PROBLEM 1. Suppose $U$ is $\{0, 1\}$ valued with $\mathbb{P}(U = 0) = \mathbb{P}(U = 1) = 1/2$. Suppose we have a distortion measure $d$ given by 
\[
d(u, v) = \begin{cases} 
0 & \text{if } u = v, \\
1 & \text{if } (u, v) = (1, 0), \\
\infty & \text{if } (u, v) = (0, 1)
\end{cases}
\]
I.e., we never want to represent a 0 with a 1. Find $R(D)$.

PROBLEM 2. Suppose $\mathcal{U} = \mathcal{V}$ are additive groups with group operation $\oplus$. (E.g., $\mathcal{U} = \mathcal{V} = \{0, \ldots, K - 1\}$, with modulo $K$ addition.) Suppose the distortion measure $d(u, v)$ depends only on the difference between $u$ and $v$ and is given by $g(u - v)$. Let $\phi(D)$ denote $\max H(Z) : E[g(Z)] <= D$.

a) Show that $\phi(D)$ is convex.

b) Let $(U, V)$ be such that $E[d(U, V)] <= D$. Show that $I(U; V) >= H(U) - \phi(D)$ by justifying
\[
I(U; V) = H(U) - H(U|V) = H(U) - H(U - V|V) >= H(U) - H(U - V) >= H(U) - \phi(D).
\]

c) Show that $R(D) >= H(U) - \phi(D)$.

d) Assume now that $U$ is uniform on $\mathcal{U}$. Show that $R(D) = H(U) - \phi(D)$.

PROBLEM 3. Suppose $\mathcal{U} = \mathcal{V} = \mathbb{R}$, the set of real numbers, and $d(u, v) = (u - v)^2$. Show that for any $U$ with variance $\sigma^2$, $R(D)$ satisfies
\[
H(X) - (1/2) \log(2\pi \epsilon \sigma^2) <= R(D) <= (1/2) \log(\sigma^2/D).
\]

PROBLEM 4. Consider a two-way communication system where two parties communicate via a common output they both can observe and influence. Denote the common output by $Y$, and the signals emitted by the two parties by $x_1$ and $x_2$ respectively. Let $p(y|x_1, x_2)$ model the memoryless channel through which the two parties influence the output.

We will consider feedback-free block codes, i.e., we will use encoding and decoding functions of the form
\[
\begin{align*}
\text{enc}_1 : \{1, \ldots, 2^{nR_1}\} &\rightarrow \mathcal{X}_1^n \\
\text{dec}_1 : \mathcal{Y}^n \times \{1, \ldots, 2^{nR_1}\} &\rightarrow \{1, \ldots, 2^{nR_2}\} \\
\text{enc}_2 : \{1, \ldots, 2^{nR_2}\} &\rightarrow \mathcal{X}_2^n \\
\text{dec}_2 : \mathcal{Y}^n \times \{1, \ldots, 2^{nR_2}\} &\rightarrow \{1, \ldots, 2^{nR_1}\}
\end{align*}
\]
with which the parties encode their own message and decode the other party’s messages. (Note that when a party is decoding the other party’s message, it can make use of the knowledge of its own message).

We will say that the rate pair $(R_1, R_2)$ is achievable, if for any $\epsilon > 0$, there exist encoders and decoders with the above form for which the average error probability is less than $\epsilon$.

Consider the following ‘random coding’ method to construct the encoders:

(i) Choose probability distributions $p_j$ on $\mathcal{X}_j$, $j = 1, 2$.

(ii) Choose $\{\text{enc}_1(m_1)i : m_1 = 1, \ldots, 2^{nR_1}, i = 1, \ldots, n\}$ i.i.d., each having distribution as $p_1$. Similarly, choose $\{\text{enc}_2(m_2)i : m_2 = 1, \ldots, 2^{nR_2}, i = 1, \ldots, n\}$ i.i.d., each having distribution as $p_2$, independently of the choices for $\text{enc}_1$. 


For the decoders we will use typicality decoders:

(i) Set \( p(x_1, x_2, y) = p_1(x_1)p_2(x_2)p(y|x_1, x_2) \). Choose a small \( \epsilon > 0 \) and consider the set \( T \) of \( \epsilon \)-typical \( (x_1^n, x_2^n, y^n) \)'s with respect to \( p \).

