Homework 6 : 29 March 2018
Calcul Quantique

Exercise 1 *Effet des imperfections sur l'algorithme de Simon*

1. L'état juste après les deux premières portes est :

$$\begin{aligned} H_0 \otimes H_1 (|0\rangle \otimes |0\rangle \otimes |0\rangle) &= H_0 |0\rangle \otimes H_1 |0\rangle \otimes |0\rangle \\ &= \frac{1}{2} \big(|0\rangle + (-1)^0 e^{i\varphi_0} |1\rangle\big) \otimes \big(|0\rangle + (-1)^0 e^{i\varphi_1} |1\rangle\big) \otimes |0\rangle \\ &= \frac{1}{2} \big(|00\rangle + e^{i\varphi_1} |01\rangle + e^{i\varphi_0} |10\rangle + e^{i\varphi_0 + i\varphi_1} |11\rangle\big) \otimes |0\rangle \end{aligned}$$

2. Après U_f :

$$\begin{aligned} &\frac{1}{2} \big\{U_f (|00\rangle \otimes |0\rangle) + e^{i\varphi_1} U_f (|01\rangle \otimes |0\rangle) + e^{i\varphi_0} U_f (|10\rangle \otimes |0\rangle) + e^{i\varphi_0 + i\varphi_1} U_f (|11\rangle \otimes |0\rangle)\big\} \\ &\quad = \frac{1}{2} \big\{|00\rangle \otimes |0\rangle + e^{i\varphi_1} |01\rangle \otimes |0\rangle + e^{i\varphi_0} |10\rangle \otimes |1\rangle + e^{i\varphi_0 + i\varphi_1} |11\rangle \otimes |0\rangle\big\}, \end{aligned}$$

où on a utilisé $f(00) = f(01) = 0$ et $f(10) = f(11) = 1$.

Enfin appliquons les deux dernières portes de Hadamard $H \otimes H$ aux deux premiers qu-bits :

$$\begin{aligned} &\frac{1}{2} (H \otimes H |00\rangle) \otimes |0\rangle + \frac{1}{2} e^{i\varphi_1} (H \otimes H |01\rangle) \otimes |0\rangle + \\ &\qquad + \frac{1}{2} e^{i\varphi_0} (H \otimes H |10\rangle) \otimes |1\rangle + \frac{1}{2} e^{i(\varphi_0 + \varphi_1)} (H \otimes H |11\rangle) \otimes |1\rangle \\ &= \frac{1}{4} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes |0\rangle \\ &\quad + \frac{1}{4} e^{i\varphi_1} (|00\rangle - |01\rangle + |10\rangle - |11\rangle) \otimes |0\rangle \\ &\quad + \frac{1}{4} e^{i\varphi_0} (|00\rangle + |01\rangle - |10\rangle - |11\rangle) \otimes |1\rangle \\ &\quad + \frac{1}{4} e^{i(\varphi_0 + \varphi_1)} (|00\rangle - |01\rangle - |10\rangle + |11\rangle) \otimes |1\rangle . \end{aligned}$$

L' état juste avant la mesure est :

$$\begin{aligned} &\frac{1}{4} \big\{\big(1 + e^{i\phi_1}\big) |00\rangle + \big(1 - e^{i\phi_1}\big) |01\rangle + \big(1 + e^{i\phi_1}\big) |10\rangle + \big(1 - e^{i\phi_1}\big) |11\rangle\big\} \otimes |0\rangle \\ &+ \frac{e^{i\varphi_0}}{4} \big\{\big(1 + e^{i\phi_1}\big) |00\rangle + \big(1 - e^{i\phi_1}\big) |01\rangle + \big(1 + e^{i\phi_1}\big) |10\rangle + \big(1 - e^{i\phi_1}\big) |11\rangle\big\} \otimes |1\rangle \end{aligned}$$

3. Mesure des deux premiers qu-bits :

$$\begin{aligned}
\text{Prob}[00] &= \frac{1}{16}\left|1+e^{i\varphi_1}\right|^2 + \frac{1}{16}\left|1+e^{i\varphi_1}\right|^2 = \frac{1}{2}\cos^2\frac{\varphi_1}{2},\\
\text{Prob}[01] &= \frac{1}{16}\left|1-e^{i\varphi_1}\right|^2 + \frac{1}{16}\left|1-e^{i\varphi_1}\right|^2 = \frac{1}{2}\sin^2\frac{\varphi_1}{2},\\
\text{Prob}[10] &= \frac{1}{16}\left|1+e^{i\varphi_1}\right|^2 + \frac{1}{16}\left|1+e^{i\varphi_1}\right|^2 = \frac{1}{2}\cos^2\frac{\varphi_1}{2},\\
\text{Prob}[11] &= \frac{1}{16}\left|1-e^{i\varphi_1}\right|^2 + \frac{1}{16}\left|1-e^{i\varphi_1}\right|^2 = \frac{1}{2}\sin^2\frac{\varphi_1}{2}.
\end{aligned}$$

4. Pour toujours trouver un vecteur dans $H^\perp = \{(0,0);(1,0)\}$, il est nécessaire et suffisant de prendre $\varphi_1 = 0$ et φ_0 quelconque :

$$\begin{aligned}
&\begin{cases}\text{Prob}[00] = \frac{1}{2}\\ \text{Prob}[10] = \frac{1}{2}\end{cases} &&\Longrightarrow \text{Prob}[(0,0)\text{ ou }(1,0)] = 1,\\
&\begin{cases}\text{Prob}[01] = 0\\ \text{Prob}[11] = 0\end{cases} &&\Longrightarrow \text{Prob}[(0,1)\text{ ou }(1,1)] = 0.
\end{aligned}$$

En général, avec $\varphi_1 \neq 0$, on a

$$\begin{aligned}
\text{Prob}[00\text{ ou }10] &= \frac{1}{2}\cos^2\frac{\varphi_1}{2} + \frac{1}{2}\cos^2\frac{\varphi_1}{2} = \cos^2\frac{\varphi_1}{2},\\
\text{Prob}[01\text{ ou }11] &= \frac{1}{2}\sin^2\frac{\varphi_1}{2} + \frac{1}{2}\sin^2\frac{\varphi_1}{2} = \sin^2\frac{\varphi_1}{2}.
\end{aligned}$$