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Handout 17 Information Theory and Coding
Homework 7 Nov. 7, 2017

PROBLEM 1. A source produces independent, equally probable symbols from an alphabet (a_1, a_2) at a rate of one symbol every 3 seconds. These symbols are transmitted over a binary symmetric channel which is used once each second by encoding the source symbol a_1 as 000 and the source symbol a_2 as 111. If in the corresponding 3 second interval of the channel output, any of the sequences 000,001,010,100 is received, a_1 is decoded; otherwise, a_2 is decoded. Let $\epsilon < 1/2$ be the channel crossover probability.

(a) For each possible received 3-bit sequence in the interval corresponding to a given source letter, find the probability that a_1 came out of the source given that received sequence.

(b) Using part (a), show that the above decoding rule minimizes the probability of an incorrect decision.

(c) Find the probability of an incorrect decision (using part (a) is not the easy way here).

(d) If the source is slowed down to produce one letter every $2n + 1$ seconds, a_1 being encoded by $2n + 1$ 0's and a_2 being encoded by $2n + 1$ 1's. What decision rule minimizes the probability of error at the decoder? Find the probability of error as $n \to \infty$.

PROBLEM 2. Consider two discrete memoryless channels. The first channel has input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$; the second channel has input alphabet $\mathcal{Y}$ and output alphabet $\mathcal{Z}$. The first channel is described by the conditional probabilities $P_1(y|x)$ and the second channel by $P_2(z|y)$. Let the capacities of these channels be C_1 and C_2. Consider a third memoryless channel described by probabilities

$$P_3(z|x) = \sum_{y \in \mathcal{Y}} P_2(z|y) P_1(y|x), \quad x \in \mathcal{X},\, z \in \mathcal{Z}.$$

(a) Show that the capacity C_3 of this third channel satisfies

$$C_3 \le \min\{C_1, C_2\}.$$

(b) A helpful statistician preprocesses the output of the first channel by forming $\tilde{Y} = g(Y)$. He claims that this will strictly improve the capacity.

(b1) Show that he is wrong.

(b2) Under what conditions does he not strictly decrease the capacity?

PROBLEM 3. Consider a random source $\mathcal{S}$ of information, and let W be a random variable which represents the first L symbols $U_1, \dots, U_L$ of this source, i.e., $W = U_1^L$. We want to transmit the value of W using a memoryless stationary channel as follows:

- At time $t = 1$, we send $X_1 = f_1(W)$ through the channel.

- At time $t = i+1 \le n$, we send $X_{i+1} = f_i(W, Y^i)$ through the channel. $Y_1, \dots, Y_i$ are the output of the channel at times $t = 1, \dots, i$ respectively,

$f_1, \dots, f_n$ are n mappings that constitute the encoder. Clearly, this is a communication system with feedback as we are using the value of Y^i in the computation of X_{i+1}.

In the previous problem, we gave an example which satisfies $I(X^n; Y^n) > nC$ and $I(W; Y^n) \le nC$. Show that the inequality $I(W; Y^n) \le nC$ always holds by justifying each of the following equalities and inequalities:

$$
\begin{aligned}
I(W; Y^n) &\overset{(a)}{=} \sum_{i=1}^{n} I(W; Y_i | Y^{i-1}) \overset{(b)}{\le} \sum_{i=1}^{n} I(W, Y^{i-1}; Y_i) \overset{(c)}{\le} \sum_{i=1}^{n} I(W, X_i, X^{i-1}, Y^{i-1}; Y_i) \\
&\overset{(d)}{=} \sum_{i=1}^{n} I(X_i, X^{i-1}, Y^{i-1}; Y_i) \overset{(e)}{=} \sum_{i=1}^{n} I(X_i; Y_i) \overset{(f)}{\le} nC.
\end{aligned}
$$

Since $I(W; Y^n)$ represents the amount of information that is shared with the receiver, the inequality $I(W; Y^n) \le nC$ shows that feedback does not increase the capacity.

Problem 4. *Channels with memory have higher capacity.* Consider a binary symmetric channel with $Y_i = X_i \oplus Z_i$, where $\oplus$ is mod 2 addition, and $X_i, Y_i \in \{0, 1\}$.

Suppose that $\{Z_i\}$ has constant marginal probabilities $\Pr\{Z_i = 1\} = p = 1 - \Pr\{Z_i = 0\}$, but that $Z_1, Z_2, \dots, Z_n$ are not necessarily independent. Assume that $(Z_1, \dots, Z_n)$ is independent of the input $(X_1, \dots, X_n)$. Let $C = \log 2 - H(p, 1-p)$. Show that

$$\max_{p_{X_1, X_2, \dots, X_n}} I(X_1, X_2, \dots, X_n; Y_1, Y_2, \dots, Y_n) \ge nC.$$

Problem 5. Consider two discrete memoryless channels. The input alphabet, output alphabet, transition probabilities and capacity of the k'th channel is given by $\mathcal{X}_k$, $\mathcal{Y}_k$, p_k and C_k respectively ($k = 1, 2$). The channels operate independently. A communication system has access to both channels, that is, the effective channel between the transmitter and receiver has input alphabet $\mathcal{X}_1 \times \mathcal{X}_2$, output alphabet $\mathcal{Y}_1 \times \mathcal{Y}_2$ and transition probabilities $p_1(y_1|x_1)p_2(y_2|x_2)$. Find the capacity of this channel.

Problem 6. Let P_1 and P_2 be two channels of input alphabet $\mathcal{X}_1$ and $\mathcal{X}_2$ and of output alphabet $\mathcal{Y}_1$ and $\mathcal{Y}_2$ respectively. Consider a communication scheme where the transmitter chooses the channel (P_1 or P_2) to be used and where the receiver knows which channel were used. This scheme can be formalized by the channel P of input alphabet $\mathcal{X} = (\mathcal{X}_1 \times \{1\}) \cup (\mathcal{X}_2 \times \{2\})$ and of output alphabet $\mathcal{Y} = (\mathcal{Y}_1 \times \{1\}) \cup (\mathcal{Y}_2 \times \{2\})$, which is defined as follows:

$$P(y, k' | x, k) = \begin{cases} P_k(y|x) & \text{if } k' = k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = (X_k, K)$ be a random variable in $\mathcal{X}$ which will be the input distribution to the channel P, and let $Y = (Y_k, K) \in \mathcal{Y}$ be the output distribution. Define X_1 as being the random variable in $\mathcal{X}_1$ obtained by conditioning X_k on $K = 1$. Similarly define X_2, Y_1 and Y_2. Let α be the probability that $K = 1$.

(a) Show that $I(X; Y) = h_2(\alpha) + \alpha I(X_1; Y_1) + (1-\alpha) I(X_2; Y_2)$.

(b) What is the input distribution X that achieves the capacity of P?

(c) Show that the capacity C of P satisfies $2^C = 2^{C_1} + 2^{C_2}$, where C_1 and C_2 are the capacities of P_1 and P_2 respectively.