PROBLEM 1.

(a) Note that under either hypotesis (X_1, X_2) is real valued. With $\theta = 0$

$$(Y_1, Y_2) = (X_1, X_2) + (Z_1, Z_2) = (X_1 + \Re\{Z_1\}, X_2 + \Re\{Z_2\}) + j(\Im\{Z_1\}, \Im\{Z_2\})$$

Since the imaginary parts of the Z's are independent of the real parts we see (by invoking Fischer-Neyman) that the real part of (Y_1, Y_2), i.e., T, is a sufficient statistic. From the above observation, the log-likelihood ratio is given by

$$\begin{aligned}\sigma^{-2}\big[(\Re\{Y_1\}^2 + (\Re\{Y_2\} - \sqrt{\mathcal{E}})^2 - (\Re\{Y_1\} - \sqrt{\mathcal{E}})^2 - \Re\{Y_2\}^2\big]&\\ = 2\sqrt{\mathcal{E}}\sigma^{-2}\Re\{Y_1 - Y_2\} = 2\sqrt{\mathcal{E}}\sigma^{-2}U,&\end{aligned}$$

so we see that U is also a sufficient statistic.

(b) From (a) we see that the MAP rule is to decide 0 if $U > 0$ and 1 else. Note that $U = (X_1 - X_2) + Z$ where $Z = \Re\{Z_1\} - \Re\{Z_2\}$ is a Gaussian with zero mean and variance σ^2, (the sum of the variances of the real parts of Z_1 and Z_2). When 0 is sent $(X_1 - X_2) = \sqrt{\mathcal{E}}$ and when 1 is sent $(X_1 - X_2) = -\sqrt{\mathcal{E}}$. Thus, the probability of error is $Q(\sqrt{\mathcal{E}/\sigma^2})$.

(c) When θ is $\pi/2$, the observation is

$$(Y_1, Y_2) = (\Re\{Z_1\}, \Re\{Z_2\}) + j(X_1 + \Im\{Z_1\}, X_2 + \Im\{Z_2\}).$$

Conseqently the real part of the observation used in (b) is independent of the transmitted codeword. The error probability is thus 1/2.

(d) With θ uniform random variable, the probability density of the observation

$$f_{Y_1Y_2|H}(y_1, y_2|0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{(\pi\sigma^2)^2} \exp\big\{-\frac{1}{\sigma^2}\big(|y_1 - e^{j\theta}\sqrt{\mathcal{E}}|^2 + |y_2|^2\big)\big\}\, d\theta,$$

With ϕ_1 denoting the phase of y_1,

$$|y_1 - e^{j\theta}\sqrt{\mathcal{E}}|^2 + |y_2|^2 = |y_1|^2 + |y_2|^2 + \mathcal{E} - 2\sqrt{\mathcal{E}}|y_1|\cos(\theta - \phi_1),$$

thus

$$f_{Y_1Y_2|H}(y_1y_2|0) = \frac{1}{(\pi\sigma^2)^2}\exp\big\{-\sigma^{-2}(|y_1|^2 + |y_2|^2 + \mathcal{E})\big\}\frac{1}{2\pi}\int_0^{2\pi} g(2\sqrt{\mathcal{E}}|y_1|/\sigma^2, \theta, \phi_1),$$

with $g(\cdot)$ as in the hint. By the hint, the integral of g depends only on the value of $2\sqrt{\mathcal{E}}|y_1|/\sigma^2$ and consequently, $f_{Y_1Y_2|H}(y_1, y_2|0)$ depends only on $(|y_1|, |y_2|)$. The case for $f_{Y_1Y_2|H}(y_1, y_2|1)$ is analogous, only by swapping y_1 and y_2.

(e) By part (d) the likelihood ratio is given by

$$\frac{f_{Y_1Y_2|H}(y_1, y_2|0)}{f_{Y_1Y_2|H}(y_1, y_2|1)} = \frac{I_0\big(2\sqrt{\mathcal{E}}|y_1|/\sigma^2\big)}{I_0\big(2\sqrt{\mathcal{E}}|y_2|/\sigma^2\big)}$$

and the MAP rule will decide in favor of 0 if the numerator is larger than the denominator (and vice versa). By the hint I_0 is increasing in its argument when the argument is non-negavite, so the MAP rule is to decide 0 if $|y_1| > |y_2|$ and to decide 1 otherwise.

The situation analyzed in this problem is equivalent to the following: one of two orthogonal, complex baseband signals w_{E0} or w_{E1}, each with energy $\mathcal{E}$ is sent over an AWGN channel by upconverting them to the passband. At the receiver, the received signal is downcoverted to the baseband, and passed through two matched filters: one matched to w_{E0} and the other to w_{E1}. The output of the filters are Y_1 and Y_2. The quantity θ is the phase difference between the transmitters upconverter and the receivers downconverter oscillators. Parts (a) and (b) analyze the case when the two oscillators are completely in-phase (known as 'coherent'); part (d) considers the case of uncoherent reception where the phase difference is completely random.

Problem 2.

(a) Since ψ is real and symmetric the receiver is passing the received signal through the filter $\psi^*(-t)$ which is the matched filter. The output of the matched filter should thus be sampled at integer times, $\dots, -2, -1, 0, 1, 2, \dots$.

(b) With ψ_j denoting the ψ time-shifted by j, we have $w = \sum_j c_j \psi_j$, and $Y_j = \langle w + N, \psi_j \rangle$. As $\{\psi_j\}$ form an orthnormal collection we see that

$$Y_j = c_j + Z_j$$

where $\{Z_j\}$ is an i.i.d. collection of zero mean Gaussians with variance $N_0/2$.

