Problem 1. Assume that $X_1$ and $X_2$ are independent random variables that are uniformly distributed in the interval $[0, 1]$. Compute the probability of the following events.

(a) $0 \leq X_1 - X_2 \leq \frac{1}{3}$.

(b) $X_1^3 \leq X_2 \leq X_1^2$.

(c) $X_2 - X_1 = \frac{1}{2}$.

(d) $(X_1 - \frac{1}{2})^2 + (X_2 - \frac{1}{2})^2 \leq \left(\frac{1}{2}\right)^2$.

(e) Given that $X_1 \geq \frac{1}{4}$, compute the probability that $(X_1 - \frac{1}{2})^2 + (X_2 - \frac{1}{2})^2 \leq \left(\frac{1}{2}\right)^2$.

Hint: For each event, identify the corresponding region inside the unit square.

Problem 2. Find the following probabilities.

(a) A box contains $m$ white and $n$ black balls. Suppose $k$ balls are drawn. Find the probability of drawing at least one white ball.

(b) We have two coins; the first is fair and the second is two-headed. We pick one of the coins at random, toss it twice, and obtain heads both times. Find the probability that the coin is fair.

Problem 3. Assume that $X$ and $Y$ are random variables with joint density function

$$f_{X,Y}(x, y) = \begin{cases} A, & 0 \leq x < y \leq 1 \\ 0, & \text{otherwise}. \end{cases}$$

(a) Are $X$ and $Y$ independent?

(b) Compute the value of $A$.

(c) Find the density function of $Y$. Do this first by arguing geometrically then compute it formally.

(d) Find $\mathbb{E}[X|Y = y]$. Hint: Argue geometrically

(e) The $\mathbb{E}[X|Y = y]$ found in (d) is a function of $y$, call it $f(y)$. Find $\mathbb{E}[f(Y)]$. This is $\mathbb{E}[\mathbb{E}[X|Y]]$.

(f) Find $\mathbb{E}[X]$ from the definition. Verify that $\mathbb{E}[X]$ is equal to $\mathbb{E}[\mathbb{E}[X|Y]]$ that you have found in (e). Is this a coincidence?

Problem 4. Let $Z_1$ and $Z_2$ be i.i.d. zero-mean Gaussian random variables, i.e., the pdf of $Z_i$, $i = 1, 2$ is

$$f_Z(z) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{z^2}{2\sigma^2}}$$

for some $\sigma > 0$. Define

$$X := \frac{Z_1}{\sqrt{Z_1^2 + Z_2^2}} \quad \text{and} \quad Y := \frac{Z_2}{\sqrt{Z_1^2 + Z_2^2}}.$$ 

Prove that $(X, Y)$ is a uniformly chosen point on the unit circle.
Problem 5.

(a) Let $X$ and $Y$ be two continuous real-valued random variables with joint probability density function $f_{X,Y}$. Show that if $X$ and $Y$ are independent, they are also uncorrelated.

(b) Consider two independent and uniformly distributed random variables $U \in \{0, 1\}$ and $V \in \{0, 1\}$. Assume that $X$ and $Y$ are defined as follows: $X = U + V$ and $Y = |U - V|$. Are $X$ and $Y$ independent? Compute the covariance of $X$ and $Y$. What do you conclude?

Problem 6. Assume you pick a point on the surface of the unit sphere (i.e. the sphere centered at the origin with radius 1) uniformly at random and $(X, Y, Z)$ denotes its coordinates (in 3D space). Compute $E[X^2]$.

Problem 7. Assume the random variable $X$ has an exponential distribution given by $f_X(x) = e^{-x}$ when $x \geq 0$. Similarly, $\hat{X}$ is exponentially distributed with $f_{\hat{X}}(\hat{x}) = 2e^{-2\hat{x}}$ for $\hat{x} \geq 0$.

(a) For what values of $x$ do we have $f_X(x) \leq f_{\hat{X}}(x)$?

(b) Calculate $\mathbb{P}(f_X(X) \leq f_{\hat{X}}(X))$.

(c) Calculate $\mathbb{P}(f_X(\hat{X}) \geq f_{\hat{X}}(\hat{X}))$. 