(ii) For decoder 1: given \( y^n \) and the correct \( m_1 \), \( \text{dec}_1 \) will declare \( \hat{m}_2 \) if it is the unique \( m_2 \) for which \( (\text{enc}_1(m_1), \text{enc}_2(m_2), y^n) \in T \). If there is no such \( m_2 \), \( \text{dec}_1 \) outputs 0. (Similar description applies to Decoder 2.)

(a) Given that \( m_1 \) and \( m_2 \) are the transmitted messages, show that

\[ (\text{enc}_1(m_1), \text{enc}_2(m_2), Y^n) \in T \]  

with high probability.

(b) Given that \( m_1 \) and \( m_2 \) are the transmitted messages, and \( \hat{m}_1 \neq m_1 \) what is the probability distribution of \( (\text{enc}_1(\hat{m}_1), \text{enc}_2(m_2), Y^n) \)?

(c) Under the assumptions in (b) show that the

\[ \Pr\{(\text{enc}_1(\hat{m}_1), \text{enc}_2(m_2), Y^n) \in T\} \leq 2^{-nI(X_1;X_2|Y) + O(\delta)}. \]

(d) Show that all rate pairs satisfying

\[ R_1 \leq I(X_1; Y X_2), \quad R_2 \leq I(X_2; Y X_1) \]

for some \( p(x_1, x_2) = p(x_1)p(x_2) \) are achievable.

(e) For the case when \( X_1, X_2, Y \) are all binary and \( Y \) is the product of \( X_1 \) and \( X_2 \), show that the achievable region is strictly larger than what we can obtain by ‘half duplex communication’ (i.e., the set of rates that satisfy \( R_1 + R_2 \leq 1 \)).

**Problem 5.** In class, when proving the ‘good news’ part of the rate distortion theorem we stated that for any given \( u^n \) that is typical with respect to \( p_U \), when \( V^n \) has i.i.d. \( p_V \) components, the probability that \( (u^n, V^n) \) is typical with respect to \( p_{UV} \) is approximately \( 2^{-nI(U;V)} \), but gave a heuristic argument for it. In this problem we will give a proof.

To that end fix \( p_{UV} \), and suppose \( u^n \in T(p_{U}, n, \delta) \). For each \( u \in \mathcal{U} \), let \( J(u) = \{1 \leq i \leq n : u_i = u\} \) be the indices for which \( u_i \) equals \( u \). Suppose \( V^n \) has i.i.d. components each distributed according to \( p_V \).

(a) Let \( V(u) = (V_i : i \in J(u)) \) denote the subvector of \( V^n \) by considering only the indices in \( J(u) \). Note that \( V(u) \) is of dimension \( n(u) = |J(u)| \) with i.i.d. components each distributed according to \( p_V \). What is the probability that \( V(u) \) belongs to \( T(p_{V|U=u}, n(u), \delta) \), expressed in the form \( 2^{-n(u)(F(\cdot) + O(\delta))} \)?

(b) Using (a), show that the probability that \( V(u) \in T(p_{V|U=u}, n(u), \delta) \) for every \( u \in \mathcal{U} \)

equals \( 2^{-n(F + O(\delta))} \) where

\[ F = \sum_u \frac{n(u)}{n} D(p_{V|U=u}\|p_V). \]

(c) Using the fact that \( u^n \) is in \( T(n, p_U, \delta) \), show that \( F \) in (b) equals

\[ \sum_u p_U(u) \sum_v p_{V|U=u}(v|u) \log \frac{p_{V|U=u}(v|u)}{p_V(v)} + O(\delta), \]

and conclude that the probability we found in (b) equals \( 2^{-n(I(U;V) + O(\delta))} \).
(d) Show that when $u^n \in T(n, p_U, \delta)$ and $V(u) \in T(n(u), p_{V|U=u}, \delta)$ for every $u \in \mathcal{U}$, we will necessarily have $(u^n, V^n)$ belonging to $T(n, p_{UV}, 2\delta + \delta^2)$.

(e) Conclude that for any $1 \geq \delta' \geq 3\delta$, for any $u^n \in T(n, p_U, \delta)$, with $V^n$ i.i.d. $p_V$, the probability that $(u^n, V^n) \in T(n, p_{UV}, \delta')$ is at least $2^{-n(I(U;V) + O(\delta))}$. 