Exercise Set 3 : 8 March 2018
Calcul Quantique

Exercise 1 *Production of Bell states*

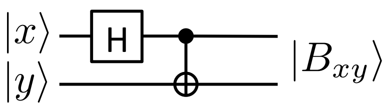
a) Direct computation gives

$$\begin{aligned}(CNOT)(H \otimes I)\,|x\rangle \otimes |y\rangle &= (CNOT)H\,|x\rangle \otimes |y\rangle \\ &= (CNOT)\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^x\,|1\rangle\right) \otimes |y\rangle \\ &= \frac{1}{\sqrt{2}}CNOT\,|0,y\rangle + \frac{(-1)^x}{\sqrt{2}}CNOT\,|1,y\rangle \\ &= \frac{1}{\sqrt{2}}\,|0,y\rangle + \frac{(-1)^x}{\sqrt{2}}\,|1,y\oplus 1\rangle\end{aligned}$$

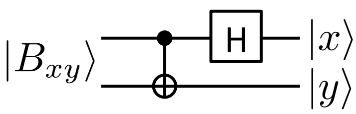
More explicitly, we enumerate all the cases :

$$\begin{aligned}(CNOT)(H \otimes I)\,|00\rangle &= (CNOT)\frac{1}{\sqrt{2}}\big(\,|00\rangle + |10\rangle\,\big) = \frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big) = |B_{00}\rangle \\ (CNOT)(H \otimes I)\,|01\rangle &= (CNOT)\frac{1}{\sqrt{2}}\big(\,|01\rangle + |11\rangle\,\big) = \frac{1}{\sqrt{2}}\big(\,|01\rangle + |10\rangle\,\big) = |B_{01}\rangle \\ (CNOT)(H \otimes I)\,|10\rangle &= (CNOT)\frac{1}{\sqrt{2}}\big(\,|00\rangle - |10\rangle\,\big) = \frac{1}{\sqrt{2}}\big(\,|00\rangle - |11\rangle\,\big) = |B_{10}\rangle \\ (CNOT)(H \otimes I)\,|11\rangle &= (CNOT)\frac{1}{\sqrt{2}}\big(\,|00\rangle - |11\rangle\,\big) = \frac{1}{\sqrt{2}}\big(\,|01\rangle - |10\rangle\,\big) = |B_{11}\rangle\end{aligned}$$

b) The circuit corresponding to $|B_{xy}\rangle = (CNOT)(H \otimes I)|x\rangle \otimes |y\rangle$:



c) The circuit corresponding to $|x\rangle \otimes |y\rangle = (H \otimes I)(CNOT)|B_{xy}\rangle$:



Exercise 2 *Construction of a multi-control-U.*

We show the quantum state at each stage of the circuit.

$$\begin{aligned}
\text{Input}: &\ |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |0\rangle \otimes |0\rangle \otimes |c_t\rangle \\
\text{After the 1st Toffoli gate}: &\ |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |0\rangle \otimes |c_t\rangle \\
\text{After the 2nd Toffoli gate}: &\ |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |c_1 \cdot c_2 \cdot c_3\rangle \otimes |c_t\rangle \\
\text{After the contolled-}U\text{ gate}: &\ |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |c_1 \cdot c_2 \cdot c_3\rangle \otimes U^{c_1c_2c_3} |c_t\rangle \\
\text{After the 3rd Toffoli gate}: &\ |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |0\rangle \otimes U^{c_1c_2c_3} |c_t\rangle \\
\text{After the 4th Toffoli gate}: &\ |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |0\rangle \otimes |0\rangle \otimes U^{c_1c_2c_3} |c_t\rangle
\end{aligned}$$

Exercise 3 *Construction of the Toffoli gate from a control-NOT (Indication : long calculation).*

Note that $(CNOT)\,|x\rangle \otimes |y\rangle = |x\rangle \otimes |x \oplus y\rangle$ can also be represented as $|x\rangle \otimes X^x\,|y\rangle$, where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is one of the Pauli gates (see Chapter 3). Therefore, the circuit outputs the tensor product state $|\psi\rangle$ given by

$$|\psi\rangle = T\,|c_1\rangle \otimes SX^{c_1}T^\dagger X^{c_1}T^\dagger\,|c_2\rangle \otimes HTX^{c_1}T^\dagger X^{c_2}TX^{c_1}T^\dagger X^{c_2}H\,|t\rangle\,.$$

We then verify explicitly all the cases of c_1 and c_2. The calculation largely uses the fact that all the quantum gates here are unitary (*e.g.*, $TT^\dagger = T^\dagger T = I$) ; in particular, the gates X and H are involutory, *i.e.*, $X^2 = H^2 = I$.