(c) From (b) the decision rule is to decide

$$\hat{c}_j = \begin{cases} 0 & Y_j > 0 \\ 1 & \text{else,} \end{cases}$$

and the error probability is thus $Q(\sqrt{2\mathcal{E}/N_0})$.

(d) Let $\tilde{\psi}_j$ denote ψ shifted by $j - 0.1$. Then

$$Y_j = \langle w + N, \tilde{\psi}_j \rangle = 0.9c_j + 0.1c_{j-1} + Z_j$$

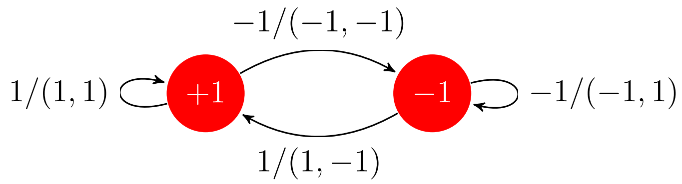
(e) With the decision rule still as in (c), there are two equally likely possibilities either $c_{j-1} = c_j$ or $c_{j-1} = -c_j$. In the first case the error probability is $Q(\sqrt{2\mathcal{E}/N_0})$, in the second the error probability is $Q(0.8\sqrt{2\mathcal{E}/N_0})$. The bit error probability is thus

$$\tfrac{1}{2}\big[Q(\sqrt{2\mathcal{E}/N_0}) + Q(0.8\sqrt{2\mathcal{E}/N_0})\big].$$

(f) We see that the data bit sequence $\{c_j\}$ influence the received sequence $\{Y_j\}$ via $x_j = 0.9c_j + 0.1c_{j-1}$. The x's are obtained from c's via a two-state 'encoder': state at time j is c_{j-1}. The MAP rule consists of finding the c sequence whose x sequence is closest to the received Y sequence, and this can be done by the Viterbi algorithm that walks through a two-state trellis. (This is as in Problem 3 of Homework 10.)

PROBLEM 3.

(a) Using the convention (x_{2j-1}, x_{2j}), the state diagram of the encoder is



(b) We need to find T such that shifts of $\psi_{\mathcal{F}}$ by integer multiples of $1/T$ add to a constant. (This constant should then equal T to ensure the unit norm constraint). By inspection we see that $|\psi_{\mathcal{F}}|^2$ has band-edge symmetry around 2.5 kHz. Consequently $1/(2T) = 2.5$ kHz, and we find $T = 0.2$ ms. We also obtain $b = 0.2 \times 10^{-3}$.

(c) Since one coded bit is sent every T seconds, we transmit 5000 coded bits each second. Since each data bit generates two coded bits, the bit rate is 2.5 kbps.

(d) It is easy to see that $E[x_i] = 0$. For the correlations between x_n and x_m, note that

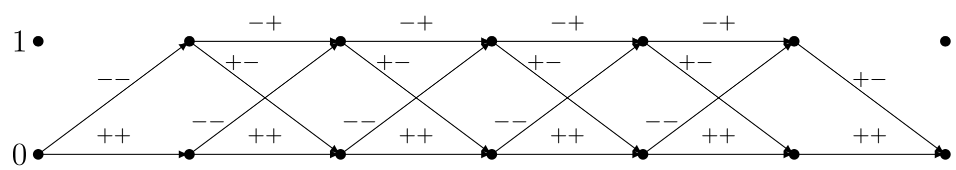
$$x_n x_m = \begin{cases} b_i b_j & n = 2i-1, m = 2j-1 \\ b_i b_{i-1} b_j & n = 2i, m = 2j-1 \\ b_i b_j b_{j-1} & n = 2i-1, m = 2j \\ b_i b_{i-1} b_j b_{j-1} & n = 2i, m = 2j. \end{cases}$$

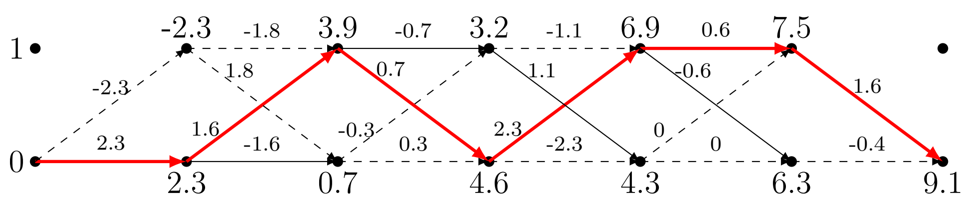
we see that unless $n = m$, there is at least one b_ℓ that occurs by itself in the expression $x_n x_m$. (E.g, $x_1 x_2 = b_1^2 b_0$ with $b_\ell = b_0$; $x_2 x_4 = b_1^2 b_0 b_2$ and we can take $\ell = 0$ or $\ell = 2$.) When we take the expection, as $E[b_\ell] = 0$, this ensures that $E[x_n x_m] = 0$, except when $n = m$ for which $E[x_n x_n] = 1$. Thus

$$K_X[k] = \begin{cases} 1 & k = 0 \\ 0 & \text{else.} \end{cases}$$

(e) Since the $\{x_k\}$ sequence is uncorrelated, the power spectral density of $w(t - \Theta)$ is given by $\mathcal{E}|\psi_{\mathcal{F}}(f)|^2/T = 5000\mathcal{E}|\psi_{\mathcal{F}}(f)|^2$.

(f) The trellis is given by





Therefore, the decoded bits are $(\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4, \hat{b}_5) = (1, -1, 1, -1, -1)$.