Exercise Set 1 : 22 February 2018
Calcul Quantique

Exercise 1 *Dirac's notation for vectors and matrices*

(a) If $|w\rangle$ is a vector and α is a scalar, then

$$\big(\alpha\,|w\rangle\,\big)^\dagger = \langle w|\,\alpha^* = \alpha^*\,\langle w|$$

(you can check this in components). Moreover, we have linearity :

$$\big(\alpha\,|v\rangle + \beta\,|w\rangle\,\big)^\dagger = \big(\alpha\,|v\rangle\,\big)^\dagger + \big(\beta\,|w\rangle\,\big)^\dagger.$$

Then we get

$$\begin{aligned}\langle v| = \big(\,|v\rangle\,\big)^\dagger &= \big(v_1\,|e_1\rangle + v_2\,|e_2\rangle + \cdots + v_N\,|e_N\rangle\,\big)^\dagger \\ &= v_1^*\,\langle e_1| + v_2^*\,\langle e_2| + \cdots + v_N^*\,\langle e_N|\,.\end{aligned}$$

(b) If $\langle v| = \sum_{i=1}^N v_i^*\,\langle e_i|$ and $|w\rangle = \sum_{j=1}^N w_j\,|e_j\rangle$, then

$$\begin{aligned}\langle v|w\rangle &= \sum_{i=1}^N\sum_{j=1}^N v_i^* w_j\,\langle e_i|e_j\rangle \\ &= \sum_{i=1}^N\sum_{j=1}^N v_i^* w_j \delta_{ij} \\ &= \sum_{i=1}^N v_i^* w_i.\end{aligned}$$

(c) Same method provided in (a).

(d) For $\vec{v} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, we have $\|\vec{v}\|^2 = \vec{v}^{T,*}\cdot\vec{v}$, so $\|\vec{v}\|^2 = \alpha^*\alpha + \beta^*\beta$. On the other hand, $\langle v|v\rangle = \alpha^*\alpha + \beta^*\beta$ also by (b).

(e) Write the ket-bra $\langle e_k|\,A\,|e_l\rangle$ as a matrix to realize that

$$|e_k\rangle\,\langle e_l| = {\scriptstyle k\text{-th pos}}\left\{\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}\right.\underbrace{\begin{pmatrix}0\cdots010\cdots0\end{pmatrix}}_{l\text{-th pos}} = \begin{array}{cc} & \begin{array}{ccccccc} & & & l & & & \end{array} \\ \begin{array}{c} \\ \\ \\ k \\ \\ \\ \\ \end{array} & \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0\\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}\end{array}$$

Thus,

$$A = \sum_{k,l} a_{kl} \left| e_k \right\rangle \left\langle e_l \right| .$$

So,

$$\begin{aligned}
\left\langle e_i \right| A \left| e_j \right\rangle &= \sum_{l,l} a_{kl} \left\langle e_i | e_k \right\rangle \left\langle e_l | e_j \right\rangle \\
&= \sum_{l,l} a_{kl} \delta_{ik} \delta_{lj} \\
&= a_{ij} .
\end{aligned}$$

(f) From the beginning of point (e), we have

$$I = \sum_{i=1}^{N} \left| e_i \right\rangle \left\langle e_i \right| .$$

Indeed, $\left| e_i \right\rangle \left\langle e_i \right|$ is the matrix with 1 at the i-th row and i-th column and zeros elsewehre. This is called the closure relation.

(g) First note that the closure relation is valid for any orthonormal basis. Indeed, if $\{\left|\varphi_i\right\rangle\}_{i=1\cdots N}$ are orthonormal, there exists a unitary basis change (a "rotation") such that

$$\begin{aligned}
\left| \varphi_i \right\rangle &= U \left| e_i \right\rangle , \\
\left\langle \varphi_i \right| &= \left\langle e_i \right| U^\dagger .
\end{aligned}$$

Then from $I = \sum_{i=1}^{N} \left| e_i \right\rangle \left\langle e_i \right|$ we get :

$$\begin{aligned}
U I U^\dagger &= \sum_{i=1}^{N} U \left| e_i \right\rangle \left\langle e_i \right| U^\dagger \\
I &= \sum_{i=1}^{N} \left| \varphi_i \right\rangle \left\langle \varphi_i \right| .
\end{aligned}$$

Now, from $\alpha_i \left| \varphi_i \right\rangle = A \left| \varphi_i \right\rangle$ we get

$$\sum_{i=1}^{N} \alpha_i \left| \varphi_i \right\rangle \left\langle \varphi_i \right| = \sum_{i=1}^{N} A \left| \varphi_i \right\rangle \left\langle \varphi_i \right| = A \sum_{i=1}^{N} \left| \varphi_i \right\rangle \left\langle \varphi_i \right| = AI = A.$$

Exercise 2 *Tensor Product in Dirac's notation*

(a) By distributivity of the tensor product (first two properties), it follows that :

$$\begin{aligned}
\left| v \right\rangle_1 \otimes \left| w \right\rangle_2 &= \left(\sum_{i=1}^{N} v_i \left| e_i \right\rangle_1 \right) \otimes \left(\sum_{j=1}^{M} w_j \left| f_j \right\rangle_2 \right) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{M} v_i w_j \left| e_i \right\rangle_1 \otimes \left| f_j \right\rangle_2 .
\end{aligned}$$

(b) Take two vectors $|e_i, f_j\rangle$ and $|e_k, f_l\rangle$ of $\mathcal{H}_1 \bigotimes \mathcal{H}_2$. Then by definition of the inner product :

$$\langle e_k, f_l | e_i, f_j\rangle = \langle e_k|e_i\rangle \langle f_l|f_j\rangle = \delta_{kl} \cdot \delta_{lj} = \delta_{(k,l);(l,j)}.$$

So this equals one if and only if $(k,l) = (i,j)$ and zero otherwise. This means that $\big\{ |e_i, f_j\rangle\, ; i = 1 \dots N; j = 1 \dots N \big\}$ is an orthonormal basis of $\mathcal{H}_1 \bigotimes \mathcal{H}_2$.
The dimension equals the number of basis vectors, so is NM, the product of $\dim \mathcal{H}_1$ and $\dim \mathcal{H}_2$.

(c) We apply the definition

$$A \otimes B\, |\Psi\rangle = \sum_{i,j} \psi_{ij} A\, |e_i\rangle_1 \otimes B\, |f_j\rangle_2$$

to $|\Psi\rangle = |e_k, f_l\rangle$. So $\psi_{ij} = 1$ for $(i,j) = (k,l)$ and 0 otherwise. This means :

$$A \otimes B\, |e_k, f_l\rangle = A\, |e_k\rangle \otimes B\, |f_l\rangle$$

and multiplying by $\langle e_i, f_j|$, we find :

$$\begin{aligned} \langle e_i, f_j|\, A \otimes B\, |e_k, f_l\rangle &= \big(\langle e_i| \otimes \langle f_j| \big)\big(A\, |e_k\rangle \otimes B\, |f_l\rangle \big) \\ &= \langle e_i|\, A\, |e_k\rangle \langle f_j|\, B\, |f_l\rangle \\ &= a_{ik} b_{jl}. \end{aligned}$$

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