

# ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

**Handout 2**  
Homework 1

Information Theory and Coding  
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PROBLEM 1. Three events  $E_1$ ,  $E_2$  and  $E_3$ , defined on the same probability space, have probabilities  $P(E_1) = P(E_2) = P(E_3) = 1/4$ . Let  $E_0$  be the event that one or more of the events  $E_1$ ,  $E_2$ ,  $E_3$  occurs.

(a) Find  $P(E_0)$  when:

- (1) The events  $E_1$ ,  $E_2$  and  $E_3$  are disjoint.
- (2) The events  $E_1$ ,  $E_2$  and  $E_3$  are statistically independent.
- (3) The events  $E_1$ ,  $E_2$  and  $E_3$  are in fact three names for the same event.

(b) Find the maximum value  $P(E_0)$  can assume when:

- (1) Nothing is known about the independence or disjointness of  $E_1$ ,  $E_2$ ,  $E_3$ .
- (2) It is known that  $E_1$ ,  $E_2$  and  $E_3$  are *pairwise independent*, i.e., that the probability of realizing both  $E_i$  and  $E_j$  is  $P(E_i)P(E_j)$ ,  $1 \leq i \neq j \leq 3$ , but nothing is known about the probability of realizing all three events together.

(c) Suppose now that events  $E_1$ ,  $E_2$  and  $E_3$  all have probability  $p$ , that they are pairwise independent, and that  $E_0$  has probability 1. Show that  $p$  has to be at least  $1/2$ .

PROBLEM 2. A dishonest gambler has a loaded die which turns up the number 1 with probability  $2/3$  and the numbers 2 to 6 with probability  $1/15$  each. Unfortunately, he has left his loaded die in a box with two honest dice and can not tell them apart. He picks one die (at random) from the box, rolls it once, and the number 1 appears. Conditional on this result, what is the probability that he picked up the loaded die? He now rolls the dice once more and it comes up 1 again. What is the possibility after this second rolling that he has picked up the loaded die?

PROBLEM 3. Suppose the random variables  $A$ ,  $B$ ,  $C$ ,  $D$  form a Markov chain:  $A \leftrightarrow B \leftrightarrow C \leftrightarrow D$ .

- (a) Is  $A \leftrightarrow B \leftrightarrow C$ ?
- (b) Is  $B \leftrightarrow C \leftrightarrow D$ ?
- (c) Is  $A \leftrightarrow (B, C) \leftrightarrow D$ ?
- (d) Is  $A \leftrightarrow B \leftrightarrow (C, D)$ ?

PROBLEM 4. Suppose the random variables  $A$ ,  $B$ ,  $C$ ,  $D$  satisfy  $A \leftrightarrow B \leftrightarrow C$ , and  $B \leftrightarrow C \leftrightarrow D$ . Does it follow from these that  $A \leftrightarrow B \leftrightarrow C \leftrightarrow D$ ?

PROBLEM 5. Let  $X$  and  $Y$  be two random variables.

- (a) Prove that the expectation of the sum of  $X$  and  $Y$ ,  $E[X + Y]$ , is equal to the sum of the expectations,  $E[X] + E[Y]$ .

- (b) Prove that if  $X$  and  $Y$  are statistically independent, then  $X$  and  $Y$  are also uncorrelated (by definition  $X$  and  $Y$  are uncorrelated if  $E[XY] = E[X]E[Y]$ ). Find an example in which  $X$  and  $Y$  are statistically dependent yet uncorrelated.
- (c) Prove that if  $X$  and  $Y$  are statistically independent, then the variance of the sum  $X + Y$  is equal to the sum of variances. Is this relationship valid if  $X$  and  $Y$  are uncorrelated but not statistically independent?

PROBLEM 6. After summer, the winter tyres of a car (with four wheels) are to be put back. However, the owner has forgotten which tyre goes to which wheel, and the tyres are installed ‘randomly’, each of the  $4! = 24$  permutations being equally likely.

- (a) What is the probability that tyre 1 is installed in its original position?
- (b) What is the probability that all the tyres are installed in their original positions?
- (c) What is the expected number of tyres that are installed in their original positions?
- (d) Redo the above for a vehicle with  $n$  wheels.
- (e) (Harder.) What is the probability that none of the wheels are installed in their original positions.

PROBLEM 7. We construct an ‘inventory’ by drawing  $n$  independent samples from a distribution  $p$ . Let  $X_1, \dots, X_n$  be the random variables that represent the drawings.

Suppose  $X$  is drawn from distribution  $p$ , independent of  $X_1, \dots, X_n$ .

- (a) What is the probability that  $X$  does not appear in the inventory?
- (b) Redo (a) for the special case when  $p$  is the uniform distribution over  $n$  items.
- (c) What happens to the probability in (b) when  $n$  gets large?