^2 D^2 T_{-1,-1} + I^2 D T_{1,-1} + I^2 D^2 T_{1,1} \\ T_{-1,1} &= I D T_{-1,1} + I D^2 T_{-1,-1} + I D T_{1,-1} + I D^2 T_{1,1}. \end{aligned}$$

Solving the system gives

$$T(I, D) = T_{1,1} \frac{D^2 I (D^6 I + D^5 I^2 - D^3 - D^4 I - D)}{-D^5 I^3 - D^4 I^2 + D^3 I + 2D^2 I^2 + D^2 I + D I^3 + D I^2 + D I - 1},$$

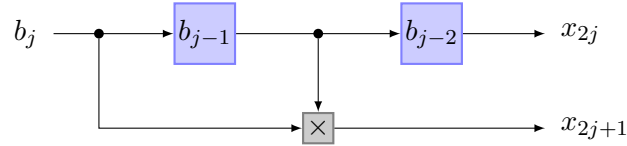
on which we can apply the formula above.

SOLUTION 3.

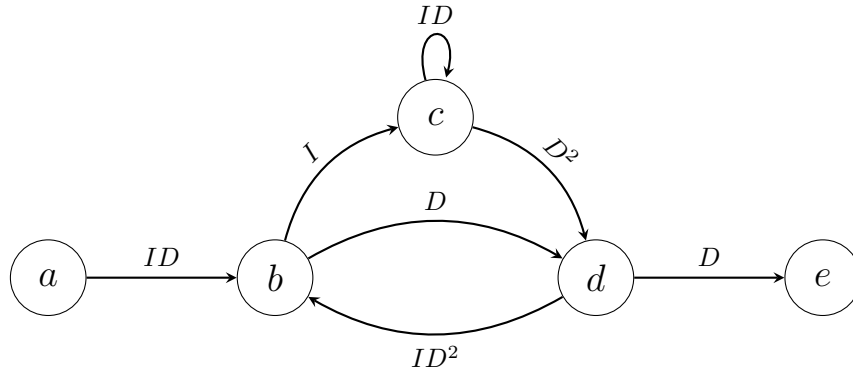
- (a) Since the state is  $(b_{j-1}, b_{j-2})$ , we need two shift registers. From the finite state machine, we can derive a table that relates the state  $(b_{j-1}, b_{j-2})$  and the current input  $b_j$  with the two outputs  $(x_{2j}, x_{2j+1})$ :

$b_j$	$b_{j-1}$	$b_{j-2}$	$x_{2j}$	$x_{2j+1}$
1	1	1	1	1
1	1	-1	-1	1
1	-1	1	1	-1
1	-1	-1	-1	-1
-1	1	1	1	-1
-1	1	-1	-1	-1
-1	-1	1	1	1
-1	-1	-1	-1	1

We can easily notice that the column of  $x_{2j}$  is the same as the column of  $b_{j-2}$ . Therefore,  $x_{2j} = b_{j-2}$ . On the other hand, we see that  $x_{2j+1} = b_{j-1}$  if  $b_j = 1$  and  $x_{2j+1} = -b_{j-1}$  if  $b_j = -1$ . Therefore  $x_{2j+1} = b_j \cdot b_{j-1}$ , which gives us the following encoder.



- (b) The detour flow graph (with respect to the all-one sequence) is given below:



We have

$$\begin{aligned}
 T_b &= T_a ID + T_d ID^2 \\
 T_c &= T_b I + T_c ID \\
 T_d &= T_c D^2 + T_b D \\
 T_e &= T_d D
 \end{aligned}$$

The solution of this system is  $T_e = T_a \frac{ID^3}{1-ID-ID^3}$ . Hence,

$$\begin{aligned}
 P_b &\leq \left. \frac{\partial T(I, D)}{\partial I} \right|_{I=1, D=z} = \left. \frac{D^3(1-ID-ID^3) + ID^3(D+D^3)}{(1-ID-ID^3)^2} \right|_{I=1, D=z} \\
 &= \frac{z^3}{(1-z-z^3)^2},
 \end{aligned}$$

where  $z = e^{-\frac{\epsilon_b}{2N_0}}$ .

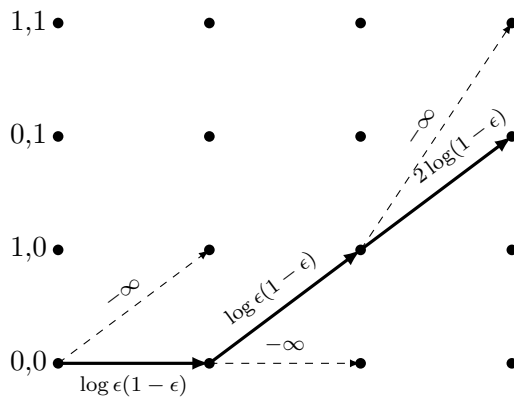
SOLUTION 4.

- (a) The decoder is the same as in the example we have seen in Chapter 6 once the following isomorphic mapping is applied:  $\{1 \rightarrow 0, -1 \rightarrow 1\}$ . Figure 6.4 shows the trellis of the encoder.
- (b) Given the observation  $y = (y_1, \dots, y_n)$ , the ML codeword is given by  $\arg \max_{x \in \mathcal{C}} p(y|x)$  where  $\mathcal{C}$  represents the set of codewords (i.e., the set of all possible paths on the trellis). Alternately, the ML codeword is given by  $\arg \max_{x \in \mathcal{C}} \sum_{i=1}^n \log p(y_i|x_i)$ .

Hence, a branch metric for the BEC is

$$\log p(y_i|x_i) = \begin{cases} \log \epsilon & \text{if } y_i = ?, \\ \log(1 - \epsilon) & \text{if } y_i = x_i, \\ -\infty & \text{if } y_i = 1 - x_i. \end{cases}$$

- (c) Given the observation  $(0, ?, ?, 1, 0, 1)$ , one can compute the branch metric in the trellis. Note that we do not need to further elaborate paths with a  $-\infty$  metric. The decoding results  $(0, 1, 0)$ .



- (d) We refer to the example shown in Chapter 6, where we have the same encoder, but a different channel. We have seen that

$$P_b \leq \frac{z^5}{(1 - 2z)^2}.$$

To determine  $z$  we use the Bhattacharyya bound, which in our case is

$$z = \sum_{y \in \{0,1,?\}} \sqrt{P(y|1)P(y|0)} = \epsilon.$$

Thus we have the following bound:

$$P_b \leq \frac{\epsilon^5}{(1 - 2\epsilon)^2}.$$