ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 35 Principles of Digital Communications
Final exam Jul. 3, 2018

3 problems, 82 points
165 minutes
2 sheet (4 pages) of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

Problem 1. (24 points) Consider a communication system with two codewords $c_0 = (\sqrt{\mathcal{E}}, 0)$ and $c_1 = (0, \sqrt{\mathcal{E}})$. One of these codewords is chosen (each with equal probability) and transmitted. With $X = (X_1, X_2)$ denoting the chosen codeword, the receiver's observation (Y_1, Y_2) is given by

$$(Y_1, Y_2) = (e^{j\theta} X_1 + Z_1,\; e^{j\theta} X_2 + Z_2)$$

where $0 \le \theta < 2\pi$ is a phase shift, and Z_1 and Z_2 are i.i.d. circularly symmetric complex Gaussian random variables and $E[|Z_1|^2] = \sigma^2$. (In other words, the real and imaginary parts of Z_1 and Z_2 are independent Gaussians with zero mean and variance $\sigma^2/2$.)

(a) (6 pts) Suppose $\theta = 0$. Is $T = (\Re\{Y_1\}, \Re\{Y_2\})$, the real part of the observation, a sufficient statistic? Is $U = \Re\{Y_1 - Y_2\}$ a sufficient statistic?

(b) (4 pts) Still supposing $\theta = 0$, find the MAP decision rule. Express the error probability in terms of a Q-function.

(c) (4 pts) Suppose now $\theta = \pi/2$, but we continue use the decision rule found in (b). What is the probability of error?

(d) (6 pts) Suppose now that θ is a random variable uniformly distributed on $[0, 2\pi)$, and independent of (Z_1, Z_2). Show that the probability density of the observation

$$f_{Y_1 Y_2|H}(y_1, y_2|i)$$

depends on (y_1, y_2) only through $(|y_1|, |y_2|)$.

Hint: Note that $g(a, \theta, \theta_0) = \exp(a\cos(\theta + \theta_0))$ is a periodic function of θ, thus $\int_0^{2\pi} g(a, \theta, \theta_0)\, d\theta$ depends only on a.

(e) (4 pts) Continuing with part (d), find the MAP decision rule.

Hint: with $I_0(a) = \int_0^{2\pi} g(a, \theta, \theta_0)\, d\theta$, it is known that I_0 is an even function, increasing in $|a|$.

PROBLEM 2. (30 points) Consider transmitting a sequence of bits over the continuous time AWGN channel of noise power spectral density $N_0/2$, using the strategy of "bit-by-bit on a pulse train." The transmitted signal is given by

$$w(t) = \sum_j c_j \psi(t - j)$$

where c_j are i.i.d. random variables taking the value $+\sqrt{\mathcal{E}}$ or $-\sqrt{\mathcal{E}}$ with equal probability, and

$$\psi(t) = \begin{cases} 1 & t \in [-1/2, 1/2) \\ 0 & \text{else.} \end{cases}$$

The receiver first passes the received signal through a filter with impulse response ψ.

(a) (4 pts) At what times should the filter output be sampled so as to minimize the bit error probability?

(b) (4 pts) Give a mathematical description of the discrete time channel model seen by the decoder. (In other words how do the samples $\{Y_j\}$ taken in (a) relate to the transmitted data $\{c_j\}$?)

(c) (4 pts) What is the bit error probability of the system above?

(d) (6 pts) Suppose now that the samples are offset from those you found in (a) by 0.1 units of time. Re-do part (b).

(e) (6 pts) Suppose the decoder is unaware of the time offset. What is the bit error probability?

(f) (6 pts) Suppose the decoder is aware of the time offset, but is constrained to work with these (incorrectly sampled) sample-values. How would you implement a decoder that finds the maximally likely transmitted sequence based on the samples $\{Y_j\}$? (Explain in words.)

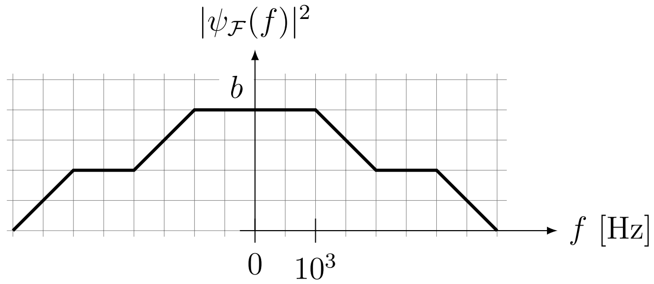
Problem 3. (28 points) Binary data $\{b_j\}$ (with b_j i.i.d., taking values in $+1$ and -1 with equal probability) is coded by a rate 1/2 convolution code as

$$x_{2j-1} = b_j, \quad x_{2j} = b_j b_{j-1}.$$

The coded bits $\{x_k\}$ are then transmitted over an AWGN channel as the signal

$$w(t) = \sum_k (\sqrt{\mathcal{E}} x_k)\psi(t - kT) \qquad (*)$$

where $\psi(t)$ is a real-valued unit-norm pulse with $|\psi_{\mathcal{F}}(f)|^2$ as follows:



(a) (4 pts) Draw the state diagram description of the convolutional encoder.

(b) (4 pts) Find T and $b = |\psi_{\mathcal{F}}(0)|^2$ so that equation $(*)$ is an orthonormal expansion with coefficients $\{\sqrt{\mathcal{E}} x_k\}$.

(c) (4 pts) How many coded bits per second are transmitted? What is the bit rate (the number of data bits transmitted per second)?

(d) (6 pts) What is the autocorrelation function of the coded bits? (That is, find $K_x[k] := E[x_i x_{i+k}] - E[x_i]E[x_{i+k}]$. [Hint: You may want to consider even/odd i separately.])

(e) (4 pts) What is the power spectral density of the signal $w(t-\Theta)$ where Θ is a random variable independent of everything else and uniformly distributed in $[0, T]$?

At the receiver the received signal $w(t) + N(t)$ is match filtered by $\psi(-t)$, sampled at instants $\{kT : k \in \mathbb{Z}\}$ to form the discrete-time sequence $\{Y_k\}$. The sequence $\{Y_k\}$ is then decoded by a Viterbi-decoder.

(f) (6 pts) Assume that the initial state of the encoder is the all-ones state. Suppose data b_1, b_2, b_3, b_4, b_5, followed by sufficiently many 1's to bring the encoder back to the all-ones state, is input to the encoder.

If the sequence $(Y_1, Y_2), (Y_3, Y_4), \dots$ is given by

$$(+1.2, +1.1), (+0.1, -1.7), (+0.5, -0.2), (-0.6, -1.7),$$
$$(-0.3, +0.3), (+0.6, -1.0), (+0.9, +0.2), (1.2, 0.9), \dots,$$

find the maximally likely sequence $\hat{b}_1, \dots, \hat{b}_5$.