For $c_1 = 0$, we have

$$\begin{aligned}
|\psi\rangle &= T\,|0\rangle \otimes ST^\dagger T^\dagger\,|c_2\rangle \otimes HTT^\dagger X^{c_2}TT^\dagger X^{c_2}H\,|t\rangle \\
&= |0\rangle \otimes |c_2\rangle \otimes H\left(TT^\dagger\right)\left(X^{c_2}\left(TT^\dagger\right)X^{c_2}\right)H\,|t\rangle \\
&= |0\rangle \otimes |c_2\rangle \otimes |t\rangle \\
&= |0\rangle \otimes |c_2\rangle \otimes |t \oplus 0 \cdot c_2\rangle
\end{aligned}$$

For $c_1 = 1$ and $c_2 = 0$, we calculate

$$XT^\dagger X = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & 1 \end{pmatrix} = e^{-i\pi/4}T$$

and therefore we have

$$\begin{aligned}
|\psi\rangle &= T\,|1\rangle \otimes SXT^\dagger XT^\dagger\,|0\rangle \otimes HTXT^\dagger TXT^\dagger H\,|t\rangle \\
&= e^{i\pi/4}\,|1\rangle \otimes S\left(XT^\dagger X\right)T^\dagger\,|0\rangle \otimes H\left(T\left(X\left(T^\dagger T\right)X\right)T^\dagger\right)H\,|t\rangle \\
&= e^{i\pi/4}\,|1\rangle \otimes e^{-i\pi/4}STT^\dagger\,|0\rangle \otimes |t\rangle \\
&= e^{i\pi/4}\,|1\rangle \otimes e^{-i\pi/4}\,|0\rangle \otimes |t\rangle \\
&= |1\rangle \otimes |0\rangle \otimes |t\rangle \\
&= |1\rangle \otimes |0\rangle \otimes |t \oplus 1 \cdot 0\rangle\,.
\end{aligned}$$

Finally, for $c_1 = c_2 = 1$, we calculate

$$\begin{aligned}\left(TXT^\dagger X\right)^2 &= \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{-i\pi/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = e^{-i\pi/2} Z, \\ HZH &= X\end{aligned}$$

and therefore we have

$$\begin{aligned}|\psi\rangle &= T\,|1\rangle \otimes SXT^\dagger XT^\dagger\,|1\rangle \otimes HTXT^\dagger XTXT^\dagger XH\,|t\rangle \\ &= e^{i\pi/4}\,|1\rangle \otimes S\left(XT^\dagger X\right)T^\dagger\,|1\rangle \otimes H\left(TXT^\dagger X\right)^2 H\,|t\rangle \\ &= e^{i\pi/4}\,|1\rangle \otimes e^{-i\pi/4} STT^\dagger\,|1\rangle \otimes e^{-i\pi/2} HZH\,|t\rangle \\ &= e^{i\pi/4}\,|1\rangle \otimes e^{i\pi/4}\,|1\rangle \otimes e^{-i\pi/2} X\,|t\rangle \\ &= |1\rangle \otimes |1\rangle \otimes X\,|t\rangle \\ &= |1\rangle \otimes |1\rangle \otimes |t \oplus 1\rangle\,.\end{aligned}$$

Exercise 4 *Unitary representation of a reversible computation.*

a) The relevant Hilbert space is $(\mathbb{C}^2)^{\otimes(n+1)}$. A unitary matrix is precisely a matrix that conserves the scalar product. Therefore, to show U_f is a unitary matrix, we show it conserves the scalar product. For any two states $|x_1, \ldots, x_n; y\rangle$ and $|\hat{x}_1, \ldots, \hat{x}_n; \hat{y}\rangle$, we have

$$\begin{aligned}&\langle x_1, \ldots, x_n; y|\, U_f^\dagger U_f\, |\hat{x}_1, \ldots, \hat{x}_n; \hat{y}\rangle \\ &\quad = \langle x_1, \ldots, x_n; y \oplus f(x_1, \ldots, x_n) | \hat{x}_1, \ldots, \hat{x}_n; \hat{y} \oplus f(\hat{x}_1, \ldots, \hat{x}_n)\rangle \\ &\quad = \begin{cases} 1 & \text{if } x_i = \hat{x}_i \text{ for all } i \text{ and } y = \hat{y} \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

which is equivalent to $\langle x_1, \ldots, x_n; y | \hat{x}_1, \ldots, \hat{x}_n; \hat{y}\rangle$.

b) When the output of f is in $\{0, 1\}^m$, so we need m storage bits $y_1, \ldots, y_m$. The relevant Hilbert space would be $(\mathbb{C}^2)^{\otimes(n+m)}$. The same calculation in (a) gives

$$\langle x_1, \ldots, x_n; y_1, \ldots y_m|\, U_f^\dagger U_f\, |\hat{x}_1, \ldots, \hat{x}_n; \hat{y}_1, \ldots, \hat{y}_m\rangle = \langle x_1, \ldots, x_n; y_1, \ldots, y_m | \hat{x}_1, \ldots, \hat{x}_n; \hat{y}_1, \ldots, \hat{y}_m\rangle\,.$$

We conclude that U_f is unitary.